Maximally Flat Functions

Consider some function $f(x)$. Say that we know the value of the function **at** x=1 is 5:

$$f(x=1)=5$$

This of course says **something** about the function $f(x)$, but it **doesn't** tell us much!

We can additionally determine the **first derivative** of this function, and likewise evaluate this derivative **at** x=1. Say that this value turns out to be **zero**:

$$\left.\frac{d f(x)}{d x}\right|_{x=1}=0$$

Note that this does not mean that the derivative of $f(x)$ is equal to zero, it merely means that the derivative of $f(x)$ is zero **at the value** $x=1$. Presumably, $d f(x)/dx$ is **non-zero** at **other** values of x.

So, we now have **two** pieces of information about the function $f(x)$. We can add to this list by continuing to take higher-order derivatives and evaluating them at the single point x=1.

Let's say that the values of **all** the derivatives (at x=1) turn out to have a zero value:

$$\left.\frac{d f^{n}(x)}{d x^{n}}\right|_{x=1}=0 \quad \text{for } n=1,2,3,\cdots,\infty$$

We say that this function is **completely flat** at the point x=1. Because **all** the derivatives are zero at x=1, it means that the function cannot change in value from that at x=1.

In other words, if the function has a value of 5 at x=1, (i.e., $f(x=1)=5$), then the function **must** have a value of 5 at **all** x !

The function $f(x)$ thus must be the **constant** function:

$$f(x)=5$$

Now let's consider the following **problem**—say some function $f(x)$ has the following form:

$$f(x)=a x^{3}+b x^{2}+c x$$

We wish to **determine** the values *a*, *b*, and *c* so that:

$$f(x=1)=5$$

and that the value of the function $f(x)$ is as **close** to a value of 5 as possible in the region where x =1.
In other words, we want the function to have the value of 5 at x=1, and to **change** from that value as **slowly** as possible as we

"move" from $x=1$.

Q: *Don't we simply want the **completely** flat function $f(x)=5$?*

A: That would be the **ideal** function for this case, but notice that solution is **not** an option. Note there are **no** values of *a*, *b*, and *c* that will make:

$$a x^3 + b x^2 + c x = 5$$

for **all** values x.

Q: *So **what** do we do?*

A: **Instead** of the completely flat solution, we can find the **maximally flat** solution!

The **maximally flat** solution comes from determining the values *a*, *b*, and *c* so that as many derivatives **as possible** are **zero** at the point x=1.

For example, we wish to make the **first derivate** equal to zero at $x=1$:

$$\begin{aligned} 0 &= \left.\frac{d f(x)}{d x}\right|_{x=1} \\ &= \left.\left(3ax^2 + 2bx + c\right)\right|_{x=1} \\ &= 3a + 2b + c \end{aligned}$$

Likewise, we wish to make the **second derivative** equal to zero at $x=1$:

$$\begin{aligned} 0 &= \left.\frac{d^2 f(x)}{d x^2}\right|_{x=1} \\ &= \left.(6ax + 2b)\right|_{x=1} \\ &= 6a + 2b \end{aligned}$$

Here we must **stop** taking derivatives, as our solution only has **three degrees of design freedom** (i.e., 3 unknowns a, b, c).

Q: *But we only have taken **two** derivatives, can't we take **one more**?*

A: **No!** We already have a **third** "design" equation: the value of the function **must** be 5 at $x=1$:

$$\begin{aligned} 5 &= f(x=1) \\ &= a(1)^3 + b(1)^2 + c(1) \\ &= a + b + c \end{aligned}$$

So, we have used the **maximally flat** criterion at $x=1$ to generate **three** equations and **three** unknowns:

$$5 = a + b + c$$

$$0 = 3a + 2b + c$$

$$0 = 6a + 2b$$

Solving, we find:

$$a = 5$$
$$b = -15$$
$$c = 15$$

Therefore, the maximally flat function (at $x=1$) is:

$$f(x) = 5x^3 - 15x^2 + 15x$$