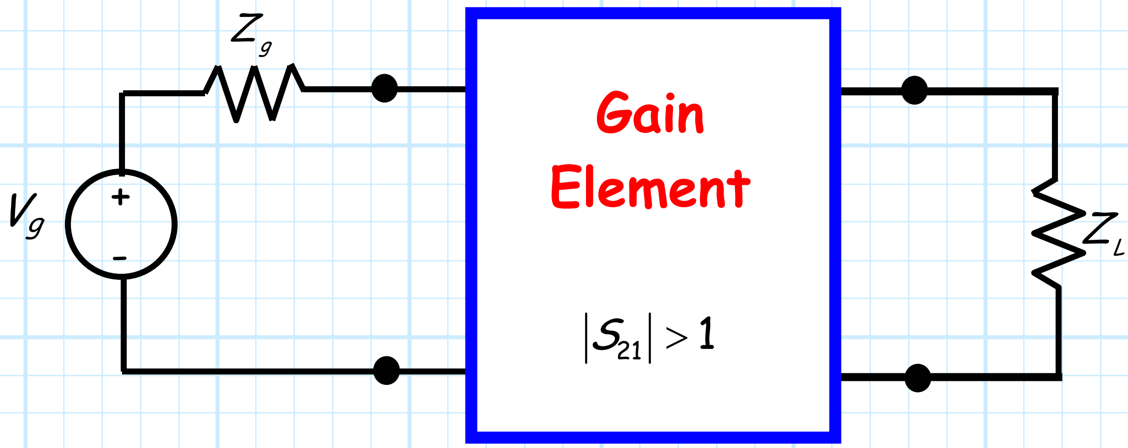
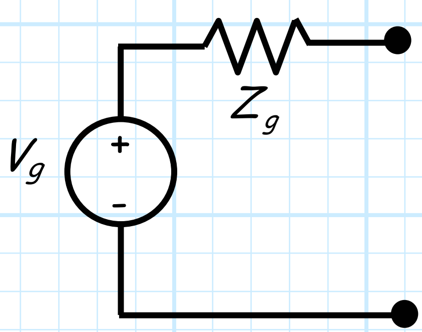
The Powers that Be

To begin our discussion of **amplifiers**, we first must define and derive a number of quantities that describe the **rate of energy flow** (i.e., power).

Consider a source and a load that are connected together by some gain element:

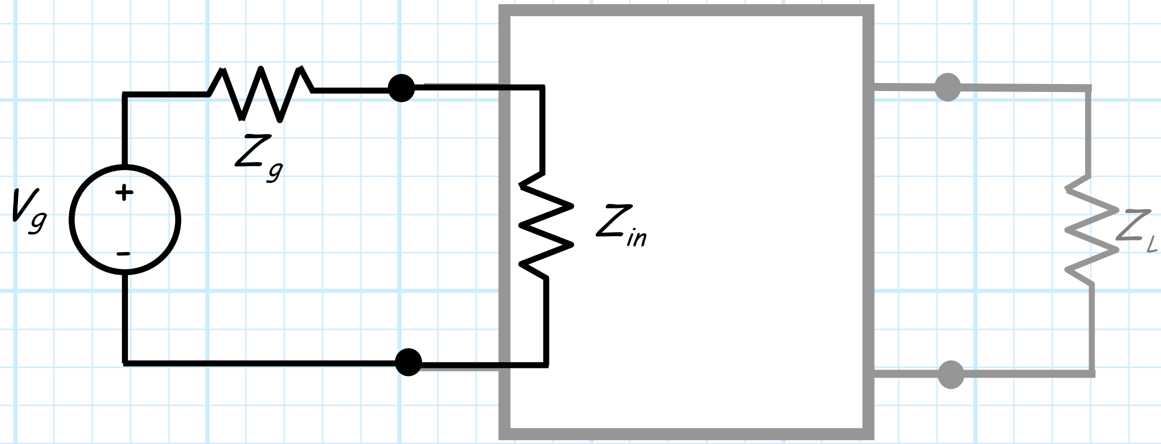


The first power we consider is the **available power** from **the source**:

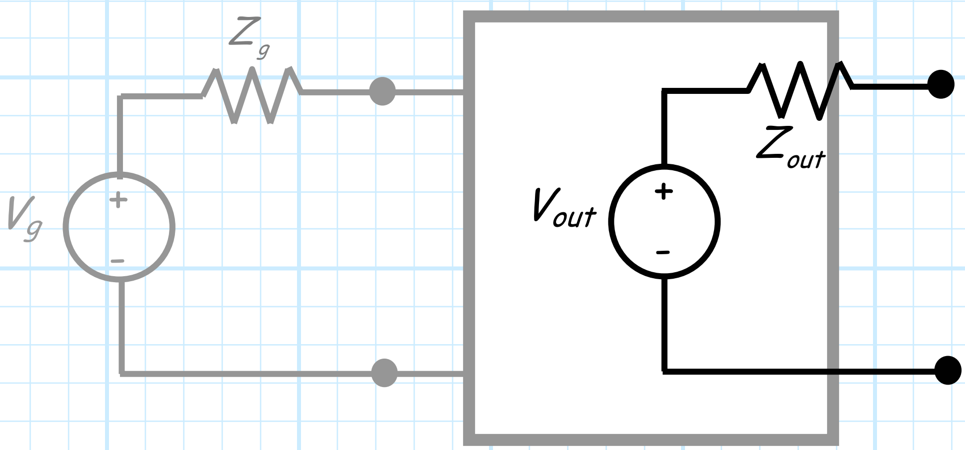


$P_{avs} \doteq$ available power from the source

We likewise consider the power P_{in} **delivered** by the source; in other words the power **absorbed** by the input impedance of the gain element with a load attached:

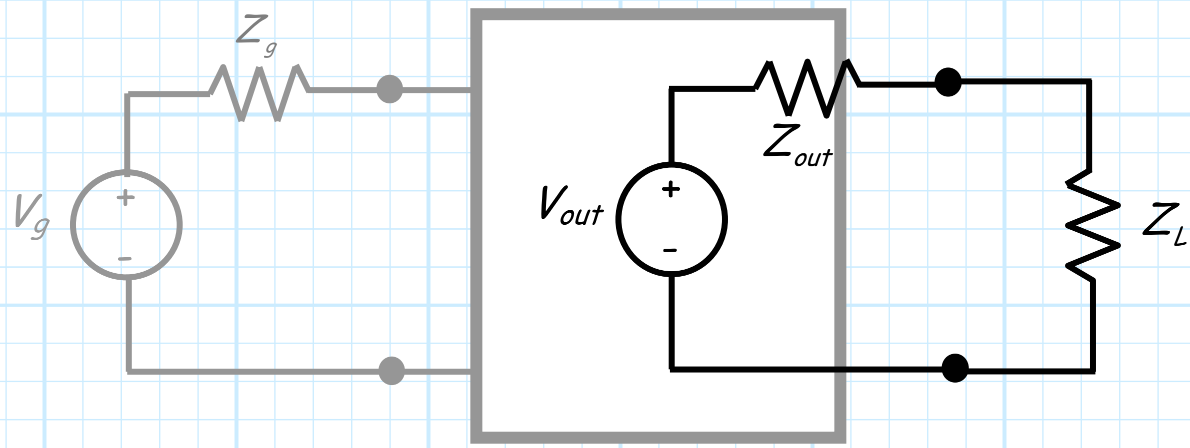


On the output, we consider the power **available** from the **output** of the gain element:



$P_{avn} \doteq$ available power from the output port

And finally, we consider the power P_L **delivered** by the output port—the power absorbed by load Z_L:



These four power quantities depend (at least in part) on the **source** parameters V_g and Z_g, **load** Z_L, and the **scattering parameters** of $S_{11}, S_{21}, S_{22}, S_{12}$ the gain element.

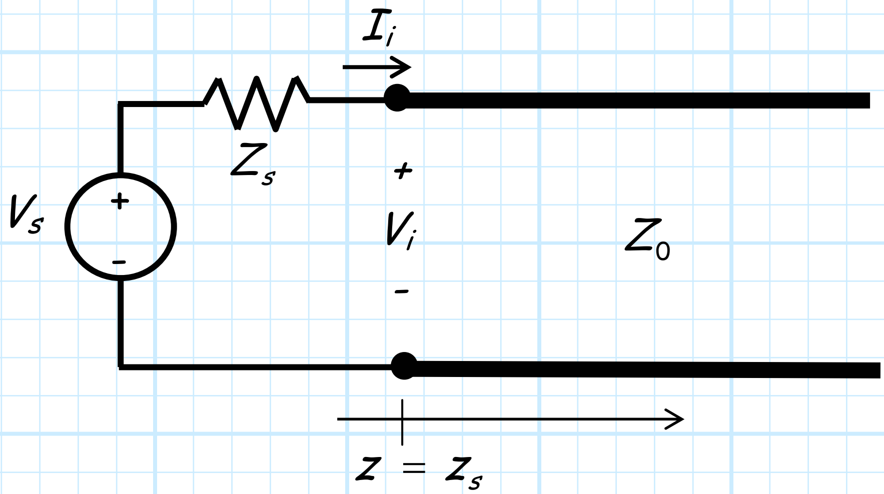
Q: *Yikes! How can we possible **determine** the power values in terms of these circuit parameters?*

A: Remember, the source, load and gain element (i.e. its scattering matrix) each are described by as set of **equations**. We simply need to **solve** these simultaneous equations!

Your text (pages 537-539) provides an algebraic solution. But you know me; I prefer to graphically solve the algebra using **signal flow graphs**!

Q: *But there's a **source** in our circuit: How do we handle that in a signal flow graph?*

A: Consider a simple source connected to a transmission line:



From **KVL** we know that:

$$V_s = V_i + Z_s I_i$$

Whereas, from the **telegraphers equations** we know that:

$$V_i = V(z = z_s) = V_0^+ e^{-j\beta z_s} + V_0^- e^{+j\beta z_s}$$

$$I_i = I(z = z_s) = \frac{V_0^+}{Z_0} e^{-j\beta z_s} - \frac{V_0^-}{Z_0} e^{+j\beta z_s}$$

Substituting the **definitions**:

$a_s \doteq V_0^- e^{+j\beta z_g}$ (complex amplitude of voltage waveincident on source)

$b_s \doteq V_0^+ e^{-j\beta z_g}$ (complex amplitude of voltage wave exiting source)

we get:

$$V_i = V(z = z_s) = b_s + a_s$$

$$I_i = I(z = z_s) = \frac{b_s}{Z_0} - \frac{a_s}{Z_0}$$

And then our **KVL** equation can be written as:

$$V_s = (b_s + a_s) + \frac{Z_s}{Z_0}(b_s - a_s)$$

And rearranging:

$$b_g = \left(\frac{Z_0}{Z_g + Z_0}\right) V_g + \Gamma_g \, a_g$$

Reluctantly defining a "reflection coeficient":

$$\Gamma_s \doteq \frac{Z_s - Z_0}{Z_s + Z_0} \qquad \text{(Doh!)}$$

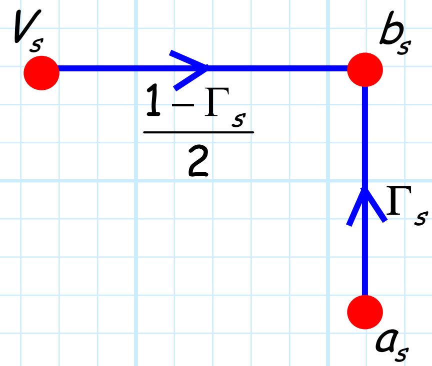
we find by rearranging:

$$\frac{Z_0}{Z_0 + Z_s} = \frac{1 - \Gamma_s}{2}$$

and so:

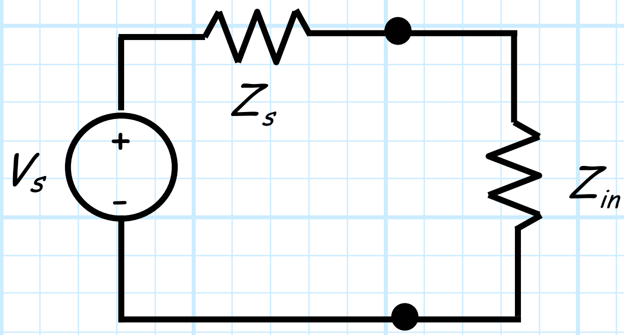
$$b_s = \left(\frac{1 - \Gamma_s}{2}\right) V_s + \Gamma_s \, a_s$$

We can express the above result graphically using a **signal-flow graph**:



$$b_s = \left(\frac{1-\Gamma_s}{2}\right) V_s + \Gamma_s \, a_s$$

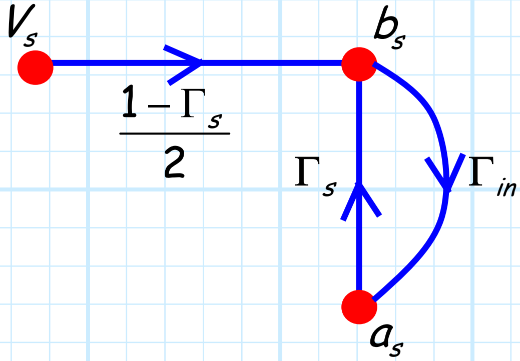
Now, consider the case where we place a **load** (e.g., the input impedance of a two port network) at this source port:



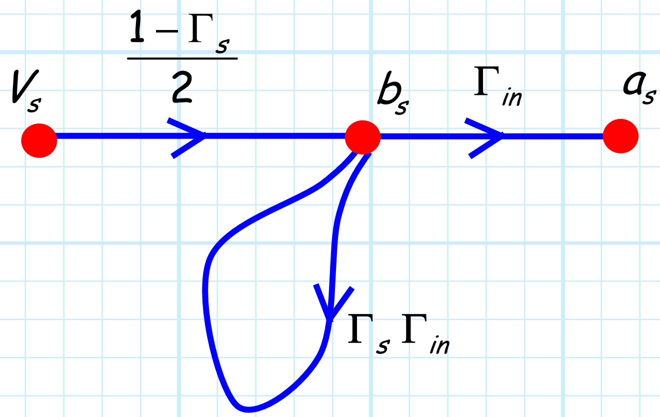
We know from transmission line theory that:

$$\Gamma_{in} = \frac{V_0^- \, e^{+j\beta z_s}}{V_0^+ \, e^{-j\beta z_s}} = \frac{a_s}{b_s} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

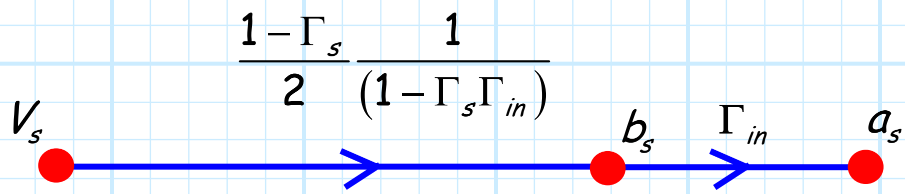
Thus, the relationship $a_s = \Gamma_{in} \, b_s$ can be added to the signal flow graph:



Using the splitting rule:



and then the self-loop rule:



we can directly conclude that:

$$b_s = V_s \frac{1-\Gamma_s}{2} \frac{1}{1-\Gamma_s \Gamma_{in}}$$

$$a_s = V_s \frac{1-\Gamma_s}{2} \frac{\Gamma_{in}}{1-\Gamma_s \Gamma_{in}}$$

Note that the power **incident** on the load can **now** be determined:

$$P_{inc} = \frac{|b_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\Gamma_s \Gamma_{in}|^2}$$

as well as the power **reflected** from the load:

$$P_{ref} = \frac{|a_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\Gamma_s \Gamma_{in}|^2} |\Gamma_{in}|^2$$

so that the power absorbed by the load (i.e. the power **delivered** by the source) is:

$$\boxed{\begin{aligned} P_{in} &= P_{inc} - P_{ref} \\ &= \frac{|b_s|^2 - |a_s|^2}{2Z_0} \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\Gamma_s \Gamma_{in}|^2} \left(1 - |\Gamma_{in}|^2\right) \\ &= \frac{|V_s|^2}{2Z_0} \left|\frac{Z_0}{Z_0 + Z_s}\right|^2 \frac{1 - |\Gamma_{in}|^2}{|1-\Gamma_s \Gamma_{in}|^2} \end{aligned}}$$

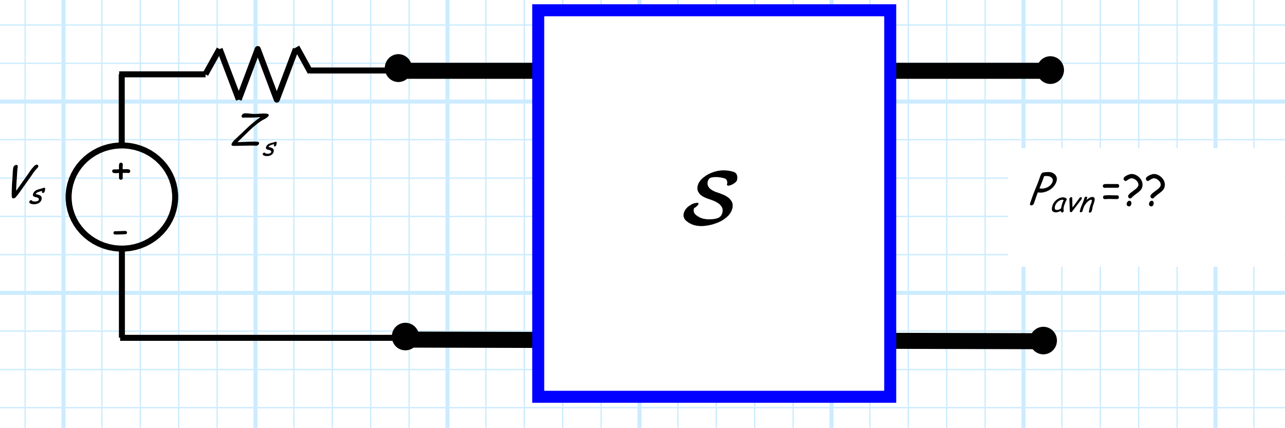
It is evident from the result above that the amount of power delivered is **dependent** on the value of **load impedance**. To maximize this power, we must find the value Γ_{in} that maximizes the term:

$$\frac{1 - |\Gamma_{in}|^2}{|1-\Gamma_s \Gamma_{in}|^2}$$

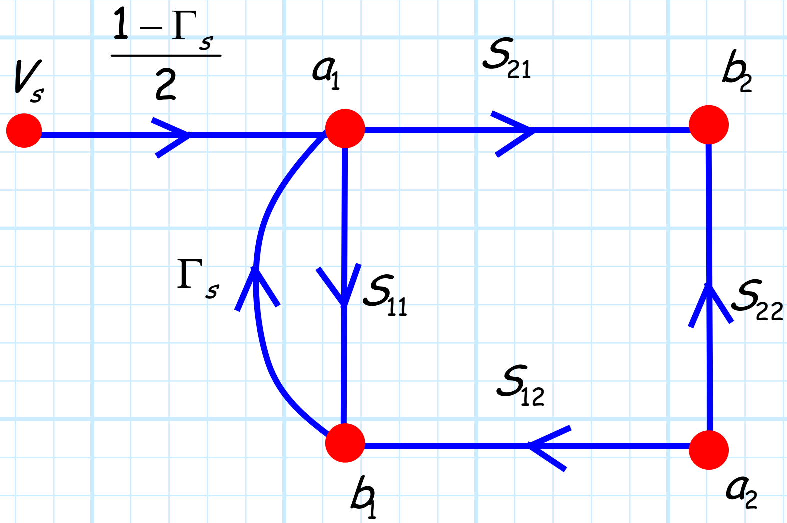
It can be shown that this term is maximized when $\Gamma_{in} = \Gamma_s^*$. No surprise here; the load must be **conjugate matched** to the source in order to maximize power transfer. This maximum value—resulting only when the load is conjugate matched to the source—is referred to as the **available power** of the source:

$$
\begin{aligned}
P_{avs} &= P_{in}\Big|_{\Gamma_{in}=\Gamma_s^*} \\
&= \frac{|V_s|^2}{8Z_0}\frac{|1-\Gamma_s|^2}{1-|\Gamma_s|^2} \\
&= \frac{|V_s|^2}{2Z_0}\left|\frac{Z_0}{Z_0+Z_s}\right|^2 \frac{1}{1-|\Gamma_s|^2} \\
&= \frac{1}{2}\;|V_s|^2\;\frac{1}{4\,Re\left\{Z_s^*\right\}}
\end{aligned}
$$

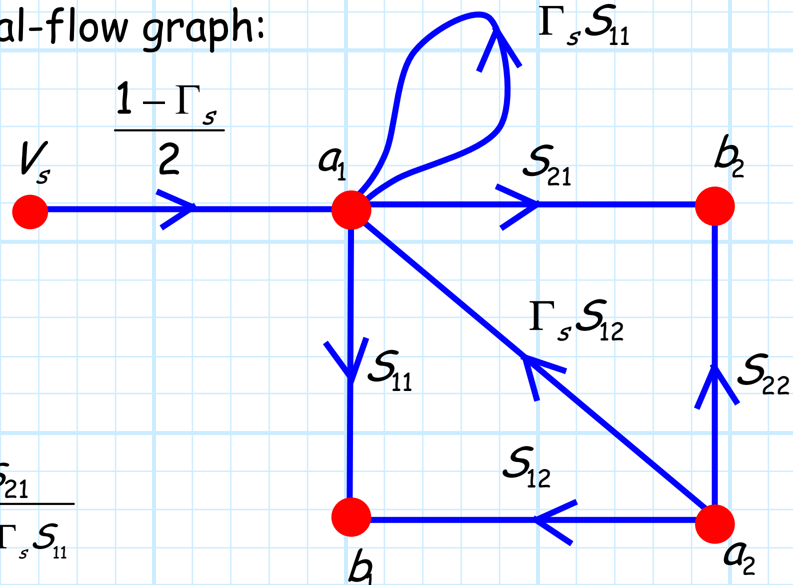
Now, consider the case where we connect some arbitrary **two-port device** to the source. We would like to determine the **available power** P_{avn} from the output port of this two-port device.

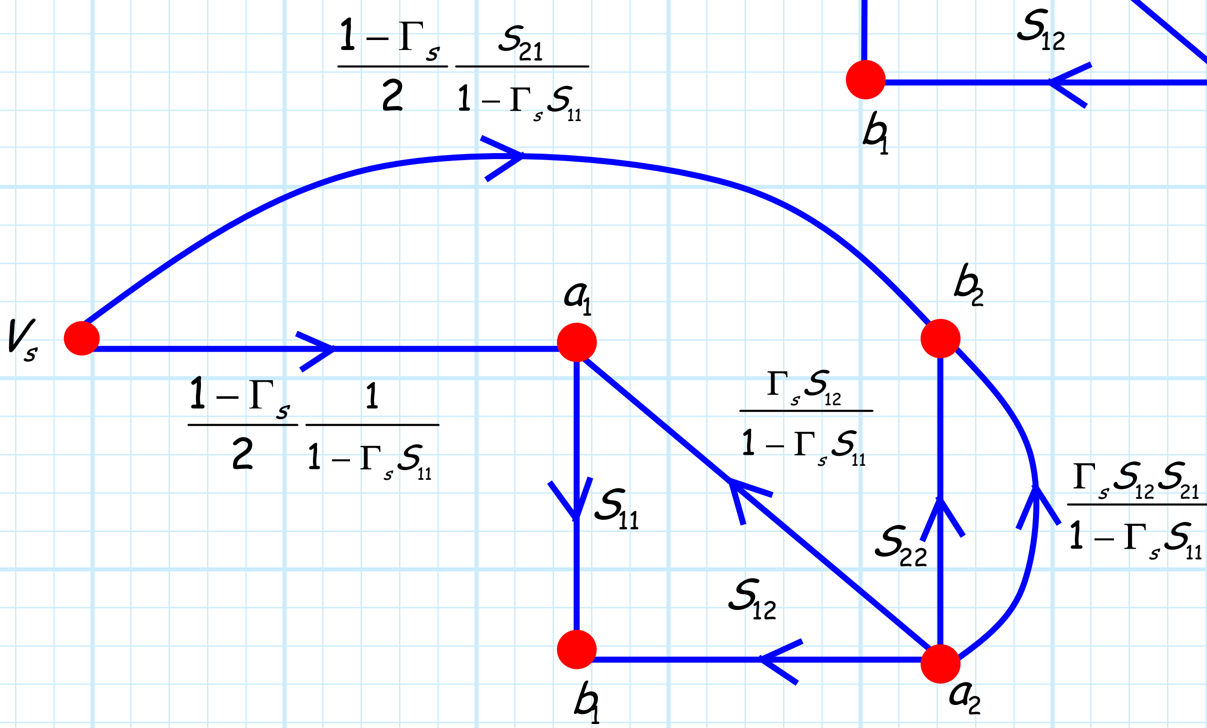


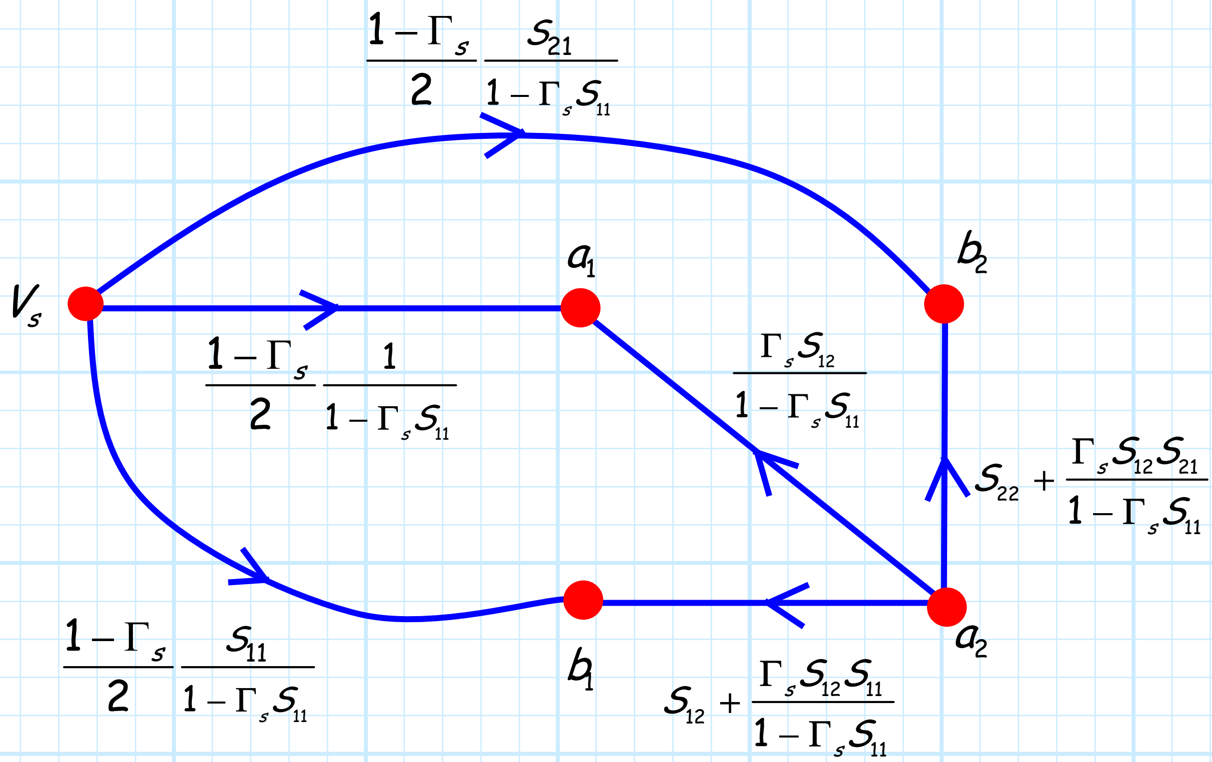
The signal-flow graph for **this** network:



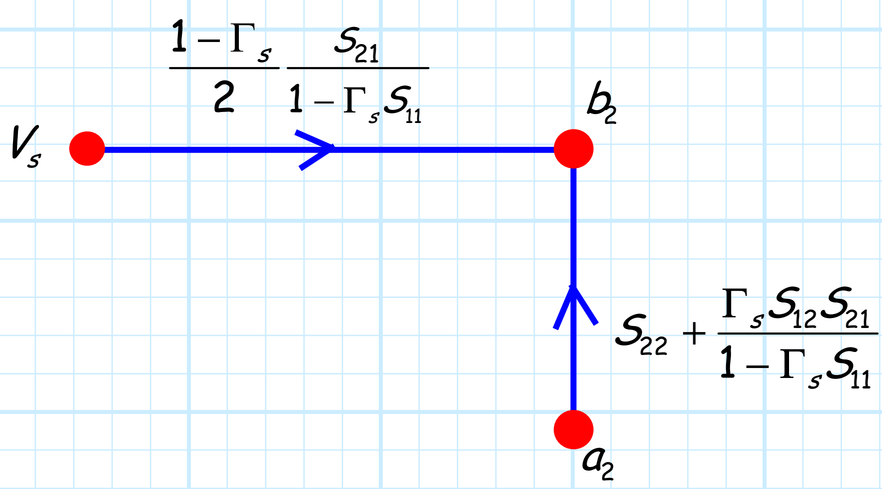
We can **reduce** this signal-flow graph:



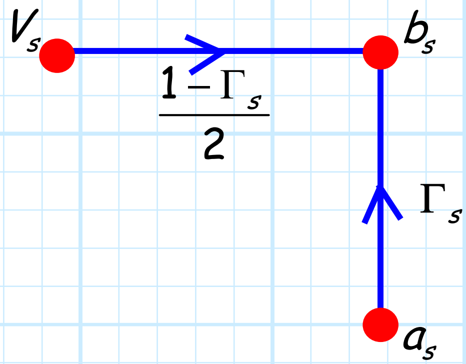




Now, for the purposes of determining the output power at port 2, we can **ignore** nodes a_1 and b_1 (in the **final** signal flow-graph above they are **terminal** nodes, no branches are **leaving** these nodes). Thus, the relevant portion of the ruduced signal flow graph is:



Notice this signal flow graph has the **same form** as the source signal-flow graph:

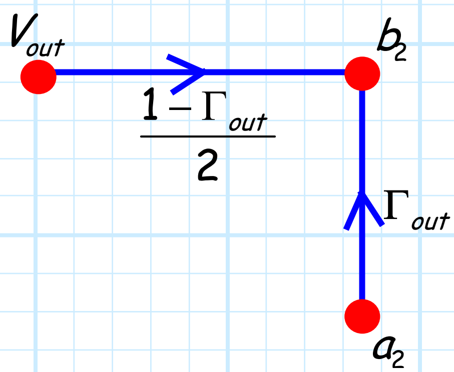


To make this comparison more specific, we **define** variables:

$$V_{out} \doteq V_s \frac{1-\Gamma_s}{1-\Gamma_{out}} \frac{S_{21}}{1-\Gamma_s S_{11}}$$

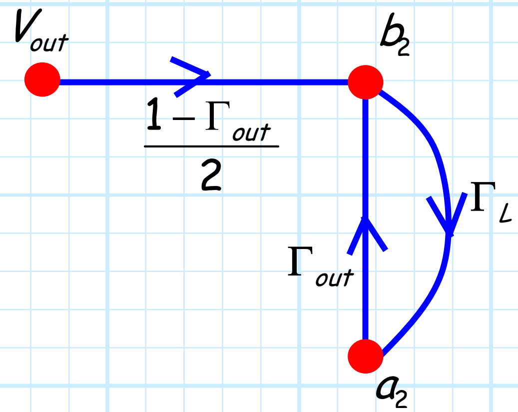
$$\Gamma_{out} \doteq S_{22} + \frac{\Gamma_s S_{12} S_{21}}{1-\Gamma_s S_{11}}$$

And thus, using these definitions, our signal flow graph can be **equivalently** written as:

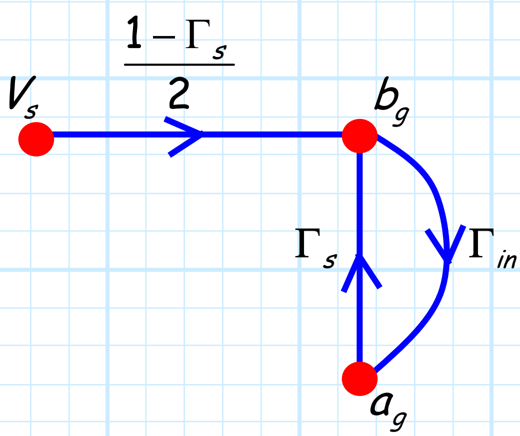


It is apparent that V_{out} and Γ_{out} define an **equivalent** source created when the original source is connected to a two-port device.

Thus, when some **load** is connected to the output of the two-port device, the signal –flow graph is:



Which has **precisely** the same form as:



As a result, the **delivered** power is **precisely** the same as the original case, with the exception that we use the **equivalent** values defined above:

$$
\boxed{
\begin{aligned}
P_L &= \frac{|b_2|^2 - |a_2|^2}{2Z_0} \\
&= \frac{|V_{out}|^2}{8Z_0} \frac{|1-\Gamma_{out}|^2}{|1-\Gamma_{out}\Gamma_L|^2}\left(1-|\Gamma_L|^2\right) \\
&= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2} \frac{|1-\Gamma_s|^2}{|1-\Gamma_{out}\Gamma_L|^2}\left(1-|\Gamma_L|^2\right)
\end{aligned}
}
$$

Likewise, the **available power** from port 2 is simply the maximum possible power absorbed by a load Γ_L. This **again** is found by maximizing the term:

$$\frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\,\Gamma_L|^2}$$

which **again** occurs when $\Gamma_L = \Gamma_{out}^*$. Thus, maximum power transfer occurs when the load is **conjugate matched** to the **equivalent** source impedance Z_{out} (Γ_{out}). As a result the **available power** from port 2 is:

$$\begin{aligned}
P_{avn} &= P_L\big|_{\Gamma_L=\Gamma_{out}^*} \\
&= \frac{|V_s|^2}{8Z_0}\frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2}\frac{|1-\Gamma_s|^2}{|1-\Gamma_{out}\,\Gamma_{out}^*|^2}\left(1-|\Gamma_{out}|^2\right) \\
&= \frac{|V_s|^2}{8Z_0}\frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2}\frac{|1-\Gamma_s|^2}{\left(1-|\Gamma_{out}|^2\right)^2}\left(1-|\Gamma_{out}|^2\right) \\
&= \frac{|V_s|^2}{8Z_0}\frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2}\frac{|1-\Gamma_s|^2}{1-|\Gamma_{out}|^2}
\end{aligned}$$