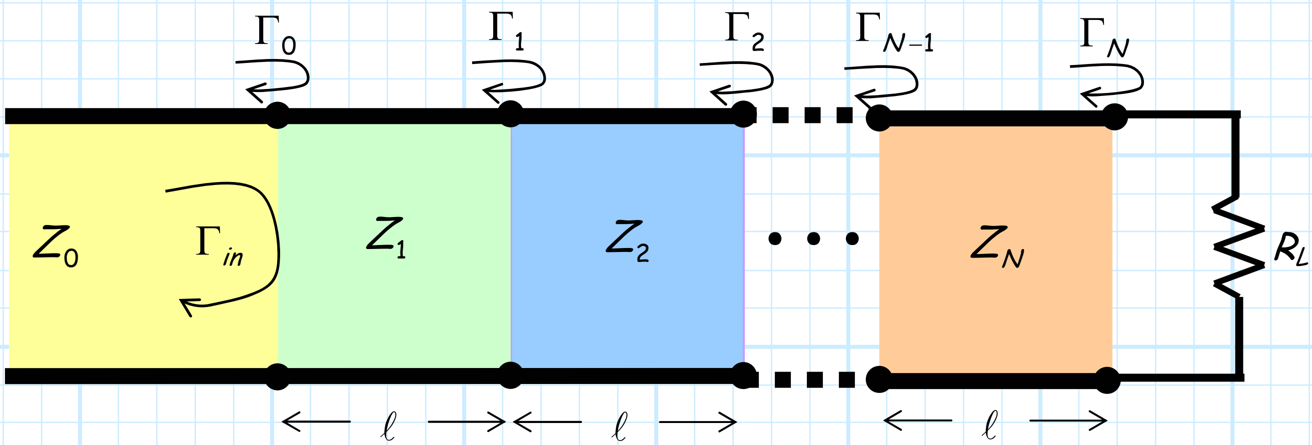
The Multi-section Transformer

Consider a sequence of N transmission line **sections**; each section has **equal length** ℓ, but **dissimilar** characteristic impedances:



Where the marginal reflection coefficients are:

$$\Gamma_0 \doteq \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad \Gamma_n \doteq \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \qquad \Gamma_N \doteq \frac{R_L - Z_N}{R_L + Z_N}$$

If the load resistance R_L is **less** than Z_0, then we should design the transformer such that:

$$Z_0 > Z_1 > Z_2 > Z_3 \cdots > Z_N > R_L$$

Conversely, if R_L is **greater** than Z_0, then we will design the transformer such that:

$$Z_0 < Z_1 < Z_2 < Z_3 \cdots < Z_N < R_L$$

In other words, we **gradually transition** from Z_0 to R_L!

Note that since R_L is **real**, and since we assume **lossless** transmission lines, all Γ_n will be **real** (this is important!).

Likewise, since we **gradually** transition from one section to another, each value:

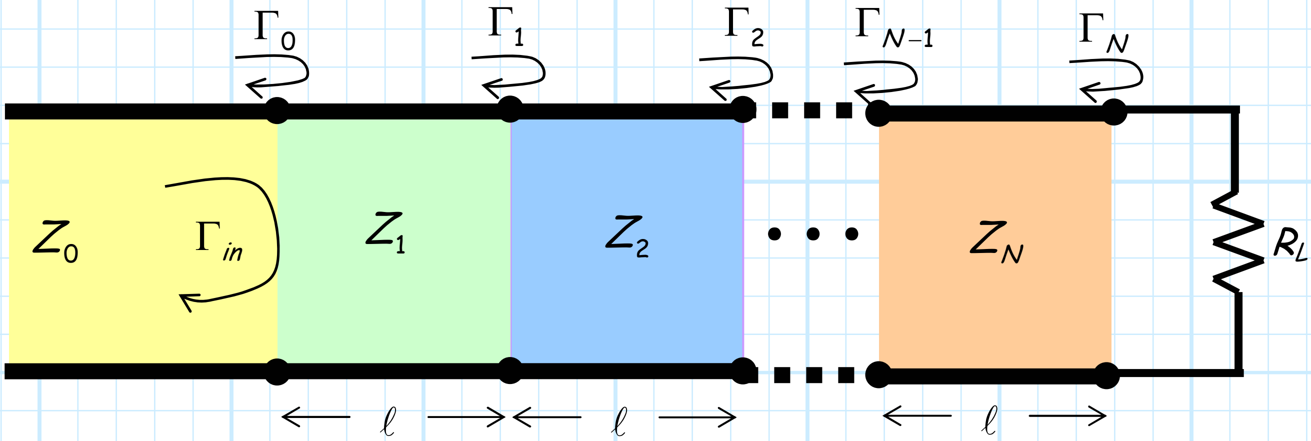
$$Z_{n+1} - Z_n$$

will be **small**.

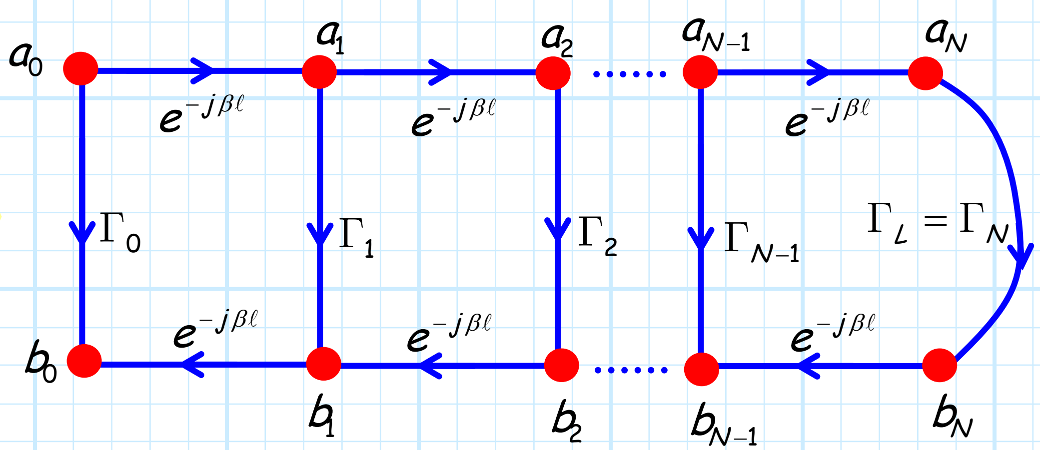
As a result, each marginal reflection coefficient Γ_n will be **real** and have a **small** magnitude.

This is also **important**, as it means that we can apply the "**theory of small reflections**" to analyze this multi-section transformer!

The theory of small reflections allows us to **approximate** the input reflection coefficient of the transformer as:



The approximate SFG when applying the theory of small reflections!



$$\begin{aligned}\frac{b_0}{a_0} &= \Gamma_{in}(\beta) \\ &\simeq \Gamma_0 + \Gamma_1\, e^{-j2\beta\ell} + \Gamma_2\, e^{-j4\beta\ell} + \cdots + \Gamma_N\, e^{-j2N\beta\ell} \\ &= \sum_{n=0}^{N} \Gamma_n\, e^{-j2n\beta\ell}\end{aligned}$$

We can alternatively express the input reflection coefficient as a function of **frequency** ($\beta\ell = \omega T$):

$$\begin{aligned}\Gamma_{in}(\omega) &= \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T} \\ &= \sum_{n=0}^{N} \Gamma_n\, e^{-j(2nT)\omega}\end{aligned}$$

where:

$$T \doteq \frac{\ell}{v_p} = \text{propagation time through 1 section}$$

We see that the function $\Gamma_{in}(\omega)$ is expressed as a **weighted** set of N **basis functions**! I.E.,

$$\Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \, \Psi(\omega)$$

where:

$$c_n = \Gamma_n \qquad \textit{and} \qquad \Psi(\omega) = e^{-j(2nT)\omega}$$

We find, therefore, that by **selecting** the proper values of basis weights c_n (i.e., the proper values of reflection coefficients Γ_n), we can **synthesize** any function $\Gamma_{in}(\omega)$ of frequency ω, provided that:

1. $\Gamma_{in}(\omega)$ is **periodic** in $\omega = 1/2T$

2. we have sufficient **number** of sections N.

Q: *What function **should** we synthesize?*

A: **Ideally**, we would want to make $\Gamma_{in}(\omega) = 0$ (i.e., the reflection coefficient is zero for all frequencies).

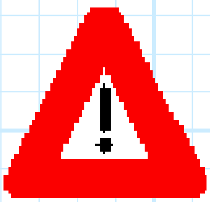
Bad news: this **ideal** function $\Gamma_{in}(\omega) = 0$ would require an **infinite** number of sections (i.e., $N = \infty$)!

Instead, we seek to find an "**optimal**" function for $\Gamma_{in}(\omega)$, given a **finite** number of N elements.

Once we determine these optimal functions, we can find the values of coefficients Γ_n (or equivalently, Z_n) that will result in a matching transformer that exhibits this **optimal** frequency response.

To **simplify** this process, we can make the transformer **symmetrical**, such that:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \quad \Gamma_2 = \Gamma_{N-2}, \quad \cdots\cdots$$



Note that this **does NOT** mean that:

$$Z_0 = Z_N, \quad Z_1 = Z_{N-1}, \quad Z_2 = Z_{N-2}, \quad \cdots\cdots$$

We find then that:

$$\begin{aligned}\Gamma(\omega) = e^{-jN\omega T}\Big[&\Gamma_0\left(e^{jN\omega T} + e^{-jN\omega T}\right) + \Gamma_1\left(e^{j(N-2)\omega T} + e^{-j(N-2)\omega T}\right) \\ &+ \Gamma_2\left(e^{j(N-4)\omega T} + e^{-j(N-4)\omega T}\right) + \cdots\Big]\end{aligned}$$

and since:

$$e^{jx} + e^{-jx} = 2\cos(x)$$

we can write for N **even**:

$$\Gamma(\omega) = 2\, e^{-jN\omega T}\left[\Gamma_0 \cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N-2n)\omega T + \cdots + \frac{1}{2}\Gamma_{N/2}\right]$$

whereas for N **odd**:

$$\Gamma(\omega) = 2\, e^{-jN\omega T}\left[\Gamma_0 \cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N-2n)\omega T + \cdots + \Gamma_{(N-1)/2} \cos \omega T\right]$$

The remaining **question** then is this: given an optimal and realizable function $\Gamma_{in}(\omega)$, **how** do we determine the necessary number of **sections** N, and **how** do we determine the **values** of all reflection coefficients Γ_n??