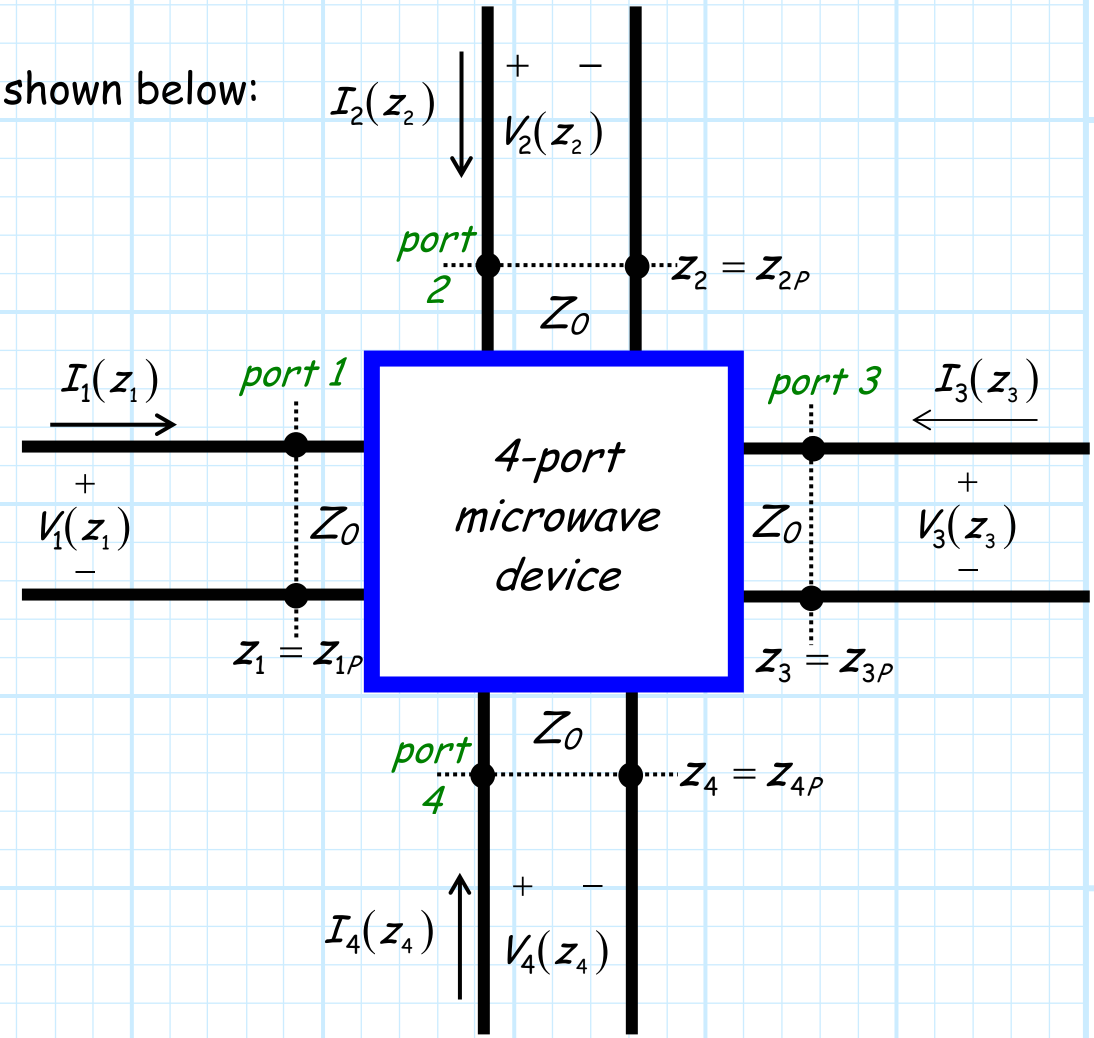
The Impedance Matrix

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **impedance matrix**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Thus, the **voltage** and **current** at port n is:

$$V_n(z_n = z_{nP}) \qquad\qquad I_n(z_n = z_{nP})$$

We can **simplify** this cumbersome notation by simply **defining** port n current and voltage as I_n and V_n:

$$V_n = V_n(z_n = z_{nP}) \qquad\qquad I_n = I_n(z_n = z_{nP})$$

For **example**, the current at port **3** would be $I_3 = I_3(z_3 = z_{3P})$.

Now, say there exists a non-zero current at **port 1** (i.e., $I_1 \neq 0$), while the current at all **other** ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).

Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21} = \frac{V_2}{I_1}$$

Likewise, the trans-impedance parameters Z_{31} and Z_{41} are:

$$Z_{31} = \frac{V_3}{I_1} \qquad \text{and} \qquad Z_{41} = \frac{V_4}{I_1}$$

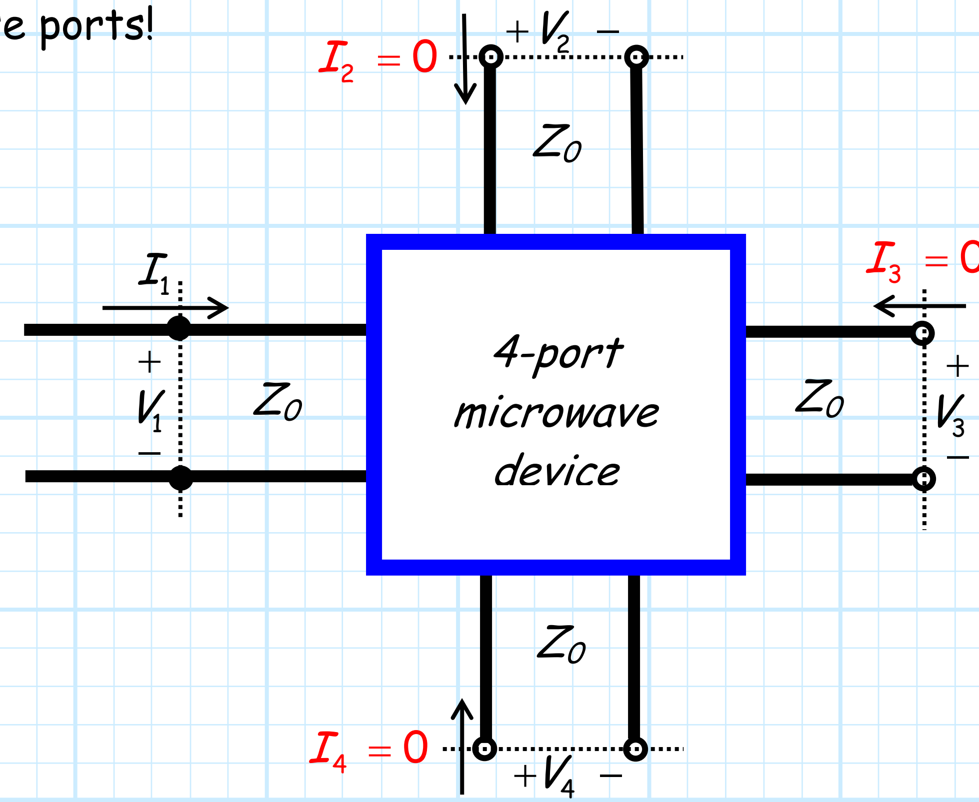
We of course could **also** define, say, trans-impedance parameter Z_{34} as the ratio between the complex values I_4 (the current into port 4) and V_3 (the voltage at port 3), given that the current at all other ports (1, 2, and 3) are zero.

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for all } k \neq n\text{)}$$

Q: *But how do we ensure that all but **one** port current is zero ?*

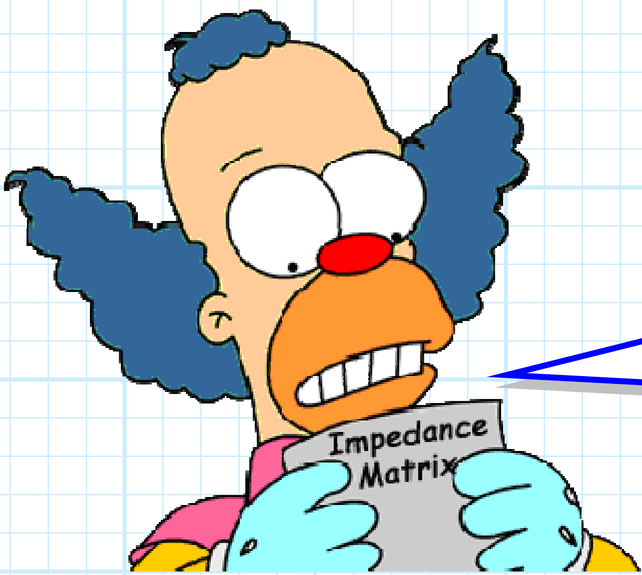
A: Place an **open** circuit **at** those ports!



Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{open}\text{)}$$



*Q: As impossible as it sounds, this handout is even more **boring** and **pointless** than any of your previous efforts. **Why** are we studying this? After all, what is the likelihood that a device will have an **open** circuit on **all** but one of its ports?!*

A: OK, say that **none** of our ports are **open-circuited**, such that we have currents **simultaneously** on **each** of the **four** ports of our device.

Since the device is **linear**, the voltage at any **one** port due to **all** the port currents is simply the coherent **sum** of the voltage at that port due to **each** of the currents!

For example, the voltage at port 3 can be determined by:

$$V_3 = Z_{34}\, I_4 + Z_{33}\, I_3 + Z_{32}\, I_2 + Z_{31}\, I_1$$

More generally, the voltage at port *m* of an *N*-port device is:

$$V_m = \sum_{n=1}^{N} Z_{mn}\, I_n$$

This expression can be written in **matrix** form as:

$$\mathbf{V} = \mathcal{Z}\, \mathbf{I}$$

Where $\mathbf{I}$ is the **vector**:

$$\mathbf{I} = \left[I_1, I_2, I_3, \cdots, I_N \right]^T$$

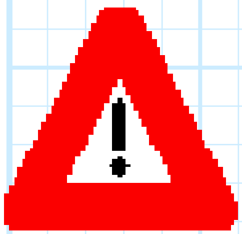
and $\mathbf{V}$ is the vector:

$$\mathbf{V} = \left[V_1, V_2, V_3, \ldots, V_N \right]^T$$

And the matrix $\mathcal{Z}$ is called the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mn} \end{bmatrix}$$

The impedance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the impedance matrix describes a multi-port device the way that Z_L describes a single-port device (e.g., a load)!



But **beware**! The values of the impedance matrix for a particular device or network, just like Z_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Z}(\omega) = \begin{bmatrix} Z_{11}(\omega) & \cdots & Z_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Z_{m1}(\omega) & \cdots & Z_{mn}(\omega) \end{bmatrix}$$