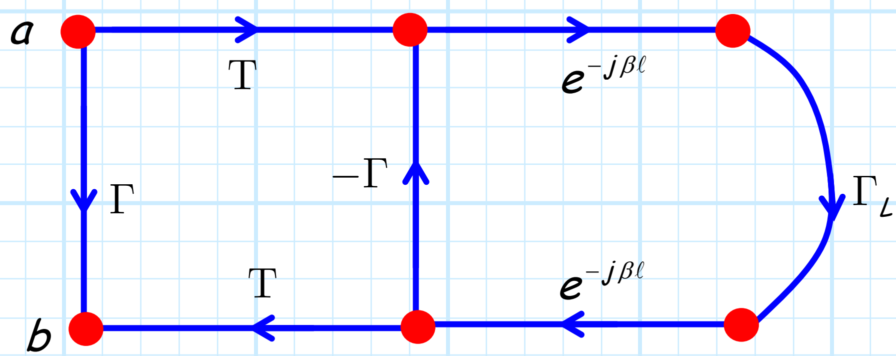
The Frequency Response of a Quarter-Wave Matching Network

Q: *You have once again provided us with **confusing** and perhaps useless information. The quarter-wave matching network has an **exact** SFG of:*

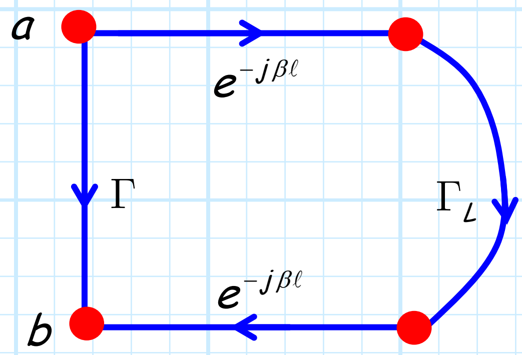


*Using our **reduction rules**, we can **quickly** conclude that:*

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \frac{\mathrm{T}^2 \, \Gamma_L \, e^{-j2\beta\ell}}{1 - \Gamma \, \Gamma_L}$$

*You could have left this **simple** and **precise** analysis **alone**—BUT **NOOO!!***

***You** had to foist upon us a long, **rambling** discussion of "the propagation series" and "direct paths" and "the theory of* *small reflections", culminating with the **approximate** (i.e., less accurate!) SFG:*



*From which we were able to conclude the **approximate** (i.e., less accurate!) result:*

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \Gamma_L \, e^{-j2\beta\ell}$$

*The **exact** result was **simple**—and **exact**! **Why** did you make us determine this **approximate** result?*

A: In a word: frequency response*.

Although the exact analysis is **about** as simple to determine as the approximation provided by the theory of small reflections, the **mathematical form** of the result is much simpler to **analyze** and/or **evaluate** (e.g., no **fractional** terms!).

Q: *What exactly would we be analyzing and/or evaluating?*

A: The **frequency response** of the matching network, for one thing.

* OK, **two** words.

Remember, all matching networks must be **lossless**, and so must be made of **reactive** elements (e.g., lossless transmission lines). The impedance of every reactive element is a **function of frequency**, and so too then is Γ_{in}.

Say we wish to determine this **function** $\Gamma_{in}(\omega)$.

Q: *Isn't* $\Gamma_{in}(\omega)=0$ *for a quarter wave matching network?*

A: Oh my gosh **no**! A properly designed matching network will typically result in a perfect match (i.e., $\Gamma_{in}=0$) at **one frequency** (i.e., the design frequency). However, if the signal frequency is **different** from this design frequency, then no match will occur (i.e., $\Gamma_{in}\neq 0$).

Recall we discussed this behavior **before**:

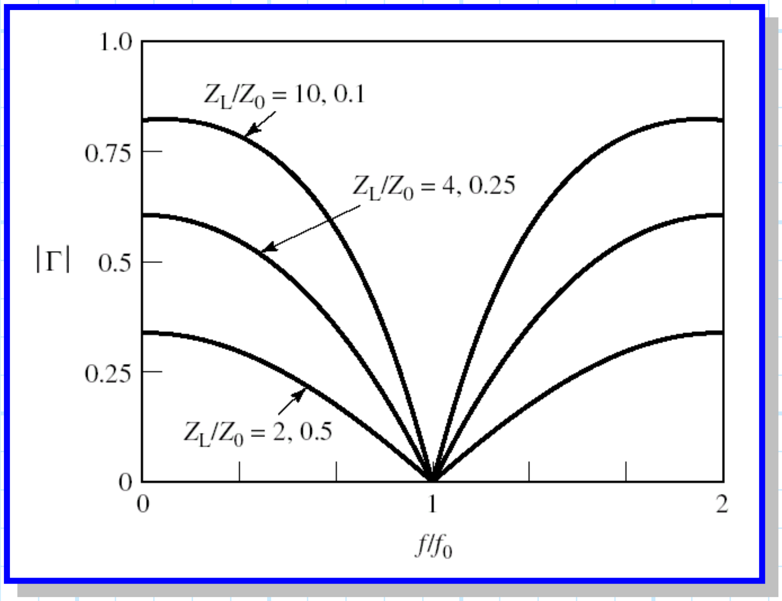


Figure 5.12 (p. 243)
Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

Q: *But **why** is the result:*

$$\Gamma_{in} = \Gamma + \frac{T^2 \, \Gamma_L \, e^{-j2\beta\ell}}{1 - \Gamma \, \Gamma_L}$$

or its approximate form:

$$\Gamma_{in} = \Gamma + \Gamma_L \, e^{-j2\beta\ell}$$

*dependent on **frequency**? I don't **see** frequency variable ω anywhere in these results!*

A: Look **closer**!

Remember that the value of spatial frequency β (in radians/meter) is dependent on the frequency ω of our eigen function (aka "the signal"):

$$\beta = \left(\frac{1}{v_p}\right)\omega$$

where **you** will recall that v_p is the propagation velocity of a wave moving along a transmission line. This velocity is a constant (i.e., $v_p = 1/\sqrt{LC}$), and so the spatial frequency β is directly proportional to the temporal frequency ω.

Thus, we can rewrite:

$$\beta \ell = \frac{\omega \ell}{v_p} = \omega T$$

Where $T = \ell / v_p$ is the **time** required for the wave to **propagate** a distance ℓ down a transmission line.

As a result, we can write the input reflection coefficient as a function of **spatial frequency** β:

$$\Gamma_{in}(\beta) = \Gamma + \Gamma_L \, e^{-j2\beta\ell}$$

Or equivalently as a function of **temporal frequency** ω:

$$\Gamma_{in}(\omega) = \Gamma + \Gamma_L \, e^{-j2\omega T}$$

Frequently, the reflection coefficient is simply written in terms of the **electrical length** θ of the transmission line, which is simply the **difference in relative phase** between the wave at the beginning and end of the length ℓ of the transmission line.

$$\beta\ell = \theta = \omega T$$

So that:

$$\Gamma_{in}(\theta) = \Gamma + \Gamma_L \, e^{-j2\theta}$$

Note we can simply insert the value $\theta = \beta\ell$ into the expression above to get $\Gamma_{in}(\beta)$, or insert $\theta = \omega T$ into the expression to get $\Gamma_{in}(\omega)$.

Now, we know that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, so the reflection coefficient function can be written as:

$$\Gamma_{in}(\theta) = \Gamma_L \left(1 + e^{-j2\theta}\right)$$

Note that: $$1 = e^{j0} = e^{-j(\theta-\theta)} = e^{-j\theta}\, e^{+j\theta}$$

And that: $$e^{-j2\theta} = e^{-j(\theta+\theta)} = e^{-j\theta}\, e^{-j\theta}$$

And so:

$$\begin{aligned}\Gamma_{in}(\theta) &= \Gamma_L \left(1 + e^{-j2\theta}\right) \\ &= \Gamma_L \left(e^{-j\theta}\, e^{+j\theta} + e^{-j\theta}\, e^{-j\theta}\right) \\ &= \Gamma_L\, e^{-j\theta} \left(e^{+j\theta} + e^{-j\theta}\right) \\ &= \Gamma_L\, e^{-j\theta} \left(2\cos\theta\right)\end{aligned}$$

Where we have used **Euler's equation** to determine that:

$$e^{+j\theta} + e^{-j\theta} = 2\cos\theta$$

Now, let's determine the **magnitude** of our result:

$$\left|\Gamma_{in}(\theta)\right| = \left|\Gamma_L\right| \left|e^{-j\theta}\right| 2 \left|\cos\theta\right| = 2\left|\Gamma_L\right| \left|\cos\theta\right|$$

Note that $\left|\Gamma_{in}(\theta)\right|$ is **zero-valued** only when $\cos\theta = 0$. This of course occurs when $\theta = \pi/2$:

$$\left|\Gamma_{in}(\theta)\right|_{\theta=\pi/2} = 2\left|\Gamma_L\right| \left|\cos \pi/2\right| = 0$$

In other words, a **perfect match** occurs when $\theta = \pi/2$!!

Q: *What the heck does this mean?*

A: Remember, $\theta = \beta \ell$. Thus if $\theta = \pi/2$:

$$\ell = \frac{\theta}{\beta} = \frac{\pi/2}{2\pi/\lambda} = \frac{\lambda}{4} \quad !!$$

As we (should have) suspected, the match occurs at the frequency whose wavelength is equal to **four times** the matching (Z_1) transmission line length, i.e. $\lambda = 4\ell$.

In other words, a perfect match occurs at the **frequency** where $\ell = \lambda/4$.

Note the **physical** length ℓ of the transmission line does **not** change with frequency, but the signal **wavelength** does:

$$\lambda = \frac{v_p}{f}$$

Q: *So, at precisely what **frequency** does a quarter-wave transformer with length ℓ provide a **perfect** match?*

A: Recall also that $\theta = \omega T$, where $T = \ell / v_p$. Thus, for $\theta = \pi/2$:

$$\theta = \frac{\pi}{2} = \omega T \qquad \Rightarrow \qquad \omega = \frac{\pi}{2}\frac{1}{T} = \frac{\pi}{2}\frac{v_p}{\ell}$$

This frequency is called the **design frequency** of the matching network—it's the frequency where a **perfect** match occurs. We denote this as frequency ω_0, which has wavelength λ_0, i.e.:

$$\omega_0 = \frac{\pi}{2T} = \pi \frac{v_p}{2\ell} \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{4T} = \frac{v_p}{4\,\ell} \qquad \lambda_0 = \frac{v_p}{f_0} = 4 v_p T = 4\ell$$

Given this, yet **another way** of expressing $\theta = \beta\ell$ is:

$$\theta = \beta\ell = \frac{\omega}{v_p}\left(\pi \frac{v_p}{2\omega_0}\right) = \pi \frac{\omega}{2\omega_0} = \pi \frac{f}{2f_0}$$

Thus, we conclude:

$$\left|\Gamma_{in}(f)\right| = 2\left|\Gamma_L\right| \left|\cos\left(\pi \tfrac{f}{2f_0}\right)\right|$$

From this result we can determine (approximately) the **bandwidth** of the quarter-wave transformer!

First, we must **define** what we mean by bandwidth. Say the **maximum** acceptable level of the reflection coefficient is value Γ_m. This is an arbitrary value, set by **you** the microwave engineer (typical values of Γ_m range from 0.05 to 0.2).

We will denote the frequencies where this maximum value Γ_m occurs f_m. In other words:

$$\left|\Gamma_{in}(f = f_m)\right| = \Gamma_m = 2\left|\Gamma_L\right| \left|\cos\left(\pi \tfrac{f_m}{2f_0}\right)\right|$$

There are **two solutions** to this equation, the first is:

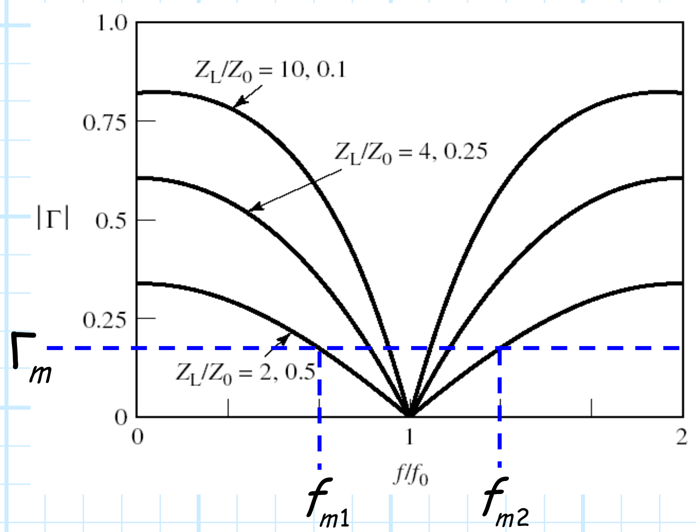
$$f_{m1} = \frac{2f_0}{\pi} \cos^{-1}\left(\frac{\Gamma_m}{2\left|\Gamma_L\right|} \right)$$

And the second:

$$f_{m2} = \frac{2f_0}{\pi} \cos^{-1}\left(-\frac{\Gamma_m}{2\left|\Gamma_L\right|} \right)$$

Important note! Make sure $\cos^{-1}x$ is expressed in **radians**!

You will find that $f_{m1} < f_0 < f_{m2}$ so, the values f_{m1} and f_{m2} define the **lower** and **upper** limits on matching network **bandwidth**.



All this analysis was brought to you by the **"simple" mathematical form** of $\Gamma_{in}(f)$ that resulted from the theory of small reflections!