The Chebyshev Matching Transformer

An **alternative** to Binomial (Maximally Flat) functions (and there are **many** such alternatives!) are **Chebyshev** polynomials.

Pafnuty Chebyshev

1821 -1894

Chebyshev solutions can provide functions $\Gamma(\omega)$ with **wider bandwidth** than the Binomial case—albeit at the "expense" of **passband ripple**.

It is evident from the plot below that the Chebychev response is **far** from maximally **flat**! Instead, a Chebyshev matching network exhibits a "**ripple**" in its passband. Note the magnitude of this ripple never exceeds some **maximum** value Γ_m (within the **pass-band**).

The two frequencies at which the value $|\Gamma(\omega)|$ **does** increase beyond Γ_m define the **bandwidth** of the matching network. We denote these frequencies $\omega_m = 2\pi f_m$ (the plot above shows the locations of the frequencies for ***N* =4**).

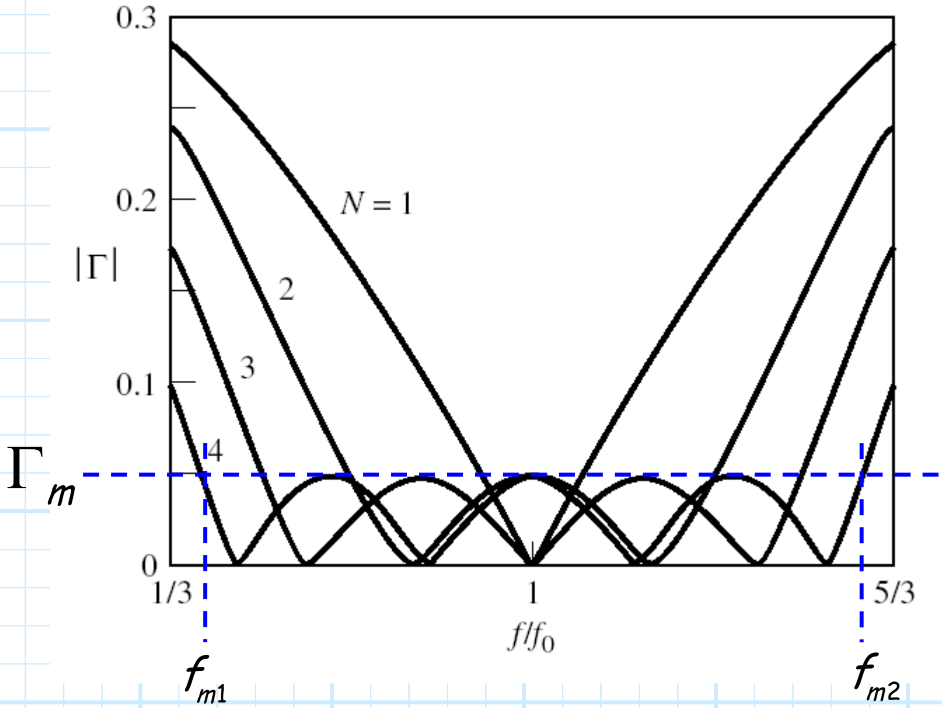


Figure 5.17 (p. 255)
Reflection coefficient magnitude versus frequency for the Chebyshev multisection matching transformers of Example 5.7.

→ Note that the **bandwidth** defined by f_m **increases** as the **number of sections** N is increased.

→ Note also that the reflection coefficient in **not necessarily zero** at the design frequency f_0 !!!

Instead, we find:

$$\left|\Gamma(f=f_0)\right| = \begin{cases} 0 \text{ for odd-valued } N \\ \Gamma_m \text{ for even-valued } N \end{cases}$$

Now, Chebyshev transformers are **symmetric**, i.e.:

$$\Gamma_0 = \Gamma_N , \;\; \Gamma_1 = \Gamma_{N-1} , \text{ etc.}$$

Recall we can express the multi-section function $\Gamma(\theta)$ (where $\theta = \omega T = \beta\ell$) in a **simpler form** when the transformer is symmetric:

$$\Gamma(\theta) = 2\, e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + G(\theta)\right]$$

where:

$$G(\theta) = \begin{cases} \dfrac{1}{2}\Gamma_{N/2} & \text{for } N \text{ even} \\ \Gamma_{(N-1)/2} \cos\theta & \text{for } N \text{ odd} \end{cases}$$

Now, the reflection coefficient of a **Chebyshev** matching network has the form:

$$\Gamma(\theta) = A\, e^{-jN\theta}\, T_N\left(\frac{\cos\theta}{\cos\theta_m}\right)$$
$$= A\, e^{-jN\theta}\, T_N\left(\cos\theta \sec\theta_m\right)$$

where $\theta_m = \omega_m T$

The function $T_N(\cos\theta \sec\theta_m)$ is a **Chebyshev polynomial** of order N.

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

We can determine **higher**-order Chebyshev polynomials using the **recursive formula**:

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

Inserting the **substitution**:

$$x = \cos\theta \sec\theta_m$$

into the Chebyshev polynomials above (and then applying a few **trig identities**) gives the results shown in equations 5.60 on page 252 of **your** book:

$$T_1(\cos\theta\sec\theta_m) = \cos\theta\sec\theta_m$$

$$\begin{aligned} T_2(\cos\theta\sec\theta_m) &= \sec^2\theta_m(1+\cos 2\theta)-1 \\ &= \sec^2\theta_m \cos 2\theta + (\sec^2\theta_m - 1) \end{aligned}$$

$$\begin{aligned} T_3(\cos\theta\sec\theta_m) &= \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m\cos\theta \\ &= \sec^3\theta_m \cos 3\theta + (3\sec^2\theta_m - 3)\sec\theta_m \cos\theta \end{aligned}$$

$$\begin{aligned} T_4(\cos\theta\sec\theta_m) &= \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) \\ &\quad -4\sec^2\theta_m(\cos 2\theta + 1) + 1 \\ &= \sec^4\theta_m \cos 4\theta \\ &\quad + 4\sec^2\theta_m(\sec^2\theta_m - 1)\cos 2\theta \\ &\quad + (3\sec^4\theta_m - 4\sec^2\theta_m + 1) \end{aligned}$$

Note that these polynomials have a $\cos N\theta$ term, a $\cos(N-2)\theta$ term, $\cos(N-4)\theta$ term, etc.—**just** like the **symmetric** multi-section transformer function!

For example, a 4-section **Chebyshev** matching network will have the form:

$$\begin{aligned} \Gamma(\theta) &= A e^{-j4\theta} T_4(\cos\theta\sec\theta_m) \\ &= A e^{-j4\theta}\Big[\sec^4\theta_m \cos 4\theta \\ &\quad + 4\sec^2\theta_m(\sec^2\theta_m - 1)\cos 2\theta \\ &\quad + (3\sec^4\theta_m - 4\sec^2\theta_m + 1)\Big] \end{aligned}$$

While the **general form** of a 4-section matching transformer is a polynomial with these **same** terms:

$$\Gamma_4(\theta) = 2\,e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \frac{1}{2}\Gamma_2\right]\Big|_{N=4}$$

$$= 2\,e^{-j4\theta}\left[\Gamma_0 \cos 4\theta + \Gamma_1 \cos(4-2)\theta + \frac{1}{2}\Gamma_2\right]$$

$$= 2\,e^{-j4\theta}\left[\Gamma_0 \cos 4\theta + \Gamma_1 \cos 2\theta + \frac{1}{2}\Gamma_2\right]$$

Thus, we can **determine** the values of marginal reflection coefficients $\Gamma_0, \Gamma_1, \Gamma_2$ by simply **equating** the **3 terms** of the two previous expressions:

$$2\,e^{-j4\theta}\Gamma_0 \cos 4\theta = A\,e^{-j4\theta} \sec^4\theta_m \cos 4\theta$$

$$\Rightarrow \Gamma_0 = \frac{1}{2} A \sec^4\theta_m$$

$$2\,e^{-j4\theta}\Gamma_1 \cos 2\theta = A\,e^{-j4\theta}\, 4 \sec^2\theta_m \left(\sec^2\theta_m - 1\right)\cos 2\theta$$

$$\Rightarrow \Gamma_1 = A\, 2 \sec^2\theta_m \left(\sec^2\theta_m - 1\right)$$

$$2\,e^{-j4\theta}\frac{1}{2}\Gamma_2 = A\,e^{-j4\theta}\left(3\sec^4\theta_m - 4\sec^2\theta_m + 1\right)$$

$$\Rightarrow \Gamma_2 = A\left(3\sec^4\theta_m - 4\sec^2\theta_m + 1\right)$$

And because it's **symmetric**, we also know that $\Gamma_3 = \Gamma_1$ and $\Gamma_4 = \Gamma_0$.

Now, we can **again** (i.e., as we did in the binomial matching network) determine the values of characteristic impedance Z_n, using the **iterative** (approximate) relationship:

$$Z_{n+1} = Z_n \exp\left[2\Gamma_n\right]$$

Q: *But what about the value of* A *?*

A: Also using the same **boundary condition** analysis that we used for the binomial function, we find from our **transmission line** knowledge that for **any** multi-section matching network, at $\theta = 0$:

$$\Gamma(\theta = 0) = \frac{R_L - Z_0}{R_L + Z_0}$$

Likewise, a **Chebyshev** matching network will have the specific value at $\theta = 0$ of:

$$\begin{aligned}\Gamma(\theta = 0) &= A\, e^{-jN(0)}\, T_N\left(\sec\theta_m \cos(0)\right) \\ &= A\, T_N\left(\sec\theta_m\right)\end{aligned}$$

These two results must of course be **equal**, and equating them allows us to solve for A:

$$A = \frac{R_L - Z_0}{R_L + Z_0}\,\frac{1}{T_N\left(\sec\theta_m\right)}$$

Here again (just like the binomial case) we will find it advantageous to use the approximation:

$$\frac{R_L - Z_0}{R_L + Z_0} \approx \frac{1}{2} \ln\left(\frac{R_L}{Z_0}\right)$$

So that the value A is **approximately**:

$$A \approx \frac{1}{2} \frac{\ln\left(R_L / Z_0\right)}{T_N\left(\sec\theta_m\right)}$$

(A can be negative!)

Q: *Gosh, both the values of marginal reflection coefficients Γ_n and value A depend explicitly on $\sec\theta_m$. Just what is this value, and how do we determine it?*

A: Recall that $\theta_m = \omega_m T$, where T is the propagation time through one section (i.e., $T = \ell / v_p$) and $\omega_m = 2\pi f_m$ defines the bandwidth associated with ripple value Γ_m. Thus, for a given ripple Γ_m and value N, we can find $\sec\theta_m$ by solving this equations

$$\begin{aligned}
\Gamma_m &= \left|\Gamma\left(\theta = \theta_m\right)\right| \\
&= \left|A\, e^{-jN\theta_m}\, T_N\left(\sec\theta_m \cos\theta_m\right)\right| \\
&= |A| \left|e^{-jN\theta_m}\right| \left|T_N\left(\sec\theta_m \cos\theta_m\right)\right| \\
&= |A| \left|T_N(1)\right|
\end{aligned}$$

Q: *Yikes! The value $\sec\theta_m$ disappeared from this equation; how can we use this to determine $\sec\theta_m$?*

A: Don't forget the value A !! Inserting this into the expression:

$$\begin{aligned}\Gamma_m &= \left|\Gamma\left(\theta=\theta_m\right)\right| \\ &= |A|\,\left|T_N(1)\right| \\ &= \frac{1}{2}\frac{\left|\ln\left(R_L/Z_0\right)\right|}{\left|T_N\left(\sec\theta_m\right)\right|}\left|T_N(1)\right|\end{aligned}$$

One **property** of Chebyshev polynomials is that:

$$\left|T_N(1)\right| = 1 \quad \text{for **all** values } N$$

Therefore, we **conclude**:

$$\Gamma_m = \left|\Gamma\left(\theta=\theta_m\right)\right| = \frac{1}{2}\frac{\left|\ln\left(R_L/Z_0\right)\right|}{\left|T_N\left(\sec\theta_m\right)\right|}$$

And so rearranging:

$$\left|T_N\left(\sec\theta_m\right)\right| = \frac{1}{2\Gamma_m}\left|\ln\left(R_L/Z_0\right)\right|$$

Note that R_L, Z_0 and Γ_m are design parameters, thus we can use the above to determine $\left|T_N\left(\sec\theta_m\right)\right|$, and thus $\sec\theta_m$!!!

Q: *Um, I'm not at all clear about how one determines $sec\,\theta_m$ from the value $\left|T_N\left(sec\,\theta_m\right)\right|$. Can you be more specific??*

A: Perhaps I should.

Recall that the Chebyshev polynomials we use in the design of this matching transformer were of the form $T_N\left(cos\,\theta\, sec\,\theta_m\right)$, i.e.:

$$\Gamma(\theta) = A\, e^{-jN\theta}\, T_N\left(cos\,\theta\, sec\,\theta_m\right)$$

Note the value $\left|T_N\left(sec\,\theta_m\right)\right|$ is just the magnitude of the Chebyshev $T_N\left(cos\,\theta\, sec\,\theta_m\right)$ when evaluated at $\cos\theta = 1$, which is the case when $\theta = 0$!!

$$\left|T_N\left(cos\,\theta\, sec\,\theta_m\right)\right|_{\theta=0} = \left|T_N\left(cos\,0\, sec\,\theta_m\right)\right| = \left|T_N\left(sec\,\theta_m\right)\right|$$

Now, **it can be shown that** (← a phrase professors use while in hand-waving mode!) for all values θ **outside the passband** of the matching network, the general form of the Chebyshev polynomial can **also** be written as:

$$T_N\left(sec\,\theta_m\, cos\,\theta\right) = cosh\left[N\, cosh^{-1}\left(sec\,\theta_m\, cos\,\theta\right)\right]$$

Note the value $\theta = 0 = \omega T$ means that the frequency $\omega = 0$. This frequency is most definitely **outside** the passband, and thus according to the **above** expression:

$$T_N\left(\sec\theta_m \cos\left(\theta=0\right)\right)=T_N\left(\sec\theta_m\right)$$
$$=\cosh\left[N\cosh^{-1}\left(\sec\theta_m\right)\right]$$

Likewise,

Q: *I know I **should** remember exactly what function* **cosh** *is, but I don't. Can you help **refresh** my memory?*

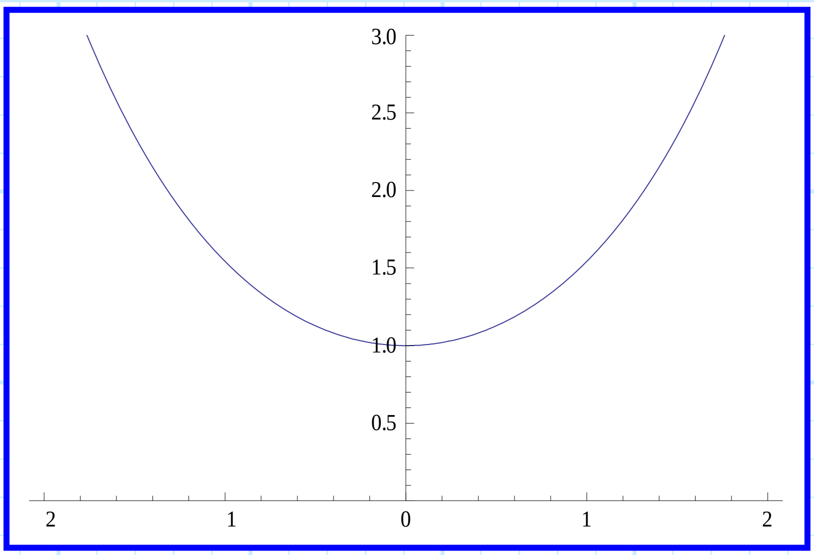
A: The function *cosh* is the **hyperbolic cosine**. Recall that **cosine** (the "regular" kind) can be expressed as:

$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$$

Similarly, the **hyperbolic cosine** is:

$$\cosh x=\frac{e^{x}+e^{-x}}{2}$$

Plotting $\cosh x$ from $x=-2$ to $x=+2$:



Hopefully it is apparent to **you**, from the above expression for $\cosh x$, that it results in a real value that is always greater than or equal to one (provided x is real):

$$\cosh x \geq 1 \qquad for \; -\infty < x < \infty$$

Thus $\left|\cosh x\right| = \cosh x$.

Q: *What about* $\cosh^{-1} y$ *?*

A: The function $\cosh^{-1} y$ is the **inverse** hyperbolic cosine, ***aka*** the hyperbolic **arccosine** (arcosh *y*). Note that this value is defined only for $y \geq 1$, and is specifically:

$$\cosh^{-1} y = \pm \ln\left(y + \sqrt{y^2 - 1}\right) \qquad for \; y \geq 1$$

Note that there is always **two solutions** (positive and negative) for the inverse hyperbolic cosine!

You will find that most scientific **calculators** support the *cosh* and *cosh*$^{-1}$ functions as well.

Anyway, combining the previous results, we find:

$$\left|T_N\left(sec\,\theta_m\right)\right| = \frac{1}{2\Gamma_m}\left|ln\frac{R_L}{Z_0}\right| = \left|cosh\left[N\,cosh^{-1}\left(sec\,\theta_m\right)\right]\right|$$

Therefore:

$$\frac{1}{2\Gamma_m}\left|ln\frac{R_L}{Z_0}\right| = cosh\left[N\, cosh^{-1}\left(sec\,\theta_m\right)\right]$$

Now (finally!) we have a form that can be manipulated algebraically. Solving the above equation for $sec\,\theta_m$:

$$sec\,\theta_m = \pm cosh\left[\frac{1}{N}\, cosh^{-1}\left(\frac{1}{2\Gamma_m}\left|ln\frac{R_L}{Z_0}\right|\right)\right]$$

Of course, the specific values of θ_m can be determined with the $\sec^{-1}$ (i.e., arcsecant) function.

Note there are two solutions for $\sec^{-1}$ (one for the plus sign and one for the minus); The two solutions are related as:

$$\theta_1 = \pi - \theta_2$$

Now, we can convert the values of θ_m into specific bandwidth frequencies f_{m1} and f_{m2}.

Since $\theta = \omega T$, we find:

$$\omega_{m1} = \frac{1}{T}\theta_m = \frac{v_p}{\ell}\theta_1$$

And similarly:

$$\omega_{m2} = \frac{1}{T}\theta_m = \frac{v_p}{\ell}\theta_2$$

Note that there are **two** solutions θ_m for this equation—**one** value of θ_m (θ_1)will be **less** than $\pi/2$ (defining the **lower** passband frequency), while the **other** (θ_2) will be **greater** than $\pi/2$ (defining the **upper** passband frequency).

Moreover, we find that the **two** values of θ_m will be **symmetric** about the value $\pi/2$! For example, if the **lower** value θ_1 is $\pi/2-\pi/10$, then the **upper** value θ_2 will be $\pi/2+\pi/10$.

Q: *So??*

A: This means that the **center** of the passband will be defined by the value $\theta=\pi/2$—and the center of the passband is our **design frequency** ω_0! In other words, since $\omega_0 T=\pi/2$:

$$\omega_0=\frac{\pi}{2}\frac{1}{T}=\frac{\pi}{2}\frac{v_p}{\ell}$$

Thus, we set the center (i.e., design) frequency by **selecting** the proper value of **section length** ℓ. Note the above expression is **precisely** the same result obtained for the **Binomial** matching network, and thus we have precisely the **same** design rule!

That **design rule** is, set the section lengths ℓ such that they are a **quarter wavelength** at the **design frequency** ω_0:

$$\ell = \frac{\lambda_0}{4}$$

where $\lambda_0 = v_p / \omega_0$.

Summarizing, the Chebyshev matching network design **procedure** is:

1. **Determine** the value N required to meet the bandwidth and ripple Γ_m requirements.

2. Determine the **Chebychev function** $\Gamma(\theta) = A\, e^{-jN\theta}\, T_N\left(\cos\theta \sec\theta_m\right)$.

3. Determine all Γ_n by **equating terms** with the symmetric multisection transformer expression:

$$\Gamma(\theta) = 2\, e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + G(\theta)\right]$$

4. **Calculate** all Z_n using the approximation:

$$\Gamma_n = \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n}$$

5. Determine section **length** $\ell = \lambda_0 / 4$.