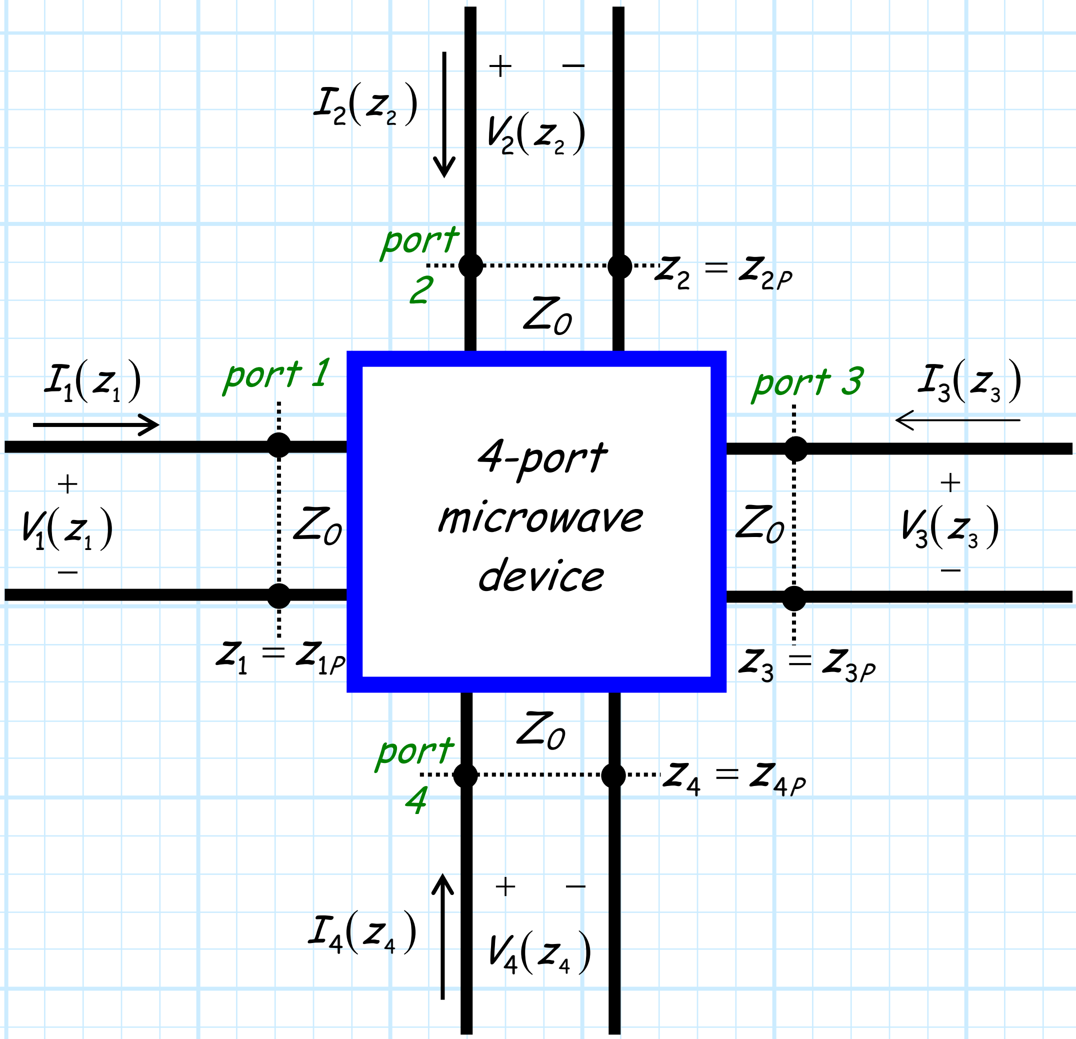
The Admittance Matrix

Consider again the **4-port** microwave device shown below:

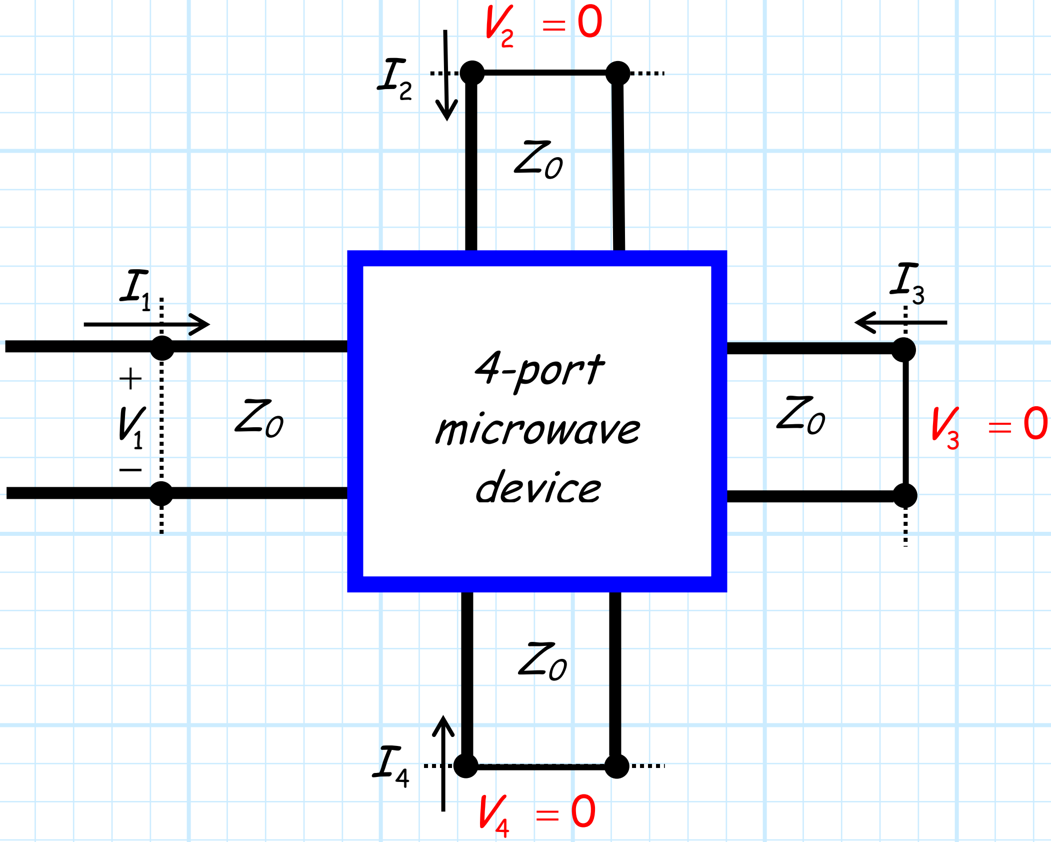


In addition to the Impedance Matrix, we can fully characterize this linear device using the **Admittance Matrix**.

The elements of the Admittance Matrix are the **trans-admittance** parameters Y_{mn}, defined as:

$$Y_{mn} = \frac{I_m}{V_n} \qquad \text{(given that } V_k = 0 \text{ for all } k \neq n\text{)}$$

Note here that the **voltage** at all but one port **must** be equal to **zero**. We can ensure that by simply placing a **short circuit** at these zero voltage ports!



Note that $Y_{mn} \neq 1/Z_{mn}$!

Now, we can thus **equivalently** state the definition of trans-admittance as:

$$\boxed{Y_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{short - circuited}\text{)}}$$

Just as with the trans-impedance values, we can use the trans-admittance values to evaluate general circuit problems, where **none** of the ports have zero voltage.

Since the device is **linear**, the current at any **one** port due to **all** the port currents is simply the coherent **sum** of the currents at that port due to **each** of the port voltages!

For example, the **current** at port 3 can be determined by:

$$I_3 = Y_{34}\, V_4 + Y_{33}\, V_3 + Y_{32}\, V_2 + Y_{31}\, V_1$$

More **generally**, the current at port m of an N-port device is:

$$\boxed{I_m = \sum_{n=1}^{N} Y_{mn}\, V_n}$$

This expression can be written in **matrix** form as:

$$\mathbf{I} = \mathcal{Y}\,\mathbf{V}$$

Where $\mathbf{I}$ is the **vector**:

$$\mathbf{I} = \left[I_1, I_2, I_3, \cdots, I_N\right]^T$$

and $\mathbf{V}$ is the vector:

$$\mathbf{V} = \left[V_1, V_2, V_3, \ldots, V_N\right]^T$$

And the matrix $\mathcal{Y}$ is called the **admittance matrix**:

$$\mathcal{Y} = \begin{bmatrix} Y_{11} & \ldots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{m1} & \ldots & Y_{mn} \end{bmatrix}$$

The admittance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the admittance matrix describes a multi-port device the way that Y_L describes a single-port device (e.g., a load)!

But **beware**! The values of the admittance matrix for a particular device or network, just like Y_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Y}(\omega) = \begin{bmatrix} Y_{11}(\omega) & \ldots & Y_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Y_{m1}(\omega) & \cdots & Y_{mn}(\omega) \end{bmatrix}$$

Q: *You said earlier that $Y_{mn} \neq 1/Z_{mn}$. Is there any* ***relationship*** *between the admittance and impedance matrix of a given device?*

A: I don't know! Let's see if **we** can figure it out.

Recall that we can determine the inverse of a matrix. Denoting the matrix inverse of the admittance matrix as $\mathcal{Y}^{-1}$, we find:

$$\begin{aligned} \mathbf{I} &= \mathcal{Y}\,\mathbf{V} \\ \mathcal{Y}^{-1}\,\mathbf{I} &= \mathcal{Y}^{-1}\left(\mathcal{Y}\,\mathbf{V}\right) \\ \mathcal{Y}^{-1}\,\mathbf{I} &= \left(\mathcal{Y}^{-1}\,\mathcal{Y}\right)\mathbf{V} \\ \mathcal{Y}^{-1}\,\mathbf{I} &= \mathbf{V} \end{aligned}$$

Meaning that:

$$\mathbf{V} = \mathcal{Y}^{-1}\,\mathbf{I}$$

But, we likewise know that:

$$\mathbf{V} = \mathcal{Z}\,\mathbf{I}$$

By comparing the two previous expressions, we can conclude:

$$\boxed{\mathcal{Z} = \mathcal{Y}^{-1} \qquad \text{and} \qquad \mathcal{Z}^{-1} = \mathcal{Y}}$$