Symmetric Circuit Analysis

Consider the following D_1 symmetric two-port device:

Q: Yikes! The plane of reflection symmetry slices through two resistors. What can we do about that?

A: Resistors are easily split into two equal pieces: the 200Ω resistor into two 100Ω resistors in series, and the 50Ω resistor as two 100Ω resistors in parallel.
Recall that the **symmetry** of this 2-port device leads to **simplified** network matrices:

\[
\mathbf{S} = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{11} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{11} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{21} \\ Y_{21} & Y_{11} \end{bmatrix}
\]

**Q:** Yes, but can circuit symmetry likewise simplify the procedure of determining these elements? In other words, can symmetry be used to **simplify** circuit analysis?

**A:** You bet!

First, consider the case where we **attach sources** to circuit in a way that **preserves** the circuit **symmetry**:
But remember! In order for symmetry to be preserved, the source values on both sides (i.e, $I_s, V_s, Z_0$) must be identical!

Now, consider the voltages and currents within this circuit under this symmetric configuration:
Since this circuit possesses bilateral (reflection) symmetry (1 → 2, 2 → 1), symmetric currents and voltages must be equal:

\[
\begin{align*}
V_1 &= V_2 \\
V_{1a} &= V_{2a} \\
V_{1b} &= V_{2b} \\
V_{1c} &= V_{2c}
\end{align*}
\]

\[
\begin{align*}
I_1 &= I_2 \\
I_{1a} &= I_{2a} \\
I_{1b} &= I_{2b} \\
I_{1c} &= I_{2c} \\
I_{1d} &= I_{2d}
\end{align*}
\]

**Q:** Wait! This can’t possibly be correct! Look at currents \(I_{1a}\) and \(I_{2a}\), as well as currents \(I_{1d}\) and \(I_{2d}\). From KCL, this must be true:

\[
\begin{align*}
I_{1a} &= -I_{2a} \\
I_{1d} &= -I_{2d}
\end{align*}
\]

Yet you say that this must be true:

\[
\begin{align*}
I_{1a} &= I_{2a} \\
I_{1d} &= I_{2d}
\end{align*}
\]
There is an obvious contradiction here! There is no way that both sets of equations can simultaneously be correct, is there?

A: Actually there is! There is one solution that will satisfy both sets of equations:

\[ I_{1a} = I_{2a} = 0 \quad I_{1d} = I_{2d} = 0 \]

The currents are zero!

If you think about it, this makes perfect sense! The result says that no current will flow from one side of the symmetric circuit into the other.

If current did flow across the symmetry plane, then the circuit symmetry would be destroyed—one side would effectively become the “source side”, and the other the “load side” (i.e., the source side delivers current to the load side).

Thus, no current will flow across the reflection symmetry plane of a symmetric circuit—the symmetry plane thus acts as an open circuit!

The plane of symmetry thus becomes a virtual open!
Q: So what?

A: So what! This means that our circuit can be split apart into two separate but identical circuits. Solve one half-circuit, and you have solved the other!

\[ V_1 = V_2 = V_s \]
\[ V_{1a} = V_{2a} = 0 \]
\[ V_{1b} = V_{2b} = V_s / 2 \]
\[ V_{1c} = V_{2c} = V_s / 2 \]

\[ I_1 = I_2 = V_s / 200 \]
\[ I_{1a} = I_{2a} = 0 \]
\[ I_{1b} = I_{2b} = V_s / 200 \]
\[ I_{1c} = I_{2c} = V_s / 200 \]
\[ I_{1d} = I_{2d} = 0 \]
Now, consider another type of symmetry, where the sources are equal but opposite (i.e., 180 degrees out of phase).

Or,
Or,

This situation still preserves the symmetry of the circuit—somewhat. The voltages and currents in the circuit will now possess odd symmetry—they will be equal but opposite (180 degrees out of phase) at symmetric points across the symmetry plane.

\[
\begin{align*}
V_1 &= -V_2 \\
V_{1a} &= -V_{2a} \\
V_{1b} &= -V_{2b} \\
V_{1c} &= -V_{2c}
\end{align*}
\]

\[
\begin{align*}
I_1 &= -I_2 \\
I_{1a} &= -I_{2a} \\
I_{1b} &= -I_{2b} \\
I_{1c} &= -I_{2c} \\
I_{1d} &= -I_{2d}
\end{align*}
\]
Perhaps it would be easier to \textbf{redefine} the circuit variables as:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{circuit_diagram.png}
\end{figure}

\begin{align*}
V_1 &= V_2 \\
V_{1a} &= V_{2a} \\
V_{1b} &= V_{2b} \\
V_{1c} &= V_{2c}
\end{align*}

\begin{align*}
I_1 &= I_2 \\
I_{1a} &= I_{2a} \\
I_{1b} &= I_{2b} \\
I_{1c} &= I_{2c} \\
I_{1d} &= I_{2d}
\end{align*}

\textbf{Q: But wait! Again I see a problem. By KVL it is evident that:}

\[ V_{1c} = -V_{2c} \]

\textbf{Yet you say that} \( V_{1c} = V_{2c} \) \textbf{must be true!}

\textbf{A: Again, the solution to both equations is zero!}

\[ V_{1c} = V_{2c} = 0 \]
For the case of **odd symmetry**, the symmetric plane must be a plane of **constant potential** (i.e., constant voltage)—just like a short circuit!

Thus, for odd symmetry, the symmetric plane forms a **virtual short**.

This greatly simplifies things, as we can again break the circuit into **two** independent and (effectively) identical circuits!