Special Values of Load Impedance

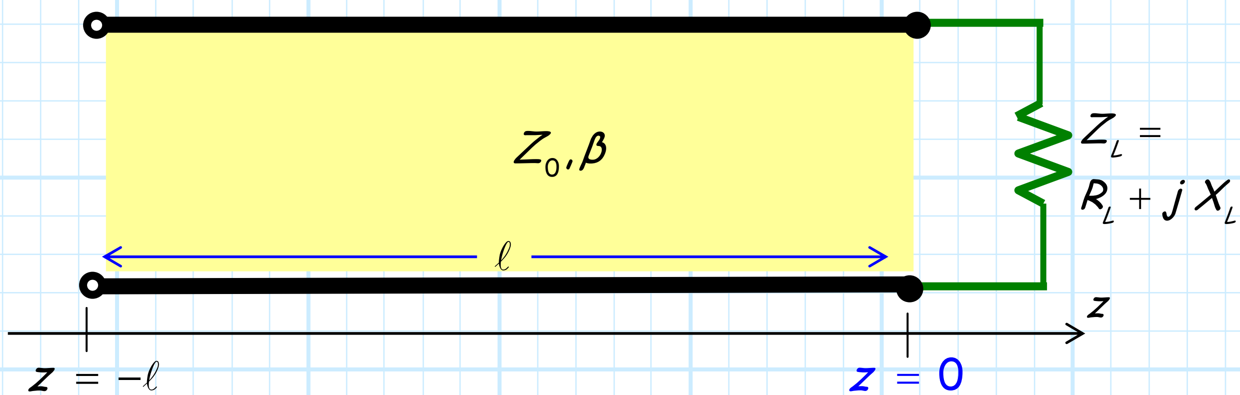
It's interesting to note that the load Z_L enforces a boundary condition that explicitly determines **neither** $V(z)$ nor $I(z)$—but **completely** specifies **line impedance** $Z(z)$!

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L e^{+j\beta z}} = Z_0 \frac{Z_L \cos\beta z - j Z_0 \sin\beta z}{Z_0 \cos\beta z - j Z_L \sin\beta z}$$

Likewise, the load boundary condition leaves $V^+(z)$ and $V^-(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(z)$!

$$\Gamma(z) = \Gamma_L e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j2\beta z}$$

Let's look at some **specific** values of load impedance $Z_L = R_L + j X_L$ and see what **functions** $Z(z)$ and $\Gamma(z)$ result!



We assume that the load is located at $z = 0$ ($\therefore \ \Gamma_L = \Gamma_0$).

The matched case

In this case $Z_L = Z_0$—the **load** impedance is **numerically** equal to the **characteristic** impedance of the transmission line. Assuming the line is **lossless,** then Z_0 is **real**, and thus:

$$R_L = Z_0 \qquad \text{and} \qquad X_L = 0$$

It is evident that the resulting load reflection coefficient is **zero**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

As a result, we find that the **reflected wave is zero**, as is the reflection coefficient function:

$$V^+(z) = V_0^+ e^{-j\beta z} \qquad V^-(z) = 0 \qquad \Gamma(z) = 0$$

Thus, the **total** voltage and current along the transmission line is simply voltage and current of the **incident** wave, and the line impedance is simply Z_0 at all z:

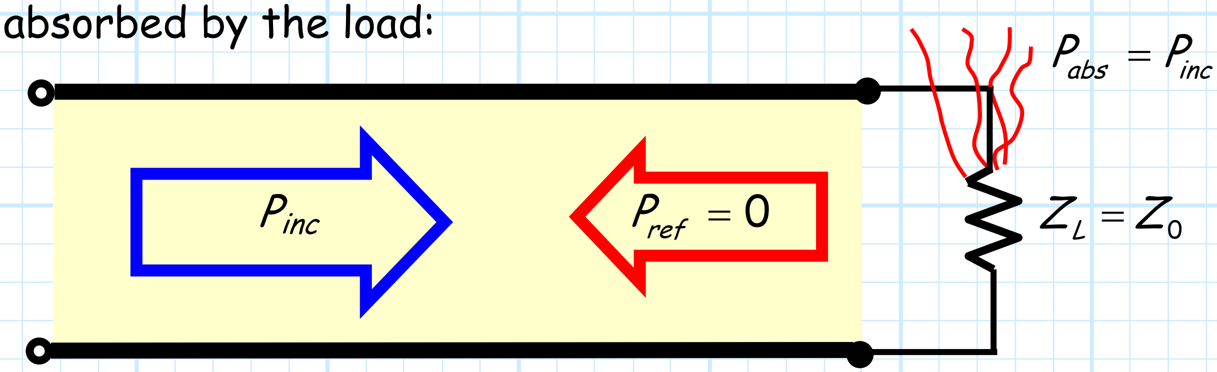
$$V(z) = V^+(z) = V_0^+ e^{-j\beta z} \qquad I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} \qquad Z(z) = \frac{V(z)}{I(z)} = Z_0$$

Power flow in the matched condition

Note from these results we can conclude that out **boundary conditions** are satisfied:

$$Z(z=0)=Z_0=Z_L \qquad \text{and} \qquad \Gamma(z=0)=\Gamma_0=0 \quad !!!$$

Note that since $\Gamma_L=0$, this is a case where the **reflected power is zero**, and **all** the incident power is absorbed by the load:



Q: *Is there any other load for which this is true?*

A: Nope, $Z_L=Z_0$ is the **only** one!

We call this condition (when $Z_L=Z_0$) the **matched** condition, and the load $Z_L=Z_0$ a **matched load**.

A short-circuit load

A device with **no** impedance ($Z_L = 0$) is called a **short** circuit! I.E.:

$$R_L = 0 \quad \text{and} \quad X_L = 0$$

In this case, the **voltage** across the load—and thus the voltage at the end of the transmission line—is **zero**:

$$V_L = Z_L\, I_L = 0 \qquad \text{and} \qquad V(z=0) = 0$$

Note that this does **not** mean that the **current** is zero!

$$I_L = I(z=0) \neq 0$$

For a **short**, the resulting load reflection coefficient is therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 = e^{j\pi}$$

A reactive result!

As a result, the **reflected** wave is equal in magnitude to the **incident** wave. The reflection coefficient function thus has **a magnitude of 1**!

$$V^{+}(z) = V_0^{+} e^{-j\beta z} \qquad V^{-}(z) = -V_0^{+} e^{+j\beta z} \qquad \Gamma(z) = \frac{V^{-}(z)}{V^{+}(z)} = -e^{j2\beta z} = e^{j(2\beta z + \pi)}$$

The reflected wave is **just** as big as the incident wave!

The total **voltage** and **current** along a shorted transmission line take an **interesting** form:

$$V(z) = -j\,2V_0^{+}\,\sin(\beta z) \qquad I(z) = \frac{2V_0^{+}}{Z_0}\cos(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of a **trigonometric** function:

$$Z(z) = \frac{V(z)}{I(z)} = -jZ_0 \tan(\beta z)$$

Note that this impedance is **purely reactive**!

Boundary conditions are confirmed

From these results we can conclude that out **boundary conditions** are satisfied:

$$Z(z=0) = -j\,Z_0 \tan(0) = 0$$

Just as we expected—a **short** circuit!

This is likewise confirmed by evaluating the voltage and current at the **end** of the line (i.e., $z = z_L = 0$):

$$V(z=0) = -j\,2V_0^+ \sin(0) = 0 \qquad\qquad I(z=0) = \frac{2V_0^+}{Z_0}\cos(0) = \frac{2V_0^+}{Z_0}$$

As expected, the voltage is **zero** at the end of the transmission line (i.e. the voltage across the **short**).

Also, the **current** at the end of the line (i.e., the current through the short) is at a **maximum**! Additionally, the **reflection coefficient** at the load is:

$$\Gamma(z=0) = -e^{j2\beta(0)} = -1 = e^{j\pi} = \Gamma_L$$

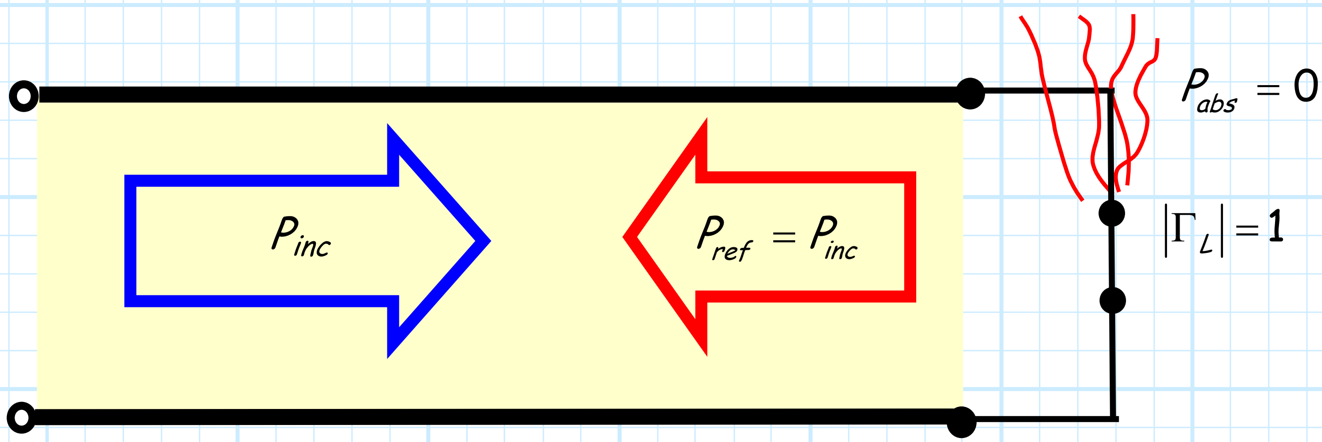
Again confirming that the **boundary conditions** are satisfied!

A short cannot absorb energy

Finally, let's determine the **power flow** associated with this short-circuit load.

Since $|\Gamma_L| = 1$, this is a case where the **absorbed** power is **zero**, and all the incident power is **reflected** by the load:

$$P_{abs} = 0 \qquad \text{and} \qquad P_{ref} = P_{inc}$$



An open-circuit load

A device with **infinite** impedance ($Z_L = \infty$) is called an **open** circuit! I.E.:

$$R_L = \infty \quad \text{and/or} \quad X_L = \pm\infty$$

In this case, the **current** through the load—and thus the current at the end of the transmission line—is **zero**:

$$I_L = \frac{V_L}{Z_L} = 0 \qquad \text{and} \qquad I\left(z = z_L\right) = 0$$

Note that this does **not** mean that the **voltage** is zero!

$$V_L = V\left(z = z_L\right) \neq 0$$

For an **open**, the resulting load reflection coefficient is:

$$\Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_0}{Z_L + Z_0} = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L} = 1 = e^{j0}$$

A reactive result!

As a result, the **reflected** wave is **equal** in magnitude to the **incident** wave. The reflection coefficient function thus has **a magnitude of 1**!

$$V^+(z) = V_0^+ \, e^{-j\beta z} \qquad V^-(z) = V_0^+ \, e^{+j\beta z} \qquad \Gamma(z) = \frac{V^-(z)}{V^+(z)} = e^{+j2\beta z}$$

The reflected wave is **just** as big as the incident wave!

The **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$V(z) = 2V_0^+ \cos(\beta z) \qquad I(z) = -j\,\frac{2V_0^+}{Z_0}\sin(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = j\,Z_0 \cot(\beta z)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are again 90° **out of phase**!

Boundary conditions are confirmed

Note from these results we can conclude that out **boundary conditions** are satisfied:

$$Z(z=0) = j\,Z_0 \cot(0) = \infty$$

Just as we expected—an **open** circuit!

This is likewise confirmed by evaluating the voltage and current at the **end** of the line (i.e., $z = z_L = 0$):

$$V(z=0) = 2V_0^+ \cos(0) = \frac{2V_0^+}{Z_0} \qquad\qquad I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin(0) = 0$$

As expected, the **current is zero** at the end of the transmission line (i.e. the current through the **open**). Likewise, the **voltage** at the end of the line (i.e., the voltage across the open) is at a **maximum**!

Additionally, the **reflection coefficient** at the load is:

$$\Gamma(z=0) = e^{j2\beta(0)} = 1 = e^{j0} = \Gamma_L$$

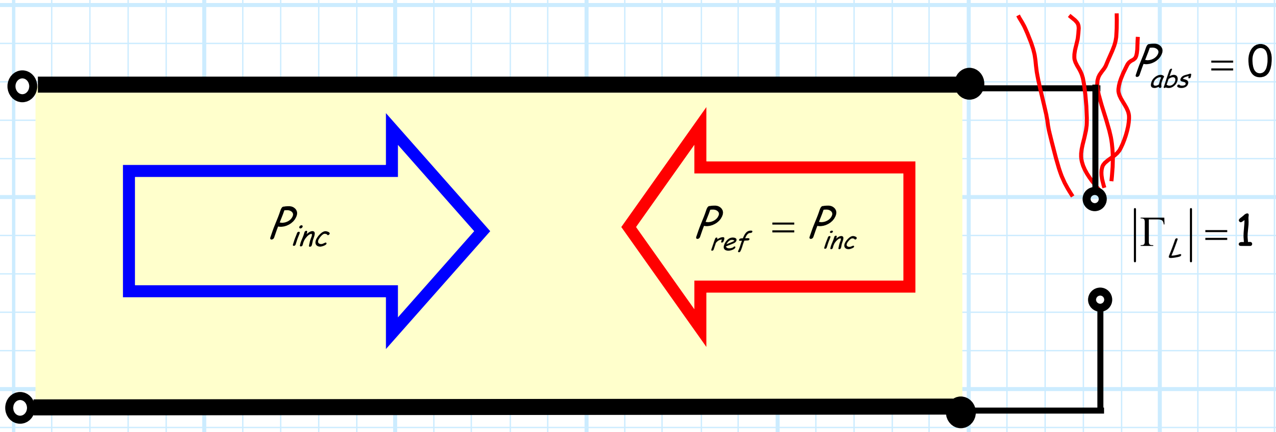
Again confirming that the **boundary conditions** are satisfied!

An open cannot absorb energy

Finally, let's determine the **power flow** associated with this open circuit load.

Since $|\Gamma_L| = 1$, this is again a case where the **absorbed** power is **zero**, and all the incident power is **reflected** by the load:

$$P_{abs} = 0 \qquad \text{and} \qquad P_{ref} = P_{inc}$$



A purely reactive load

For this case, the load impedance is **purely reactive** $Z_L = j\,X_L$ (e.g. a capacitor of inductor), and thus the resistive portion is zero:

$$R_L = 0$$

Thus, **both** the current through the load, and voltage across the load, are **non-zero**:

$$I_L = I(z = z_L) \neq 0 \qquad\qquad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient is a **complex** number.

V^+, V^- and Γ

However, we find that the magnitude of **this (reactive)** load reflection coefficient is:

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is **one**!

Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma} \qquad \text{where} \qquad \theta_\Gamma = \tan^{-1}\left[\frac{2Z_0 X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that $V_0^- = e^{j\theta_\Gamma} V_0^+$, and so for a **reactive load,** :

$$V^+(z) = V_0^+ e^{-j\beta z} \qquad V^-(z) = e^{j\theta_\Gamma} V_0^+ e^{+j\beta z} \qquad \Gamma(z) = \frac{V^-(z)}{V^+(z)} = e^{+j2\beta z}$$

The reflected wave is again **just** as big as the incident wave!

I, V, and Z

The **total** voltage and current along the transmission line are **complex** (assuming $z_L = 0$):

$$V(z) = 2V_0^+ \; e^{+j\theta_\Gamma/2} \cos(\beta z + \theta_\Gamma/2) \qquad I(z) = -j\,\frac{2V_0^+}{Z_0}\, e^{+j\theta_L/2} \sin(\beta z + \theta_L/2)$$

Meaning that the **line impedance** can again be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = j\,Z_0 \cot(\beta z + \theta_\Gamma/2)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are once again 90° out of phase!

Boundary Conditions!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that

$$V(z=0) = 2V_0^+ \cos(\theta_\Gamma/2) \qquad I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin(\theta_\Gamma/2)$$

As expected, **neither** the current **nor** voltage at the end of the line is zero.

We also note that the line impedance at the **end** of the transmission line is:

$$Z(z=0) = j\,Z_0 \cot(\theta_\Gamma/2)$$

With a little trigonometry, we can show (trust me!) that:

$$\cot(\theta_\Gamma/2) = \frac{X_L}{Z_0}$$

and therefore:

$$Z(z=0) = j\,Z_0 \cot(\theta_\Gamma/2) = j\,X_L = Z_L$$

Just as we **expected**!

Déjà vu All Over Again

Q: *Gee, a **reactive** load leads to results very **similar** to that of an **open** or **short** circuit. Is this just **coincidence**?*

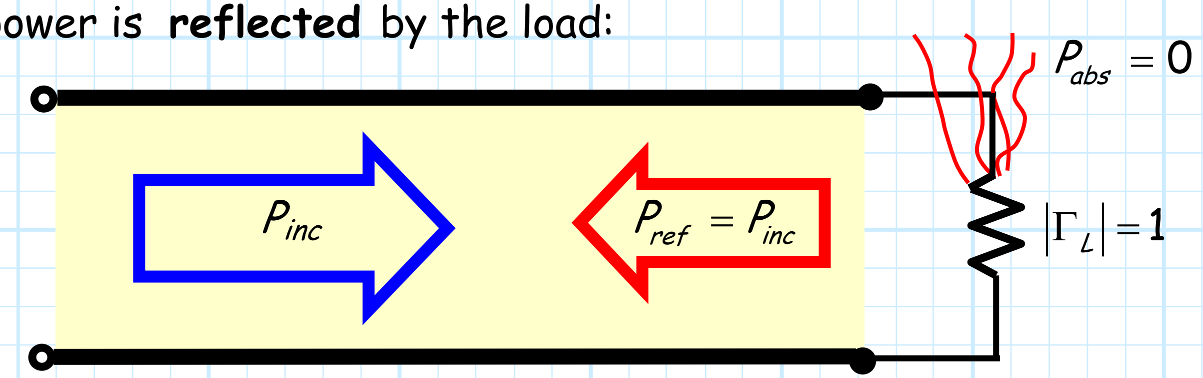
A: Hardly! An open and short **are** in fact reactive loads—they **cannot absorb power** (think about this!).

Specifically, for an **open**, we find $\theta_\Gamma = 0$, so that: $\Gamma_L = e^{j\theta_\Gamma} = 1$

Likewise, for a **short**, we find that $\theta_\Gamma = \pi$, so that: $\Gamma_L = e^{j\theta_\Gamma} = -1$

The **power flow** associated with a reactive load is the same as for an open or short.

Since $|\Gamma_L| = 1$, it is again a case where the **absorbed** power is **zero**, and all the incident power is **reflected** by the load:



Resistive Load

For this case $Z_L = R_L$, so the load impedance is **purely real** (e.g. a **resistor**), meaning its reactive portion is zero:

$$X_L = 0$$

The resulting **load reflection coefficient** is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient must be a purely **real** value!

In other words:

$$\mathrm{Re}\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \qquad \mathrm{Im}\{\Gamma_L\} = 0$$

Phase difference is either 0 or π

The magnitude is thus:

$$\left|\Gamma_L\right| = \left|\frac{R - Z_0}{R + Z_0}\right|$$

whereas the phase θ_Γ can take on one of two values:

$$\theta_\Gamma = \begin{cases} 0 & if \quad \text{Re}\left\{\Gamma_L\right\} > 0 \quad \text{(i.e., if } R_L > Z_0\text{)} \\ \pi & if \quad \text{Re}\left\{\Gamma_L\right\} < 0 \quad \text{(i.e., if } R_L < Z_0\text{)} \end{cases}$$

For this case, the impedance at the **end** of the line must be **real** ($Z\left(z = z_L\right) = R_L$).

Thus, the current and the voltage at this point are precisely **in phase**, or precisely 180 degrees **out of phase**!

The load is real; why isn't the line impedance?

However, even though the **load** impedance is real, the **line** impedance at all other points on the line is generally **complex**!

Moreover, the **general** current and voltage expressions, as well as reflection coefficient function, **cannot** be further simplified for the case where $Z_L = R_L$.

Q: *Why is that?*

*When the load was purely **imaginary** (reactive), we where able to **simply** our general expressions, and likewise deduce all sorts of interesting results!*

A: True! And here's **why**.

Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**.

In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

Remember, a lossless line is purely reactive!

* If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line).

* Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.

* However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components.

* This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

The "General" Load

Now, let's look at the **general** case $Z_L = R_L + jX_L$, where the **load** has both a **real** (resistive) and **imaginary** (reactive) component.

Q: *Haven't we* ***already*** *determined all the* ***general*** *expressions (e.g., $\Gamma_L, V(z), I(z), Z(z), \Gamma(z)$) for this general case?*

Is there ***anything*** *else left to be determined?*

A: There is **one** last thing we need to discuss.

It seems trivial, but its ramifications are **very** important!

For you see, the "general" case is **not**, in reality, quite so **general**.

Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as a (passive) **negative** resistor!

Complex arithmetic—is there anything funer?

This leads to one **very** important and **very** useful result.

Consider the **load reflection coefficient**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(R_L + jX_L) - Z_0}{(R_L + jX_L) + Z_0} = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}$$

Now let's look at the **magnitude** of this value:

$$\begin{aligned} |\Gamma_L|^2 &= \left| \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L} \right|^2 \\ &= \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \\ &= \frac{(R_L^2 - 2R_L Z_0 + Z_0^2) + X_L^2}{(R_L^2 + 2R_L Z_0 + Z_0^2) + X_L^2} \\ &= \frac{(R_L^2 + Z_0^2 + X_L^2) - 2R_L Z_0}{(R_L^2 + Z_0^2 + X_L^2) + 2R_L Z_0} \end{aligned}$$

A passive load? Then $|\Gamma|<1$!

It is apparent that since both R_L and Z_0 are **positive**, the **numerator** of the above expression must be **less** than (or equal to) the **denominator** of the above expression.

→ In other words, the magnitude of the load reflection coefficient is always **less** than or equal to one!

$$|\Gamma_L| \leq 1 \quad (\text{for } R_L \geq 0)$$

Moreover, we find that this means the reflection coefficient **function** likewise always has a magnitude **less** than or equal to one, for **all** values of position z.

$$|\Gamma(z)| \leq 1 \quad (\text{for all } z)$$

A passive load? Then the reflected wave will always be less than the incident!

Which means, of course, that the **reflected** wave will always have a magnitude **less** than that of the **incident** wave magnitude:

$$|V^-(z)| \leq |V^+(z)| \qquad \text{(for all } z\text{)}$$

Recall this result is consistent with **conservation of energy**—the reflected wave from a **passive** load **cannot** be larger than the wave incident on it.