The Powers that Be

To begin our discussion of amplifiers, we first must define and derive a number of quantities that describe the rate of energy flow (i.e., power).

Consider a source and a load that are connected together by some gain element:

The first power we consider is the available power from the source:

\[ P_{\text{avs}} \] available power from the source
We likewise consider the power $P_{in}$ delivered by the source; in other words the power absorbed by the input impedance of the gain element with a load attached:

\[ V_{g} \quad Z_{g} \quad Z_{in} \]

On the output, we consider the power available from the output of the gain element:

\[ V_{out} \quad Z_{out} \]

\[ P_{avn} \equiv \text{available power from the output port} \]

And finally, we consider the power $P_{L}$ delivered by the output port—the power absorbed by load $Z_{L}$:
These four power quantities depend (at least in part) on the source parameters $V_g$ and $Z_g$, load $Z_L$, and the scattering parameters of $S_{11}, S_{21}, S_{22}, S_{12}$ the gain element.

**Q:** Yikes! How can we possibly **determine** the power values in terms of these circuit parameters?

**A:** Remember, the source, load and gain element (i.e. its scattering matrix) each are described by a set of equations. We simply need to **solve** these simultaneous equations!

Your text (pages 537-539) provides an algebraic solution. But you know me; I prefer to graphically solve the algebra using **signal flow graphs**!

**Q:** But there’s a **source** in our circuit: How do we handle that in a signal flow graph?
A: Consider a simple source connected to a transmission line:

From KVL we know that:

\[ V_s = V_i + Z_s I_i \]

Whereas, from the telegraphers equations we know that:

\[ V_i = V(z = z_s) = V_0^+ e^{-j\beta z_s} + V_0^- e^{+j\beta z_s} \]

\[ I_i = I(z = z_s) = \frac{V_0^+}{Z_0} e^{-j\beta z_s} - \frac{V_0^-}{Z_0} e^{+j\beta z_s} \]

Substituting the definitions:

\[ a_s = V_0^- e^{+j\beta z_g} \] (complex amplitude of voltage wave incident on source)

\[ b_s = V_0^+ e^{-j\beta z_g} \] (complex amplitude of voltage wave exiting source)

we get:
\begin{align*}
V_i & = V(z = z_s) = b_s + a_s \\
I_i & = I(z = z_s) = \frac{b_s}{Z_0} - \frac{a_s}{Z_0}
\end{align*}

And then our \textbf{KVL} equation can be written as:

\begin{align*}
V_s & = (b_s + a_s) + \frac{Z_s}{Z_0} (b_s - a_s)
\end{align*}

And rearranging:

\begin{align*}
b_g & = \left( \frac{Z_0}{Z_g + Z_0} \right) V_g + \Gamma_g a_g
\end{align*}

\textbf{Reluctantly} defining a "reflection coefficient":

\begin{align*}
\Gamma_s & \equiv \frac{Z_s - Z_0}{Z_s + Z_0} \quad \text{(Doh!)}
\end{align*}

we find by rearranging:

\begin{align*}
\frac{Z_0}{Z_0 + Z_s} & = \frac{1 - \Gamma_s}{2}
\end{align*}

and so:

\begin{align*}
b_s & = \left( \frac{1 - \Gamma_s}{2} \right) V_s + \Gamma_s a_s
\end{align*}
We can express the above result graphically using a *signal-flow graph*:

\[
\begin{align*}
\Gamma_s & = \frac{a_s - b_s}{2} \\
\Gamma_s & = \frac{b_s}{2} V_s + \Gamma_s a_s
\end{align*}
\]

Now, consider the case where we place a load (e.g., the input impedance of a two port network) at this source port:

We know from transmission line theory that:

\[
\Gamma_{in} = \frac{V_0^- e^{j\beta z_s}}{V_0^+ e^{-j\beta z_s}} = \frac{a_s}{b_s} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}
\]

Thus, the relationship \( a_s = \Gamma_{in} b_s \) can be added to the signal flow graph:
Using the splitting rule:

\[
\frac{1 - \Gamma_s}{2} \quad b_s \quad \Gamma_{in} \quad a_s
\]

and then the self-loop rule:

\[
\frac{1 - \Gamma_s}{2} \left(\frac{1}{1 - \Gamma_s \Gamma_{in}}\right) \quad b_s \quad \Gamma_{in} \quad a_s
\]

we can directly conclude that:

\[
b_s = V_s \frac{1 - \Gamma_s}{2} \frac{1}{1 - \Gamma_s \Gamma_{in}}
\]

\[
a_s = V_s \frac{1 - \Gamma_s}{2} \frac{\Gamma_{in}}{1 - \Gamma_s \Gamma_{in}}
\]

Note that the power \textit{incident} on the load can now be determined:

\[
P_{inc} = \frac{|b_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{in}|^2}
\]
as well as the power **reflected** from the load:

\[
\rho_{\text{ref}} = \frac{|a_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2} |\Gamma_{\text{in}}|^2
\]

so that the power absorbed by the load (i.e. the power **delivered** by the source) is:

\[
\rho_{\text{in}} = \rho_{\text{inc}} - \rho_{\text{ref}}
\]

\[
= \frac{|b_s|^2 - |a_s|^2}{2Z_0}
\]

\[
= \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2} \left(1 - |\Gamma_{\text{in}}|^2\right)
\]

\[
= \frac{|V_s|^2}{2Z_0} \frac{Z_0}{Z_0 + Z_s} \left(1 - |\Gamma_{\text{in}}|^2\right) \frac{1 - |\Gamma_{\text{in}}|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2}
\]

It is evident from the result above that the amount of power delivered is **dependent** on the value of load impedance. To maximize this power, we must find the value \(\Gamma_{\text{in}}\) that maximizes the term:

\[
\frac{1 - |\Gamma_{\text{in}}|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2}
\]
It can be shown that this term is maximized when $\Gamma_{in} = \Gamma_s^*$. No surprise here; the load must be **conjugate matched** to the source in order to maximize power transfer. This maximum value—resulting only when the load is conjugate matched to the source—is referred to as the **available power** of the source:

$$
\begin{align*}
P_{\text{avg}} &= P_{in} \bigg|_{\Gamma_{in}=\Gamma_s^*} \\
&= \frac{|V_s|^2}{8Z_0} \frac{1 - |\Gamma_s|^2}{1 - |\Gamma_s|^2} \\
&= \frac{|V_s|^2}{2Z_0} \left| \frac{Z_0}{Z_0 + Z_s} \right|^2 \frac{1}{1 - |\Gamma_s|^2} \\
&= \frac{1}{2} \frac{|V_s|^2}{4 \text{Re}\{Z_s^*\}} \\
&= \frac{1}{2} \frac{|V_s|^2}{4 \text{Re}\{Z_s^*\}}
\end{align*}
$$

Now, consider the case where we connect some arbitrary **two-port device** to the source. We would like to determine the **available power** $P_{\text{avg}}$ from the output port of this two-port device.
The signal-flow graph for this network:

\[ \frac{1 - \Gamma_s}{2} \]

We can reduce this signal-flow graph:

\[ \frac{1 - \Gamma_s}{2} \frac{s_{21}}{1 - \Gamma_s s_{11}} \]
Now, for the purposes of determining the output power at port 2, we can ignore nodes \( a_1 \) and \( b_1 \) (in the final signal flow graph above they are terminal nodes, no branches are leaving these nodes). Thus, the relevant portion of the reduced signal flow graph is:

\[
\frac{1 - \Gamma_s}{2} \frac{s_{21}}{1 - \Gamma_s s_{11}}
\]

Notice this signal flow graph has the same form as the source signal-flow graph:
To make this comparison more specific, we define variables:

\[
V_{\text{out}} = V_s \frac{1 - \Gamma_s}{1 - \Gamma_{\text{out}}} \frac{S_{21}}{1 - \Gamma_s S_{11}}
\]

\[
\Gamma_{\text{out}} = S_{22} + \frac{\Gamma_s S_{12} S_{21}}{1 - \Gamma_s S_{11}}
\]

And thus, using these definitions, our signal flow graph can be equivalently written as:

It is apparent that \(V_{\text{out}}\) and \(\Gamma_{\text{out}}\) define an equivalent source created when the original source is connected to a two-port device.
Thus, when some load is connected to the output of the two-port device, the signal-flow graph is:

Which has precisely the same form as:

As a result, the delivered power is precisely the same as the original case, with the exception that we use the equivalent values defined above:

\[
P_L = \frac{|b_2|^2 - |a_2|^2}{2Z_0} \]

\[
= \frac{|V_{out}|^2}{8Z_0} \frac{|1 - \Gamma_{out}|^2}{|1 - \Gamma_{out} \Gamma_L|^2} \left(1 - |\Gamma_L|^2\right)
\]

\[
= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2 |1 - \Gamma_{out} \Gamma_L|^2} \left(1 - |\Gamma_L|^2\right)
\]
Likewise, the available power from port 2 is simply the maximum possible power absorbed by a load $\Gamma_L$. This again is found by maximizing the term:

$$\frac{1 - |\Gamma_L|^2}{|1 - \Gamma_{out} \Gamma_L|^2}$$

which again occurs when $\Gamma_L = \Gamma_{out}^*$. Thus, maximum power transfer occurs when the load is conjugate matched to the equivalent source impedance $Z_{out}(\Gamma_{out})$. As a result the available power from port 2 is:

$$P_{avw} = P_L|_{\Gamma_L=\Gamma_{out}^*}$$

$$\begin{align*}
\frac{|V_s|^2}{8Z_0} & \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|}{|1 - \Gamma_{out} \Gamma_{out}^*|^2} \left(1 - |\Gamma_{out}|^2\right) \\
= & \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \left(1 - |\Gamma_{out}|^2\right) \\
= & \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{1 - |\Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{1 - |\Gamma_{out}|^2}
\end{align*}$$