The Binomial Multi-Section Transformer

Recall that a **multi-section matching network** can be described using the theory of small reflections as:

$$\begin{aligned}\Gamma_{in}(\omega) &= \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T} \\ &= \sum_{n=0}^{N} \Gamma_n \, e^{-j2n\omega T}\end{aligned}$$

where:

$$T \doteq \frac{\ell}{v_p} = \text{propagation time through 1 section}$$

Note that for a multi-section transformer, we have N **degrees of design freedom**, corresponding to the N characteristic impedance values Z_n.

Q: *What should the values of Γ_n (i.e., Z_n) be?*

A: We need to define N independent **design equations**, which we can then use to solve for the N values of **characteristic impedance** Z_n.

First, we start with a single **design frequency** ω_0, where we wish to achieve a **perfect** match:

$$\Gamma_{in}(\omega = \omega_0) = 0$$

That's just **one** design equation: we need ***N* - 1** more!

These addition equations can be selected using **many** criteria—one such criterion is to make the function $\Gamma_{in}(\omega)$ **maximally flat** at the point $\omega = \omega_0$.

To accomplish this, we first consider the **Binomial Function**:

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

This function has the desirable **properties** that:

$$\begin{aligned}\Gamma(\theta = \pi/2) &= A\left(1 + e^{-j\pi}\right)^N \\ &= A(1-1)^N \\ &= 0\end{aligned}$$

and that:

$$\left.\frac{d^n\,\Gamma(\theta)}{d\theta^n}\right|_{\theta=\pi/2} = 0 \text{ for } n = 1,2,3,\cdots,N-1$$

In other words, this Binomial Function is **maximally flat** at the point $\theta = \pi/2$, where it has a value of $\Gamma(\theta = \pi/2) = 0$.

Q: *So? What does **this** have to do with our multi-section matching network?*

A: Let's **expand** (multiply out the *N* identical product terms) of the Binomial Function:

$$\begin{aligned}\Gamma(\theta) &= A\left(1+e^{-j2\theta}\right)^N \\ &= A\left(C_0^N + C_1^N e^{-j2\theta} + C_2^N e^{-j4\theta} + C_3^N e^{-j6\theta} + \cdots + C_N^N e^{-j2N\theta}\right)\end{aligned}$$

where:

$$C_n^N \doteq \frac{N!}{(N-n)!\,n!}$$

Compare this to an ***N*-section** transformer function:

$$\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T}$$

and it is obvious the two functions have **identical** forms, **provided** that:

$$\Gamma_n = A\, C_n^N \qquad \text{and} \qquad \omega T = \theta$$

Moreover, we find that this function is very **desirable** from the standpoint of the a matching network. Recall that $\Gamma(\theta)=0$ at $\theta=\pi/2$--a **perfect** match!

Additionally, the function is **maximally flat** at $\theta=\pi/2$, therefore $\Gamma(\theta)\approx 0$ over a wide range around $\theta=\pi/2$--a **wide bandwidth**!

Q: *But how does $\theta = \pi/2$ relate to frequency ω?*

A: Remember that $\omega T = \theta$, so the value $\theta = \pi/2$ corresponds to the frequency:

$$\omega_0 = \frac{1}{T}\frac{\pi}{2} = \frac{v_p}{\ell}\frac{\pi}{2}$$

This frequency (ω_0) is therefore our **design** frequency—the frequency where we have a **perfect** match.

Note that the length ℓ has an interesting **relationship** with this frequency:

$$\ell = \frac{v_p}{\omega_0}\frac{\pi}{2} = \frac{1}{\beta_0}\frac{\pi}{2} = \frac{\lambda_0}{2\pi}\frac{\pi}{2} = \frac{\lambda_0}{4}$$

In other words, a **Binomial** Multi-section matching network will have a **perfect** match at the frequency where the section lengths ℓ are a **quarter wavelength**!

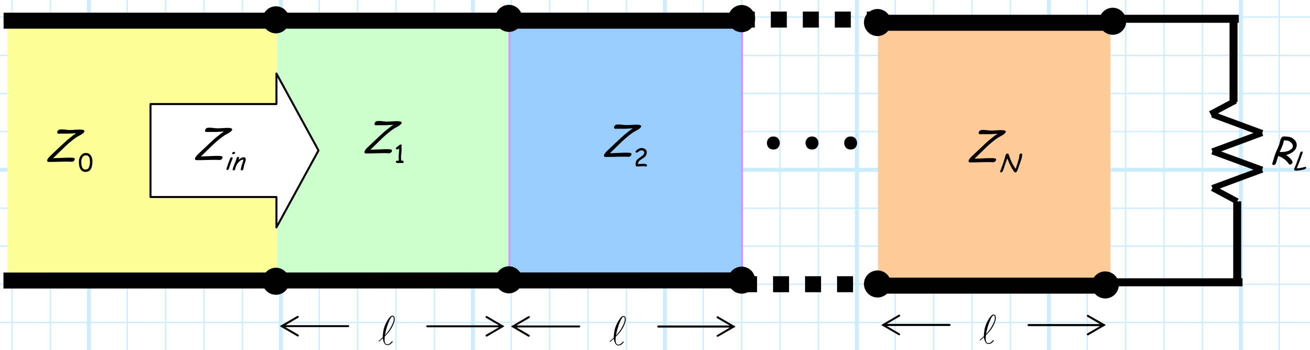
Thus, we have our **first design rule**:

Set section lengths ℓ so that they are a **quarter-wavelength** ($\lambda_0/4$) at the design frequency ω_0.

Q: *I see! And then we select all the values Z_n such that $\Gamma_n = A\, C_n^N$. But wait! **What** is the value of **A** ??*

A: *We can determine this value by evaluating a* ***boundary condition!***

Specifically, we can **easily** determine the value of $\Gamma(\omega)$ at $\omega = 0$.



Note as ω approaches **zero**, the electrical length $\beta\ell$ of each section will **likewise** approach zero. Thus, the input impedance Z_{in} will simply be equal to R_L as $\omega \rightarrow 0$.

As a result, the input reflection coefficient $\Gamma(\omega = 0)$ **must** be:

$$\Gamma(\omega = 0) = \frac{Z_{in}(\omega = 0) - Z_0}{Z_{in}(\omega = 0) + Z_0}$$

$$= \frac{R_L - Z_0}{R_L + Z_0}$$

However, we **likewise** know that:

$$\Gamma(0) = A\left(1 + e^{-j2(0)}\right)^N$$

$$= A(1+1)^N$$

$$= A\, 2^N$$

Equating the two expressions:

$$\Gamma(0) = A\,2^N = \frac{R_L - Z_0}{R_L + Z_0}$$

And therefore:

$$A = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0} \qquad \text{(}A\text{ can be negative!)}$$

We now have a form to calculate the **required marginal reflection coefficients** Γ_n:

$$\Gamma_n = A\,C_n^N = \frac{A\,N!}{(N-n)!\,n!}$$

Of course, we **also** know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

Equating the two and solving, we find that that the section characteristic impedances must satisfy:

$$Z_{n+1} = Z_n\frac{1+\Gamma_n}{1-\Gamma_n} = Z_n\frac{1+A\,C_n^N}{1-A\,C_n^N}$$

Note this is an **iterative** result—we determine Z_1 from Z_0, Z_2 from Z_1, and so forth.

> **Q:** *This result **appears** to be our second design equation. Is there some reason why you didn't draw a big blue box around it?*

> **A:** Alas, there is a **big problem** with this result.

Note that there are N+1 coefficients Γ_n (i.e., $n \in \{0,1,\cdots,N\}$) in the Binomial series, yet there are only N design degrees of freedom (i.e., there are only N transmission line sections!).

Thus, our design is a bit **over constrained**, a result that manifests itself the finally marginal reflection coefficient Γ_N.

Note from the iterative solution above, the **last** transmission line impedance Z_N is selected to satisfy the **mathematical** requirement of the **penultimate** reflection coefficient Γ_{N-1}:

$$\Gamma_{N-1} = \frac{Z_N - Z_{N-1}}{Z_N + Z_{N-1}} = A\, C_{N-1}^{N}$$

Thus the last impedance must be:

$$Z_N = Z_{N-1} \frac{1 + A\, C_{N-1}^{N}}{1 - A\, C_{N-1}^{N}}$$

But there is **one more** mathematical requirement! The last marginal reflection coefficient **must** likewise satisfy:

$$\Gamma_N = A\, C_N^N = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0}$$

where we have used the fact that $C_N^N = 1$.

But, we **just** selected Z_N to satisfy the requirement for Γ_{N-1},—we have no **physical** design parameter to satisfy this last **mathematical** requirement!

As a result, we find to our great consternation that the last requirement is not satisfied:

$$\Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq A\, C_N^N \quad !!!!!!$$

Q: *Yikes! Does this mean that the resulting matching network will **not** have the desired Binomial frequency response?*

A: That's **exactly** what it means!

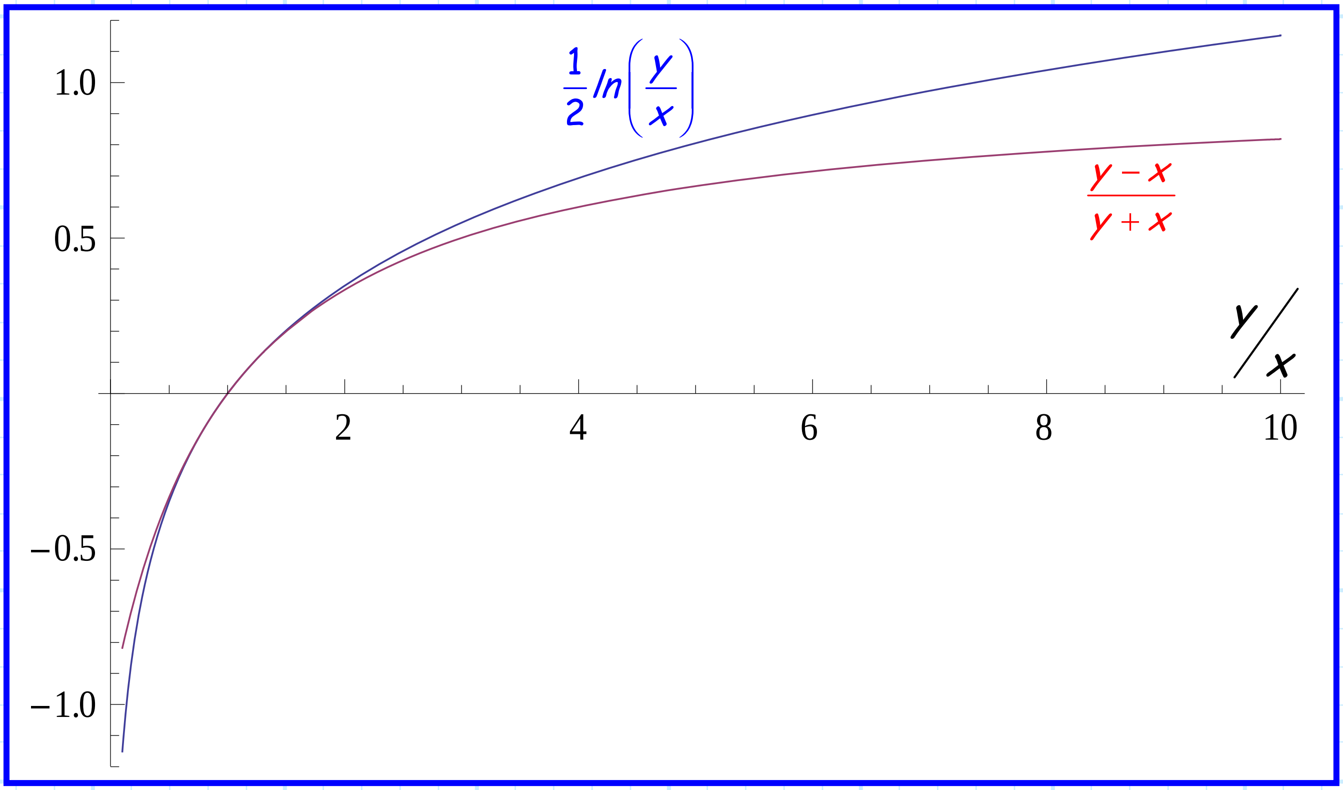
Q: *You big #%@#$%&!!!! **Why** did you **waste** all my time by discussing an over-constrained design problem that can't be built?*

A: Relax; there is a **solution** to our dilemma—albeit an **approximate** one.

You undoubtedly have previously used the **approximation**:

$$\frac{y-x}{y+x} \approx \frac{1}{2} ln\left(\frac{y}{x}\right)$$

An approximation that is especially **accurate** when $|y-x|$ is small (i.e., when $y/x \simeq 1$).



Now, we know that the values of Z_{n+1} and Z_n in a multi-section matching network are typically **very close**, such that $|Z_{n+1} - Z_n|$ is small. Thus, we use the approximation:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2} ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

Likewise, we can **also** apply this approximation (although not as accurately) to the value of A:

$$A = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0} \approx 2^{-(N+1)}\, ln\left(\frac{R_L}{Z_0}\right)$$

So, let's **start over**, only this time we'll use these **approximations**. First, determine A:

$$A \approx 2^{-(N+1)}\, ln\left(\frac{R_L}{Z_0}\right)$$

(A can be negative!)

Now use this result to calculate the **mathematically required** marginal reflection coefficients Γ_n:

$$\Gamma_n = A\, C_n^N = \frac{A\, N!}{(N-n)!\,n!}$$

Of course, we **also** know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n \approx \frac{1}{2}\, ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

Equating the two and solving, we find that that the section characteristic impedances **must** satisfy:

$$Z_{n+1} = Z_n \, exp\left[2\Gamma_n\right]$$

Now **this** is our **second design rule**. Note it is an **iterative** rule—we determine Z_1 from Z_0, Z_2 from Z_1, and so forth.

Q: *Huh? How is this any better? How does applying* ***approximate*** *math lead to a* ***better*** *design result??*

A: Applying these approximations help resolve our over-constrained problem. Recall that the over-constraint resulted in:

$$\Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq A\, C_N^N$$

But, as it turns out, these approximations leads to the happy situation where:

$$\Gamma_N \approx \frac{1}{2} ln\left(\frac{R_L}{Z_N}\right) = A\, C_N^N$$ ← **Sanity check!!**

provided that the value A is likewise the approximation given above.

Effectively, these approximations couple the results, such that each value of characteristic impedance Z_n **approximately** satisfies both Γ_n and Γ_{n+1}. Summarizing:

* If you use the **"exact"** design equations to determine the characteristic impedances Z_n, the last value Γ_N will exhibit a significant numeric error, and your design **will not** appear to be maximally flat.

* If you instead use the **"approximate"** design equations to determine the characteristic impedances Z_n, all values Γ_n will exhibit a slight error, but the resulting design **will** appear to be **maximally flat, Binomial** reflection coefficient function $\Gamma(\omega)$!

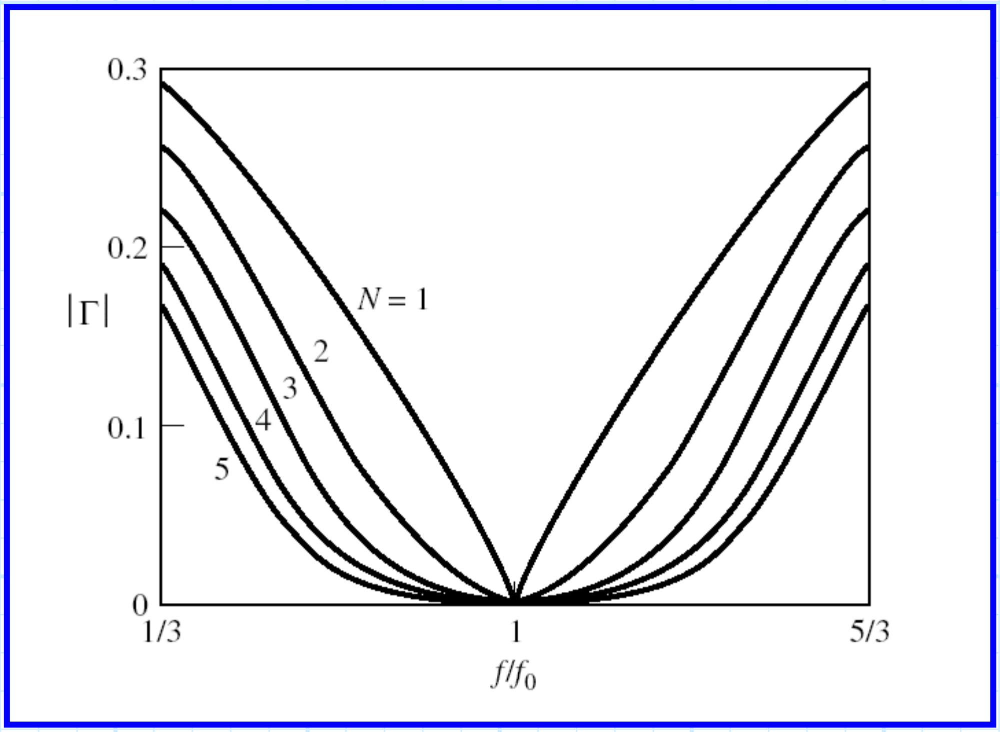


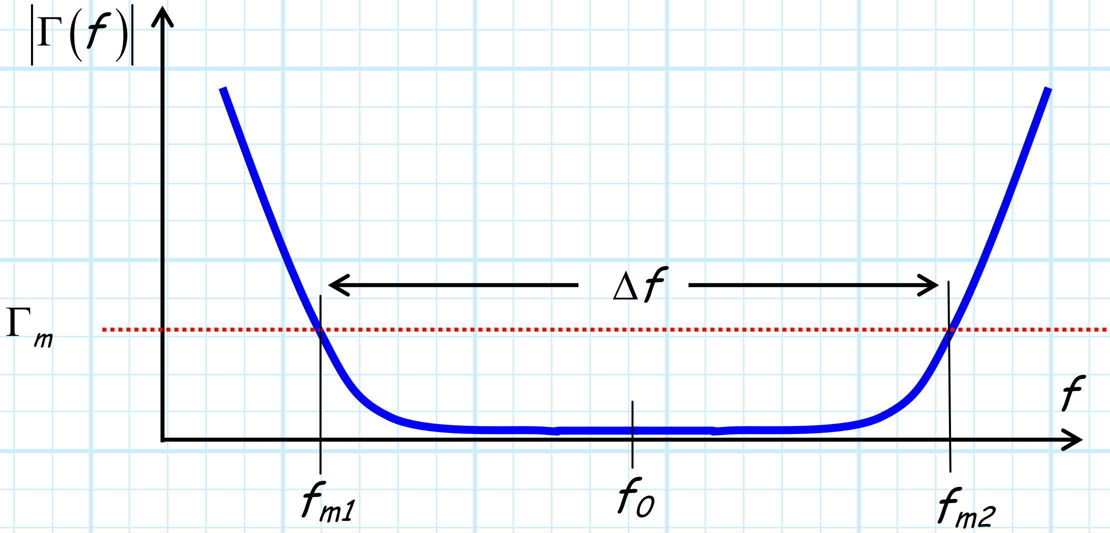
Figure 5.15 (p. 250)
*Reflection coefficient magnitude versus frequency for multisection **binomial** matching transformers of Example 5.6 Z_L = 50Ω and Z_0 = 100Ω.*

Note that as we **increase** the number of **sections**, the matching **bandwidth** increases.

Q: *Can we determine the **value** of this bandwidth?*

A: Sure! But we first must **define** what we mean by bandwidth.

As we move from the design (perfect match) frequency f_0 the value $|\Gamma(f)|$ will **increase**. At some frequency (f_m, say) the magnitude of the reflection coefficient will increase to some **unacceptably** high value (Γ_m, say). At that point, we **no longer** consider the device to be matched.



Note there are **two** values of frequency f_m—one value **less** than design frequency f_0, and one value **greater** than design frequency f_0. These two values define the **bandwidth** Δf of the matching network:

$$\Delta f = f_{m2} - f_{m1} = 2\left(f_0 - f_{m1}\right) = 2\left(f_{m2} - f_0\right)$$

Q: *So what is the **numerical** value of Γ_m?*

A: **I don't know**—it's up to **you** to decide!

Every engineer must determine what **they** consider to be an acceptable match (i.e., decide Γ_m). This decision depends on the **application** involved, and the **specifications** of the overall microwave system being designed.

However, we **typically** set Γ_m to be 0.2 or less.

Q: *OK, after we have selected Γ_m, can we determine the **two** frequencies f_m?*

A: Sure! We just have to do a little **algebra**.

We start by **rewriting** the Binomial function:

$$\begin{aligned}\Gamma(\theta) &= A\left(1+e^{-j2\theta}\right)^N \\ &= Ae^{-jN\theta}\left(e^{+j\theta}+e^{-j\theta}\right)^N \\ &= Ae^{-jN\theta}\left(e^{+j\theta}+e^{-j\theta}\right)^N \\ &= Ae^{-jN\theta}\left(2\cos\theta\right)^N\end{aligned}$$

Now, we take the **magnitude** of this function:

$$\begin{aligned}\left|\Gamma(\theta)\right| &= 2^N\left|A\right|\left|e^{-jN\theta}\right|\left|\cos\theta\right|^N \\ &= 2^N\left|A\right|\left|\cos\theta\right|^N\end{aligned}$$

Now, we **define** the values θ where $\left|\Gamma(\theta)\right| = \Gamma_m$ as θ_m. I.E., :

$$\begin{aligned}\Gamma_m &= \left|\Gamma(\theta = \theta_m)\right| \\ &= 2^N \left|A\right| \left|\cos\theta_m\right|^N\end{aligned}$$

We can now solve for θ_m (in **radians**!) in terms of Γ_m:

$$\theta_{m1} = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right] \qquad \theta_{m2} = \cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Note that there are **two solutions** to the above equation (one **less** that $\pi/2$ and one **greater** than $\pi/2$)!

Now, we can convert the values of θ_m into specific frequencies.

Recall that $\omega T = \theta$, therefore:

$$\omega_m = \frac{1}{T}\theta_m = \frac{v_p}{\ell}\theta_m$$

But recall also that $\ell = \lambda_0/4$, where λ_0 is the wavelength at the **design frequency** f_0 (not f_m!), and where $\lambda_0 = v_p/f_0$.

Thus we can conclude:

$$\omega_m = \frac{v_p}{\ell}\theta_m = \frac{4v_p}{\lambda_0}\theta_m = \left(4f_0\right)\theta_m$$

or:

$$f_m = \frac{1}{2\pi}\frac{v_p}{\ell}\theta_m = \frac{(4f_0)\,\theta_m}{2\pi} = \frac{(2f_0)\,\theta_m}{\pi}$$

where θ_m is expressed in **radians**. Therefore:

$$f_{m1} = \frac{2f_0}{\pi}\cos^{-1}\left[+\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right] \qquad f_{m2} = \frac{2f_0}{\pi}\cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Thus, the **bandwidth** of the binomial matching network can be determined as:

$$\begin{aligned}\Delta f &= 2\left(f_0 - f_{m1}\right)\\ &= 2f_0 - \frac{4f_0}{\pi}\cos^{-1}\left[+\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]\end{aligned}$$

Note that this equation can be used to determine the **bandwidth** of a binomial matching network, given Γ_m and number of sections N.

However, it can likewise be used to determine the **number of sections** N required to meet a specific bandwidth requirement!

Finally, we can list the **design steps** for a binomial matching network:

1. Determine the value N required to meet the bandwidth (Δf and Γ_m) requirements.

2. Determine the **approximate** value A from Z_0, R_L and N.

3. Determine the **marginal reflection coefficients** $\Gamma_n = A\, C_n^N$ required by the **binomial** function.

4. Determine the characteristic impedance of each section using the **iterative approximation**:

$$Z_{n+1} = Z_n \, exp\left[2\Gamma_n\right]$$

5. Perform the **sanity check**:

$$\Gamma_N \approx \frac{1}{2} ln\left(\frac{R_L}{Z_N}\right) = A\, C_N^N$$

6. Determine section **length** $\ell = \lambda_0/4$ for design frequency f_0.