Special Values of Load Impedance

It's interesting to note that the load Z_L enforces a boundary condition that explicitly determines neither $V(z)$ nor $I(z)$—but **completely** specifies **line impedance** $Z(z)$!

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L e^{+j\beta z}} = Z_0 \frac{Z_L \cos\beta z - jZ_0 \sin\beta z}{Z_0 \cos\beta z - jZ_L \sin\beta z}$$

$$\Gamma(z) = \Gamma_L e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j2\beta z}$$

Likewise, the load boundary condition leaves $V^+(z)$ and $V^-(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(z)$!

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(z)$ result!

1. $Z_L = Z_0$

In this case, the **load** impedance is **numerically** equal to the **characteristic** impedance of the transmission line. Assuming the line is **lossless**, then Z_0 is **real**, and thus:

$$R_L = Z_0 \quad \text{and} \quad X_L = 0$$

It is evident that the resulting load reflection coefficient is **zero**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is **no reflected wave** $V^-(z)$!

$$\begin{aligned} V^-(z) &= \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z} \\ &= \left(e^{-2j\beta z_L}\,(0)\,V_0^+\right)e^{+j\beta z} \\ &= 0 \end{aligned}$$

Thus, the **total** voltage and current along the transmission line is simply voltage and current of the **incident** wave:

$$V(z) = V^+(z) = V_0^+ e^{-j\beta z}$$

$$I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}$$

Meaning that the **line** impedance is likewise **numerically** equal to the **characteristic** impedance of the transmission line for **all** line position z:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0$$

And likewise, the reflection coefficient is **zero** at **all** points along the line:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0$$

We call this condition (when $Z_L = Z_0$) the **matched** condition, and the load $Z_L = Z_0$ a **matched load**.

2. $Z_L = 0$

A device with **no** impedance is called a **short** circuit! I.E.:

$$R_L = 0 \quad \text{and} \quad X_L = 0$$

In this case, the **voltage** across the load—and thus the voltage at the end of the transmission line—is **zero**:

$$V_L = Z_L\, I_L = 0 \qquad \text{and} \qquad V(z = z_L) = 0$$

Note that this does **not** mean that the **current** is zero!

$$I_L = I(z = z_L) \neq 0$$

For a **short**, the resulting load reflection coefficient is therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1$$

Meaning (assuming $z_L = 0$):

$$V_0^- = -V_0^+$$

As a result, the total **voltage** and **current** along the transmission line is simply:

$$V(z) = V_0^+\left(e^{-j\beta z} - e^{+j\beta z}\right) = -j2V_0^+ \sin(\beta z)$$

$$I(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} + e^{+j\beta z}\right) = \frac{2V_0^+}{Z_0}\cos(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of a **trigonometric** function:

$$Z(z) = \frac{V(z)}{I(z)} = -jZ_0 \tan(\beta z)$$

Note that this impedance is **purely reactive**. This means that the current and voltage on the transmission line will be everywhere 90° **out of phase**.

Hopefully, this was likewise apparent to **you** when you **observed** the expressions for $V(z)$ and $I(z)$!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that:

$$V(z=0) = -j2V_0^+ \sin(0) = 0$$

$$I(z=0) = \frac{2V_0^+}{Z_0}\cos(0) = \frac{2V_0^+}{Z_0}$$

As expected, the voltage is **zero** at the end of the transmission line (i.e. the voltage across the **short**). Likewise, the **current** at the end of the line (i.e., the current through the short) is at a **maximum**!

Finally, we note that the **line** impedance at the **end** of the transmission line is:

$$Z(z=0) = -jZ_0 \tan(0) = 0$$

Just as we expected—a **short** circuit!

Finally, the reflection coefficient **function** is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{-V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = -e^{j\beta z}$$

Note that for **this** case $|\Gamma(z)| = 1$, meaning that:

$$|V^-(z)| = |V^+(z)|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

3. $Z_L = \infty$

A device with **infinite** impedance is called an **open** circuit! I.E.:

$$R_L = \infty \quad \text{and/or} \quad X_L = \pm\infty$$

In this case, the **current** through the load—and thus the current at the end of the transmission line—is **zero**:

$$I_L = \frac{V_L}{Z_L} = 0 \qquad \text{and} \qquad I(z = z_L) = 0$$

Note that this does **not** mean that the **voltage** is zero!

$$V_L = V(z = z_L) \neq 0$$

For an **open**, the resulting load reflection coefficient is:

$$\Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_0}{Z_L + Z_0} = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L} = 1$$

Meaning (assuming $z_L = 0$):

$$V_0^- = V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$V(z) = V_0^+ \left(e^{-j\beta z} + e^{+j\beta z}\right) = 2V_0^+ \cos(\beta z)$$

$$I(z) = \frac{V_0^+}{Z_0} \left(e^{-j\beta z} - e^{+j\beta z}\right) = -j\,\frac{2V_0^+}{Z_0} \sin(\beta z)$$

Meaning that the **line impedance** can likewise be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are again 90° **out of phase**!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that

$$V(z=0) = 2V_0^+ \cos(0) = \frac{2V_0^+}{Z_0}$$

$$I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin(0) = 0$$

As expected, the **current is zero** at the end of the transmission line (i.e. the current through the **open**). Likewise, the **voltage** at the end of the line (i.e., the voltage across the open) is at a **maximum**!

Finally, we note that the **line impedance** at the **end** of the transmission line is:

$$Z(z=0) = jZ_0\, cot(0) = \infty$$

Just as we expected—an **open** circuit!

Finally, the reflection coefficient is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2\beta z}$$

Note that likewise for **this** case $|\Gamma(z)| = 1$, meaning **again** that:

$$|V^-(z)| = |V^+(z)|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

4. $Z_L = j\,X_L$

For this case, the load impedance is **purely reactive** (e.g. a capacitor of inductor), and thus the resistive portion is zero:

$$R_L = 0$$

Thus, **both** the current through the load, and voltage across the load, are **non-zero**:

$$I_L = I(z = z_L) \neq 0 \qquad\qquad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient is generally some **complex** number.

We can rewrite this value explicitly in terms of its **real** and **imaginary** part as:

$$\Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \left(\frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2}\right) + j\left(\frac{2\,Z_0\,X_L}{X_L^2 + Z_0^2}\right)$$

Yuck! This ***isn't*** *much help!*

Let's instead write this complex value Γ_L in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is **one**! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma}$$

where

$$\theta_\Gamma = tan^{-1}\left[\frac{2\,Z_0\,X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that for a **reactive load**:

$$V_0^- = e^{j\theta_\Gamma} V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$\begin{aligned} V(z) &= V_0^+ \left(e^{-j\beta z} + e^{+j\theta_L} e^{+j\beta z} \right) \\ &= V_0^+ \; e^{+j\theta_\Gamma/2} \left(e^{-j(\beta z + \theta_\Gamma/2)} + e^{+j(\beta z + \theta_\Gamma/2)} \right) \\ &= 2V_0^+ \; e^{+j\theta_\Gamma/2} \cos\left(\beta z + \theta_\Gamma/2\right) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V_0^+}{Z_0} \left(e^{-j\beta z} - e^{+j\beta z} \right) \\ &= \frac{V_0^+}{Z_0} e^{+j\theta_L/2} \left(e^{-j(\beta z + \theta_L/2)} - e^{+j(\beta z + \theta_L/2)} \right) \\ &= -j \, \frac{2V_0^+}{Z_0} e^{+j\theta_L/2} \sin\left(\beta z + \theta_L/2\right) \end{aligned}$$

Meaning that the **line impedance** can again be written in terms of trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot\left(\beta z + \theta_\Gamma/2\right)$$

Again note that this impedance is **purely reactive**— *V(z)* and *I(z)* are once again 90° out of phase!

Note at the **end** of the line (i.e., $z = z_L = 0$), we find that

$$V(z=0) = 2V_0^+ \cos\left(\theta_\Gamma/2\right)$$

$$I(z=0) = -j\,\frac{2V_0^+}{Z_0}\sin\left(\theta_\Gamma/2\right)$$

As expected, **neither** the current **nor** voltage at the end of the line are zero.

We also note that the line impedance at the **end** of the transmission line is:

$$Z(z=0) = jZ_0\,cot\left(\theta_\Gamma/2\right)$$

With a little trigonometry, we can show (trust me!) that:

$$cot\left(\theta_\Gamma/2\right) = \frac{X_L}{Z_0}$$

and therefore:

$$Z(z=0) = jZ_0\,cot\left(\theta_\Gamma/2\right) = j\,X_L = Z_L$$

Just as we **expected**!

Finally, the reflection coefficient **function** is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\theta_\Gamma} e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2\left(\beta z + \theta_\Gamma/2\right)}$$

Note that likewise for **this** case $\left|\Gamma(z)\right| = 1$, meaning **once again**:

$$\left|V^{-}(z)\right| = \left|V^{+}(z)\right|$$

In other words, the **magnitude** of each **wave** on the transmission line is the **same**—the reflected wave is **just** as big as the incident wave!

Q: *Gee, a* ***reactive*** *load leads to results very* ***similar*** *to that of an* ***open*** *or* ***short*** *circuit. Is this just* ***coincidence****?*

A: Hardly! An open and short **are** in fact reactive loads—they **cannot absorb power** (think about this!).

Specifically, for an **open**, we find $\theta_\Gamma = 0$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = 1$$

Likewise, for a **short**, we find that $\theta_\Gamma = \pi$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = -1$$

5. $Z_L = R_L$

For this case, the load impedance is **purely real** (e.g. a **resistor**), meaning its reactive portion is zero:

$$X_L = 0$$

Thus, **both** the current through the load, and voltage across the load, are **non-zero**:

$$I_L = I(z = z_L) \neq 0 \qquad\qquad V_L = V(z = z_L) \neq 0$$

The resulting **load reflection coefficient** is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient must be a purely **real** value! In other words:

$$Re\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \qquad\qquad \text{Im}\{\Gamma_L\} = 0$$

The magnitude is thus:

$$|\Gamma_L| = \left|\frac{R - Z_0}{R + Z_0}\right|$$

whereas the phase θ_Γ can take on one of two values:

$$\theta_\Gamma = \begin{cases} 0 & if \quad \text{Re}\{\Gamma_L\} > 0 \quad (\text{i.e., if } R_L > Z_0) \\ \\ \pi & if \quad \text{Re}\{\Gamma_L\} < 0 \quad (\text{i.e., if } R_L < Z_0) \end{cases}$$

For this case, the impedance at the **end** of the line must be **real** ($Z(z = z_L) = R_L$). Thus, the current and the voltage at this point are precisely **in phase**.

However, even though the **load** impedance is real, the **line** impedance at all other points on the line is generally **complex**!

Moreover, the **general** current and voltage expressions, as well as reflection coefficient function, **cannot** be further simplified for the case where $Z_L = R_L$.

Q: *Why is that? When the load was purely* ***imaginary*** *(reactive), we where able to* ***simply*** *our general expressions, and likewise deduce all sorts of interesting results!*

A: True! And here's **why**. Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**. In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

* If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line). Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.

* However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components.

This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

5. $Z_L = R_L + jX_L$

Now, let's look at the **general** case, where the **load** has both a **real** (resitive) and **imaginary** (reactive) component.

Q: *Haven't we **already** determined all the **general** expressions (e.g., $\Gamma_L, V(z), I(z), Z(z), \Gamma(z)$) for this general case? Is there **anything** else left to be determined?*

A: There is **one** last thing we need to discuss. It seems trivial, but its ramifications are **very** important!

For you see, the "general" case is **not**, in reality, quite so general. Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as a (passive) **negative** resistor!

This leads to one **very** important and useful result. Consider the **load reflection coefficient**:

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{\left(R_L + jX_L\right) - Z_0}{\left(R_L + jX_L\right) + Z_0} \\ &= \frac{\left(R_L - Z_0\right) + jX_L}{\left(R_L + Z_0\right) + jX_L}\end{aligned}$$

Now let's look at the **magnitude** of this value:

$$\begin{aligned}\left|\Gamma_L\right|^2 &= \left|\frac{\left(R_L - Z_0\right) + jX_L}{\left(R_L + Z_0\right) + jX_L}\right|^2 \\ &= \frac{\left(R_L - Z_0\right)^2 + X_L^2}{\left(R_L + Z_0\right)^2 + X_L^2} \\ &= \frac{\left(R_L^2 - 2R_L Z_0 + Z_0^2\right) + X_L^2}{\left(R_L^2 + 2R_L Z_0 + Z_0^2\right) + X_L^2} \\ &= \frac{\left(R_L^2 + Z_0^2 + X_L^2\right) - 2R_L Z_0}{\left(R_L^2 + Z_0^2 + X_L^2\right) + 2R_L Z_0}\end{aligned}$$

It is apparent that since both R_L and Z_0 are **positive**, the **numerator** of the above expression must be **less** than (or equal to) the **denominator** of the above expression.

→ In other words, the magnitude of the load reflection coefficient is always **less** than or equal to one!

$$\left|\Gamma_L\right| \leq 1 \quad (\text{for } R_L \geq 0)$$

Moreover, we find that this means the reflection coefficient **function** likewise always has a magnitude **less** than or equal to one, for **all** values of position *z*.

$$\left|\Gamma(z)\right| \leq 1 \quad (\text{for all } z)$$

Which means, of course, that the **reflected** wave will always have a magnitude **less** than that of the **incident** wave magnitude:

$$\left|V^-(z)\right| \leq \left|V^+(z)\right| \qquad (\text{for all } z)$$

We will find out later that this result is consistent with **conservation of energy**—the reflected wave from a passive load **cannot** be larger than the wave incident on it.