Richard’s Transformations

Recall the input impedances of short-circuited and open-circuited transmission line stubs.

\[ Z_{in}^s = jZ_0 \tan \beta \ell \]

\[ Z_{in}^o = -jZ_0 \cot \beta \ell \]

Note that the input impedances are purely reactive—just like lumped elements!

However, the reactance of lumped inductors and capacitors have a different mathematical form to that of transmission line stubs:

\[ Z_L = j\omega L \]
\[ Z_C = \frac{-j}{\omega C} \]
In other words, the impedance of transmission line stubs and lumped elements (capacitors and inductors) are different functions with respect to frequency. Therefore, we can say in general that, for example:

\[ Z_{in}^s \neq Z_L \quad Z_{in}^c \neq Z_C \]

However, for a given lumped element (L or C) and a given stub (with a given \( Z_0 \) and length \( \ell \)) the functions will be equal at precisely one frequency!

For example, there is one frequency—let’s call it \( \omega_c \)—that satisfies this equation for a given \( L, Z_0, \) and \( \ell \):

\[
j \omega_c L = j Z_0 \tan \beta_c \ell = j Z_0 \tan \left( \frac{\omega_c \ell}{v_p} \right)
\]

or similarly satisfies this equation:

\[
-\frac{j}{\omega_c C} = -j Z_0 \cot \beta_c \ell = -j Z_0 \cot \left( \frac{\omega_c \ell}{v_p} \right)
\]

To make things easier, let’s set the length of our transmission line stub to \( \lambda_c/8 \), where:

\[
\lambda_c = \frac{v_p}{\omega_c} = \frac{2\pi}{\beta_c}
\]
Q: Why $\ell = \lambda_c / 8$?

A: Well, for one reason, $\beta_c \ell = \pi / 4$ and therefore $\tan(\pi / 4) = 1.0$!

This of course greatly simplifies our earlier results:

$$j \omega_c L = j Z_0 \tan \left( \frac{\pi}{4} \right) = j Z_0$$

$$\frac{-j}{\omega_c C} = -j Z_0 \cot \left( \frac{\pi}{4} \right) = -j Z_0$$

Therefore, if we wish to build a short-circuited stub with the same impedance as an inductor $L$ at frequency $\omega_c$, we set the characteristic impedance of the stub transmission line to be $Z_0 = \omega_c L$:

Likewise, if we wish to build an open-circuited stub with the same impedance as an capacitor $C$ at frequency $\omega_c$, we set the characteristic impedance of the stub transmission line to be $Z_0 = 1 / (\omega c C)$:
We call these two results Richard’s Transformations.

However, it is important to remember that Richard’s Transformations do not result in perfect replacements for lumped elements—the stubs do not behave like capacitors and inductors!

Instead, the transformation is perfect—the impedances are equal—at only one frequency ($\omega_c$).

We can use Richard’s transformations to replace the inductors and capacitors of a lumped element filter design. In fact, for lowpass filter design, the frequency $\omega_c$ is the filter’s cutoff frequency.

Using these stubs to replace inductors and capacitors will result in a filter response similar to that of the lumped element design—a low pass filter with cutoff frequency $\omega_c$. 

\[ Z_c = -\frac{j}{\omega_c C} = Z_{in}^o \]

\[ Z_0 = \frac{1}{\omega_c C} \]

\[ l = \frac{\lambda_c}{8} \]
However, the behavior of the filter in the stopband will be very different from the lumped element design. For example, at the (high) frequencies where the stub length becomes a multiple of \( \lambda/2 \), the filter response will be that of \( \omega = 0 \)—near perfect transmission!

**Figure 8.37 goes here**

**Q:** So why does the filter response match the lumped element response so well in the passband?

**A:** To see why, we first note that the Taylor Series approximation for \( \tan \phi \) and \( \cot \phi \) when \( \phi \) is small (i.e., \( \phi \ll 1 \)) is:

\[
\tan \phi \approx \phi \quad \text{and} \quad \cot \phi \approx \frac{1}{\phi} \quad \text{for} \quad \phi \ll 1
\]

and \( \phi \) is expressed in radians.

The impedance of our Richard's transformation shorted stub at some arbitrary frequency \( \omega \) is:
\[ Z_{in}^s(\omega) = j Z_0 \tan\left(\frac{\lambda_c}{8}\right) \]

\[ = j (\omega c L) \tan\left(\frac{\omega \lambda_c}{v_p}\right) \]

\[ = j (\omega c L) \tan\left(\frac{\omega}{\omega_c \pi} \frac{\pi}{4}\right) \]

Therefore, when \( \omega \ll \omega_c \) (i.e., frequencies in the passband of a low-pass filter!), we can approximate this impedance as:

\[ Z_{in}^s(\omega) = j (\omega c L) \tan\left(\frac{\omega}{\omega_c \pi} \frac{\pi}{4}\right) \approx j \omega c L \left(\frac{\omega}{\omega_c} \frac{\pi}{4}\right) = j \omega c L \left(\frac{\pi}{4}\right) \quad \text{when} \ \omega \ll \omega_c \]

Compare this to a lumped inductor impedance:

\[ Z_L = j \omega L \]

Since the value \( \pi/4 \) is relatively close to one, we find that the Richard’s Transformation shorted stub has an input impedance very close to the lumped element inductor for all frequencies less than \( \omega_c \) (i.e., all frequencies of the low-pass filter passband)!

Similarly, we find that the Richard’s transformation open-circuit stub has an input impedance of approximately:
\[ Z_{in}^o(\omega) = \frac{-j}{\omega_c C} \cot\left(\frac{\omega}{\omega_c} \frac{\pi}{4}\right) \]

\[ \approx \frac{-j}{\omega_c C} \left(\frac{\omega_c}{\omega} \frac{4}{\pi}\right) \]

\[ = \frac{1}{j \omega C} \left(\frac{4}{\pi}\right) \text{ when } \omega \ll \omega_c \]

Again, when compared to the lumped element capacitor impedance:

\[ Z_c = \frac{1}{j \omega C} \]

we find that results are approximately the same for all passband frequencies (i.e., when \( \omega \ll \omega_c \)).