Special Values of Load Impedance

Let’s look at some specific values of load impedance $Z_L = R_L + jX_L$ and see what happens on our transmission line!

1. $Z_L = Z_0$

In this case, the load impedance is numerically equal to the characteristic impedance of the transmission line. Assuming the line is lossless, then $Z_0$ is real, and thus:

$$R_L = Z_0 \quad \text{and} \quad X_L = 0$$

It is evident that the resulting load reflection coefficient is zero:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is no reflected wave $V^-(z)$!

$$V^-(z) = (e^{2j\beta z_L} \Gamma_L V_0^+) e^{+j\beta z}$$

$$= (e^{2j\beta z_L} (0) V_0^+) e^{+j\beta z}$$

$$= 0$$
Thus, the total voltage and current along the transmission line is simply voltage and current of the incident wave:

\[ V(z) = V^+(z) = V_0^+ e^{-j\beta z} \]

\[ I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} \]

Meaning that the line impedance is likewise numerically equal to the characteristic impedance of the transmission line for all line position \( z \):

\[ Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0 \]

And likewise, the reflection coefficient is zero at all points along the line:

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0 \]

We call this condition (when \( Z_L = Z_0 \)) the matched condition, and the load \( Z_L = Z_0 \) a matched load.

\[ 2. \; Z_L = 0 \]

A device with no impedance is called a short circuit! I.E.:

\[ R_L = 0 \quad \text{and} \quad X_L = 0 \]
In this case, the **voltage** across the load—and thus the voltage at the end of the transmission line—is **zero**:

\[ V_L = Z_L I_L = 0 \quad \text{and} \quad V(z = z_L) = 0 \]

Note that this does **not** mean that the **current** is zero!

\[ I_L = I(z = z_L) \neq 0 \]

For a **short**, the resulting load reflection coefficient is therefore:

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \]

**Meaning** (assuming \( z_L = 0 \)): \( V_0^- = -V_0^+ \)

As a result, the total **voltage** and **current** along the transmission line is simply:

\[ V(z) = V_0^+ \left( e^{-j \beta z} - e^{+j \beta z} \right) = -j 2V_0^+ \sin(\beta z) \]

\[ I(z) = \frac{V_0^+}{Z_0} \left( e^{-j \beta z} + e^{+j \beta z} \right) = \frac{2V_0^+}{Z_0} \cos(\beta z) \]

**Meaning** that the line **impedance** can likewise be written in terms of a **trigonometric** function:
Note that this impedance is purely reactive. This means that the current and voltage on the transmission line will be everywhere 90° out of phase.

Hopefully, this was likewise apparent to you when you observed the expressions for \( V(z) \) and \( I(z) \)!

Note at the end of the line (i.e., \( z = z_L = 0 \)), we find that

\[
V(z = 0) = -j2V_0^+ \sin(0) = 0
\]

\[
I(z = 0) = \frac{2V_0^+}{Z_0} \cos(0) = \frac{2V_0^+}{Z_0}
\]

As expected, the voltage is zero at the end of the transmission line (i.e. the voltage across the short). Likewise, the current at the end of the line (i.e., the current through the short) is at a maximum!

Finally, we note that the line impedance at the end of the transmission line is:

\[
Z(z = 0) = -jZ_0 \tan(0) = 0
\]

Just as we expected—a short circuit!
Finally, the reflection coefficient function is (assuming \( z_L = 0 \)):

\[
\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{-V_0^+ e^{+j \beta z}}{V_0^+ e^{-j \beta z}} = -e^{j \beta z}
\]

Note that for this case \( |\Gamma(z)| = 1 \), meaning that:

\[
|V^-(z)| = |V^+(z)|
\]

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!

3. \( z_L = \infty \)

A device with infinite impedance is called an open circuit!
I.E.:

\[
R_L = \infty \quad \text{and/or} \quad X_L = \pm \infty
\]

In this case, the current through the load—and thus the current at the end of the transmission line—is zero:

\[
I_L = \frac{V_L}{Z_L} = 0 \quad \text{and} \quad I(z = z_L) = 0
\]

Note that this does not mean that the voltage is zero:

\[
V_L = V(z = z_L) \neq 0
\]
For an open, the resulting load reflection coefficient is:

\[ \Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_0}{Z_L + Z_0} = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L} = 1 \]

Meaning (assuming \( z_L = 0 \)):

\[ V_0^- = V_0^+ \]

As a result, the total voltage and current along the transmission line is simply (assuming \( z_L = 0 \)):

\[ V(z) = V_0^+ \left( e^{-j\beta z} + e^{+j\beta z} \right) = 2V_0^+ \cos(\beta z) \]

\[ I(z) = \frac{V_0^+}{Z_0} \left( e^{-j\beta z} - e^{+j\beta z} \right) = -j \frac{2V_0^+}{Z_0} \sin(\beta z) \]

Meaning that the line impedance can likewise be written in terms of trigonometric function:

\[ Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z) \]

Again note that this impedance is purely reactive—\( V(z) \) and \( I(z) \) are again 90° out of phase!

Note at the end of the line (i.e., \( z = z_L = 0 \)), we find that
\[ V(z=0) = 2V_0^+ \cos(0) = \frac{2V_0^+}{Z_0} \]

\[ I(z=0) = -j \frac{2V_0^+}{Z_0} \sin(0) = 0 \]

As expected, the current is zero at the end of the transmission line (i.e. the current through the open). Likewise, the voltage at the end of the line (i.e., the voltage across the open) is at a maximum!

Finally, we note that the line impedance at the end of the transmission line is:

\[ Z(z=0) = jZ_0 \cot(0) = \infty \]

Just as we expected—an open circuit!

Finally, the reflection coefficient is (assuming \( z_L = 0 \)):

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{j\beta z}}{V_0^+ e^{-j\beta z}} = e^{2j\beta z} \]

Note that likewise for this case |\( \Gamma(z) \)| = 1, meaning again that:

\[ |V^-(z)| = |V^+(z)| \]

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!
4. \( R_L = 0 \)

For this case, the load impedance is **purely reactive** (e.g. a capacitor or inductor):

\[ Z_L = jX_L \]

Thus, both the current through the load, and voltage across the load, are **non-zero**:

\[ I_L = I(z = z_L) \neq 0 \quad \text{and} \quad V_L = V(z = z_L) \neq 0 \]

The resulting load reflection coefficient is:

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0} \]

Given that \( Z_0 \) is real (i.e., the line is lossless), we find that this load reflection coefficient is generally some **complex** number.

We can rewrite this value explicitly in terms of its **real** and **imaginary** part as:

\[ \Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2} + j\left( \frac{2Z_0X_L}{X_L^2 + Z_0^2} \right) \]

*Yuck! This isn't much help!*
Let’s instead write this complex value $\Gamma_L$ in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is one! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma}$$

where

$$\theta_\Gamma = \tan^{-1}\left[\frac{2Z_0X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that for a **reactive load**:

$$V_0^- = e^{j\theta_\Gamma} V_0^+$$

As a result, the total voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$V(z) = V_0^+ \left( e^{-j\beta z} + e^{+j\theta_\Gamma} e^{+j\beta z} \right)$$

$$= V_0^+ e^{+j\theta_\Gamma / 2} \left( e^{-j(\beta z + \theta_\Gamma / 2)} + e^{+j(\beta z + \theta_\Gamma / 2)} \right)$$

$$= 2V_0^+ e^{+j\theta_\Gamma / 2} \cos(\beta z + \theta_\Gamma / 2)$$
\[ I(z) = \frac{V_0^+}{Z_0} \left( e^{-j\beta z} - e^{+j\beta z} \right) \]
\[ = \frac{V_0^+}{Z_0} e^{+j\theta_L/2} \left( e^{-j(\beta z + \theta_L/2)} - e^{+j(\beta z + \theta_L/2)} \right) \]
\[ = -j \frac{2V_0^+}{Z_0} e^{+j\theta_L/2} \sin(\beta z + \theta_L/2) \]

**Meaning that the line impedance** can again be written in terms of trigonometric function:

\[ Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z + \theta_t/2) \]

Again note that this impedance is **purely reactive**—\( V(z) \) and \( I(z) \) are once again 90° out of phase!

**Note at the end** of the line (i.e., \( z = z_L = 0 \)), we find that

\[ V(z = 0) = 2V_0^+ \cos(\theta_t/2) \]
\[ I(z = 0) = -j \frac{2V_0^+}{Z_0} \sin(\theta_t/2) \]

As expected, neither the current nor voltage at the end of the line are zero.

We also note that the line impedance at the end of the transmission line is:
\[ Z(z = 0) = jZ_0 \cot(\theta_\Gamma/2) \]

With a little trigonometry, we can show (trust me!) that:

\[ \cot(\theta_\Gamma/2) = \frac{X_L}{Z_0} \]

and therefore:

\[ Z(z = 0) = jZ_0 \cot(\theta_\Gamma/2) = jX_L = Z_L \]

Just as we expected!

Finally, the reflection coefficient function is (assuming \( z_L = 0 \)):

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{j\theta_\Gamma} e^{j\beta z}}{V_0^- e^{-j\beta z}} = e^{j2(\beta z + \theta_\Gamma/2)} \]

Note that likewise for this case \( |\Gamma(z)| = 1 \), meaning once again:

\[ |V^-(z)| = |V^+(z)| \]

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!

**Q:** Gee, a reactive load leads to results very similar to that of an open or short circuit. Is this just coincidence?
A: Hardly! An open and short are in fact reactive loads—they cannot absorb power (think about this!).

Specifically, for an open, we find $\theta_\Gamma = 0$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = 1$$

Likewise, for a short, we find that $\theta_\Gamma = \pi$, so that:

$$\Gamma_L = e^{j\theta_\Gamma} = -1$$

5. $X_L = 0$

For this case, the load impedance is purely real (e.g. a resistor):

$$Z_L = R_L$$

Thus, both the current through the load, and voltage across the load, are non-zero:

$$I_L = I(z = z_L) \neq 0 \quad \quad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that $Z_0$ is real (i.e., the line is lossless), we find that this load reflection coefficient must be a purely real value!
In other words:

$$\Re\{\Gamma_L\} = \frac{R-Z_0}{R+Z_0}$$

$$\Im\{\Gamma_L\} = 0$$

The magnitude is thus:

$$|\Gamma_L| = \frac{|R-Z_0|}{|R+Z_0|}$$

whereas the phase $\theta_{\Gamma}$ can take on one of two values:

$$\theta_{\Gamma} = \begin{cases} 0 & \text{if } \Re\{\Gamma_L\} > 0 \text{ (i.e., if } R_L > Z_0) \\ \pi & \text{if } \Re\{\Gamma_L\} < 0 \text{ (i.e., if } R_L < Z_0) \end{cases}$$

For this case, the impedance at the end of the line must be real ($Z(z = z_L) = R_L$). Thus, the current and the voltage at this point are precisely in phase.

However, even though the load impedance is real, the line impedance at all other points on the line is generally complex!

Moreover, the general current and voltage expressions, as well as reflection coefficient function, cannot be further simplified for the case where $Z_L = R_L$. 
Q: Why is that? When the load was purely imaginary (reactive), we were able to simply our general expressions, and likewise deduce all sorts of interesting results!

A: True! And here’s why. Remember, a lossless transmission line has series inductance and shunt capacitance only. In other words, a length of lossless transmission line is a purely reactive device (it absorbs no energy!).

* If we attach a purely reactive load at the end of the transmission line, we still have a completely reactive system (load and transmission line). Because this system has no resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly simplified.

* However, if we attach a purely real load to our reactive transmission line, we now have a complex system, with both real and imaginary (i.e., resistive and reactive) components. This complex case is exactly what our general expressions already describes—no further simplification is possible!

5. \( Z_L = R_L + jX_L \)

Now, let’s look at the general case, where the load has both a real (resistive) and imaginary (reactive) component.

Q: Haven’t we already determined all the general expressions (e.g., \( \Gamma_L, V(z), I(z), Z(z), \Gamma(z) \)) for this general case? Is there anything else left to be determined?
A: There is one last thing we need to discuss. It seems trivial, but its ramifications are very important!

For you see, the “general” case is not, in reality, quite so general. Although the reactive component of the load can be either positive or negative \((-\infty < X_L < \infty)\), the resistive component of a passive load must be positive \((R_L > 0)\)—there’s no such thing as negative resistor!

This leads to one very important and useful result. Consider the load reflection coefficient:

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(R_L + jX_L) - Z_0}{(R_L + jX_L) + Z_0} = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}
\]

Now let’s look at the magnitude of this value:
\[ |\Gamma_L|^2 = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L} \]
\[ = \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \]
\[ = \frac{(R_L^2 - 2R_L Z_0 + Z_0^2) + X_L^2}{(R_L^2 + 2R_L Z_0 + Z_0^2) + X_L^2} \]
\[ = \frac{(R_L^2 + Z_0^2 + X_L^2) - 2R_L Z_0}{(R_L^2 + Z_0^2 + X_L^2) + 2R_L Z_0} \]

It is apparent that since both \(R_L\) and \(Z_0\) are positive, the numerator of the above expression must be less than (or equal to) the denominator of the above expression.

- In other words, the magnitude of the load reflection coefficient is always less than or equal to one!

\[ |\Gamma_L| \leq 1 \quad \text{(for } R_L \geq 0) \]

Moreover, we find that this means the reflection coefficient function likewise always has a magnitude less than or equal to one, for all values of position \(z\).

\[ |\Gamma(z)| \leq 1 \quad \text{(for all } z) \]
Which means, of course, that the reflected wave will always have a magnitude less than that of the incident wave magnitude:

\[ |V^{-}(z)| \leq |V^{+}(z)| \quad (\text{for all } z) \]

We will find out later that this result is consistent with conservation of energy— the reflected wave from a passive load cannot be larger than the wave incident on it.