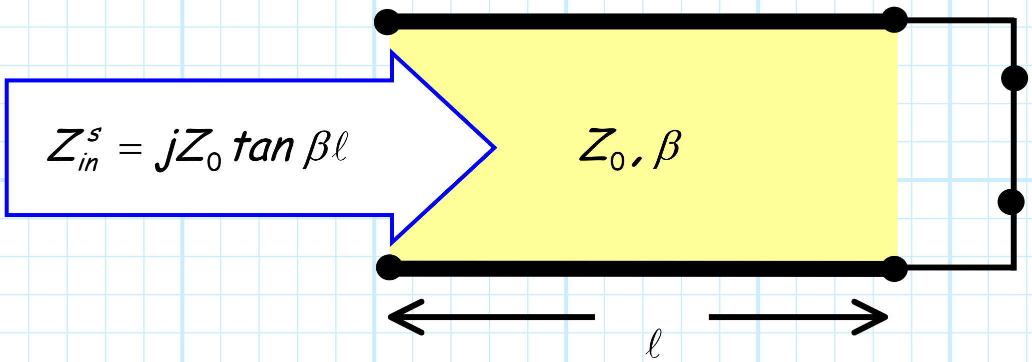
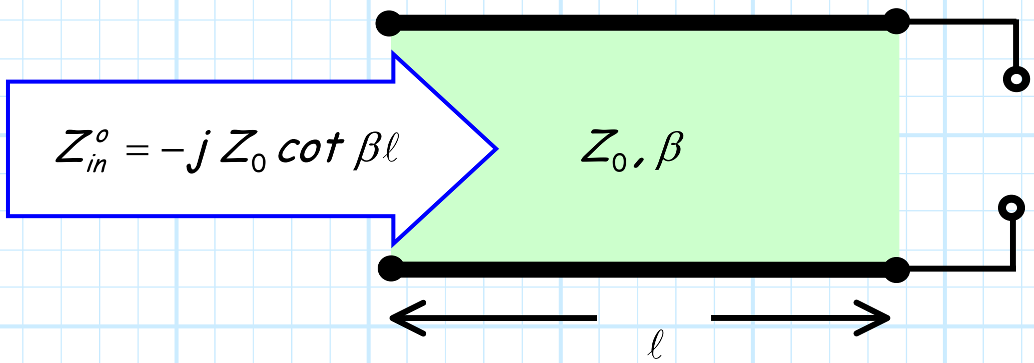
Richard's Transformations

Recall the input impedances of short-circuited and open-circuited transmission line **stubs**.





Note that the input impedances are purely **reactive**—just like **lumped** elements!

However, the reactance of lumped inductors and capacitors have a **different** mathematical form to that of transmission line stubs:

$$Z_L = j\omega L \qquad\qquad Z_C = \frac{-j}{\omega C}$$

In other words, the impedance of transmission line stubs and lumped elements (capacitors and inductors) are different functions with respect to **frequency**. Therefore, we can say in general that, for example:

$$Z_{in}^{s} \neq Z_L \qquad\qquad Z_{in}^{o} \neq Z_C$$

However, for a given lumped element (L or C) and a given stub (with a given Z_0 and length ℓ) the functions **will** be equal at precisely **one frequency**!

For example, there is one frequency—let's call it ω_c—that satisfies **this** equation for a given L, Z_0, and ℓ:

$$\begin{aligned} j\omega_c L &= j\, Z_0 \tan \beta_c \ell \\ &= j\, Z_0 \tan\left[\frac{\omega_c}{v_p}\ell\right] \end{aligned}$$

or similarly satisfies **this** equation:

$$\begin{aligned} \frac{-j}{\omega_c C} &= -j\, Z_0 \cot \beta_c \ell \\ &= -j\, Z_0 \cot\left[\frac{\omega_c}{v_p}\ell\right] \end{aligned}$$

To make things easier, let's set the **length** of our transmission line stub to $\lambda_c/8$, where:

$$\lambda_c = \frac{v_p}{\omega_c} = \frac{2\pi}{\beta_c}$$

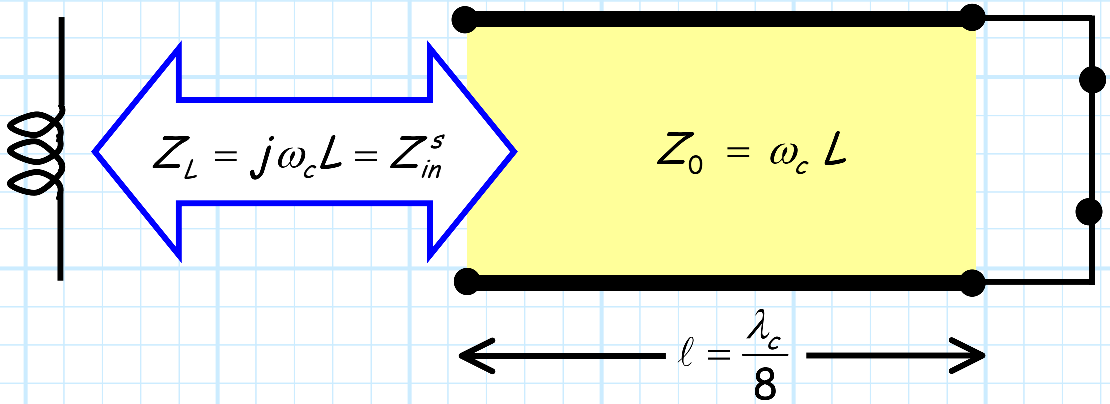
Q: *Why* $\ell = \lambda_c/8$?

A: Well, for **one** reason, $\beta_c \ell = \pi/4$ and therefore $\tan(\pi/4) = 1.0$!

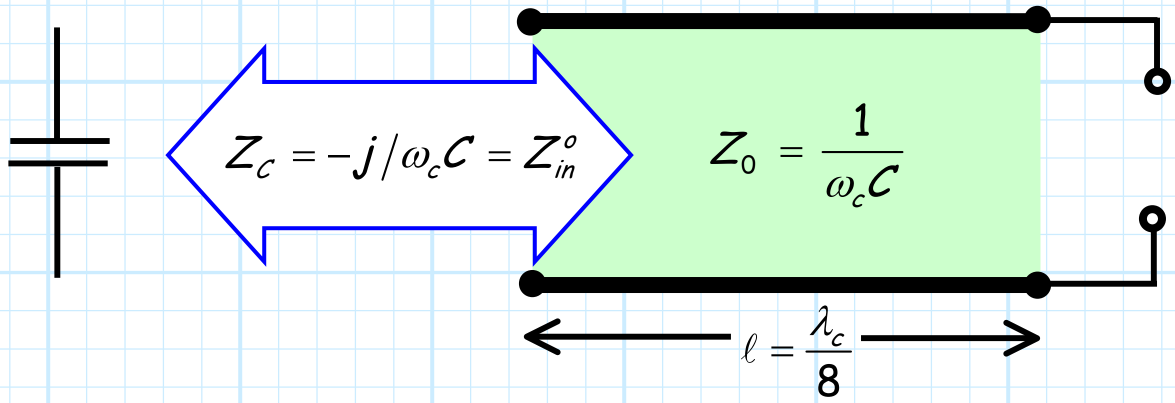
This of course greatly **simplifies** our earlier results:

$$\begin{aligned} j\omega_c L &= j\,Z_0 \tan\left(\frac{\pi}{4}\right) \\ &= j\,Z_0 \end{aligned} \qquad\qquad \begin{aligned} \frac{-j}{\omega_c C} &= -j\,Z_0 \cot\left(\frac{\pi}{4}\right) \\ &= -j\,Z_0 \end{aligned}$$

Therefore, if we wish to build a **short-circuited** stub with the **same** impedance as an **inductor** L at frequency ω_c, we set the **characteristic impedance** of the stub transmission line to be $Z_0 = \omega_c L$:



Likewise, if we wish to build an **open-circuited** stub with the **same** impedance as an **capacitor** C at frequency ω_c, we set the **characteristic impedance** of the stub transmission line to be $Z_0 = 1/\omega_c C$:



We call these two results **Richard's Transformations.**

However, it is important to remember that Richard's Transformations do **not** result in **perfect** replacements for lumped elements—the stubs **do not** behave like capacitors and inductors!

Instead, the transformation is **perfect**—the impedances are **equal**—at **only one frequency** (ω_c).

We can use Richard's transformations to replace the inductors and capacitors of a lumped element filter design. In fact, for **lowpass filter design**, the frequency ω_c is the filter's **cutoff frequency**.

Using these stubs to **replace** inductors and capacitors will result in a filter response **similar** to that of the lumped element design—a low pass filter with cutoff frequency ω_c.

However, the behavior of the filter in the **stopband** will be **very** different from the lumped element design. For example, at the (high) frequencies where the stub length becomes a **multiple** of $\lambda/2$, the filter response will be that of $\omega = 0$—near perfect **transmission**!

Figure 8.37 goes here

Q: *So **why** does the filter response match the lumped element response so **well** in the **passband**?*

A: To see why, we first note that the **Taylor Series approximation** for $\tan\phi$ and $\cot\phi$ when ϕ is small (i.e., $\phi \ll 1$) is:

$$\tan\phi \approx \phi \quad \text{and} \quad \cot\phi \approx \frac{1}{\phi} \quad \text{for} \quad \phi \ll 1$$

and ϕ is expressed in **radians**.

The **impedance** of our Richard's transformation shorted stub at some **arbitrary frequency** ω is:

$$
\begin{aligned}
Z_{in}^{s}(\omega) &= j\,Z_0 \tan\left(\beta \frac{\lambda_c}{8}\right) \\
&= j\,(\omega_c L)\tan\left(\frac{\omega}{v_p}\frac{\lambda_c}{8}\right) \\
&= j\,(\omega_c L)\tan\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right)
\end{aligned}
$$

Therefore, when $\omega \ll \omega_c$ (i.e., frequencies in the **passband** of a low-pass filter!), we can **approximate** this impedance as:

$$
\begin{aligned}
Z_{in}^{s}(\omega) &= j\,(\omega_c L)\tan\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) \\
&\approx j\,\omega_c L\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) \\
&= j\omega L\left(\frac{\pi}{4}\right) \qquad \text{when } \omega \ll \omega_c
\end{aligned}
$$

Compare this to a **lumped inductor** impedance:

$$Z_L = j\omega L$$

Since the value $\pi/4$ is **relatively** close to one, we find that the Richard's Transformation shorted stub has an input impedance **very** close to the lumped element inductor for **all** frequencies **less than** ω_c (i.e., all frequencies of the low-pass filter pass-band)!

Similarly, we find that the Richard's transformation **open-circuit** stub has an input impedance of **approximately**:

$$
\begin{aligned}
Z_{in}^{o}(\omega) &= \frac{-j}{\omega_c C} cot\left(\frac{\omega}{\omega_c}\frac{\pi}{4}\right) \\
&\approx \frac{-j}{\omega_c C}\left(\frac{\omega_c}{\omega}\frac{4}{\pi}\right) \\
&= \frac{1}{j\omega C}\left(\frac{4}{\pi}\right) \qquad \qquad when\ \omega \ll \omega_c
\end{aligned}
$$

Again, when compared to the **lumped element capacitor** impedance:

$$Z_C = \frac{1}{j\omega C}$$

we find that results are approximately the **same** for all pass-band frequencies (i.e., when $\omega \ll \omega_c$).