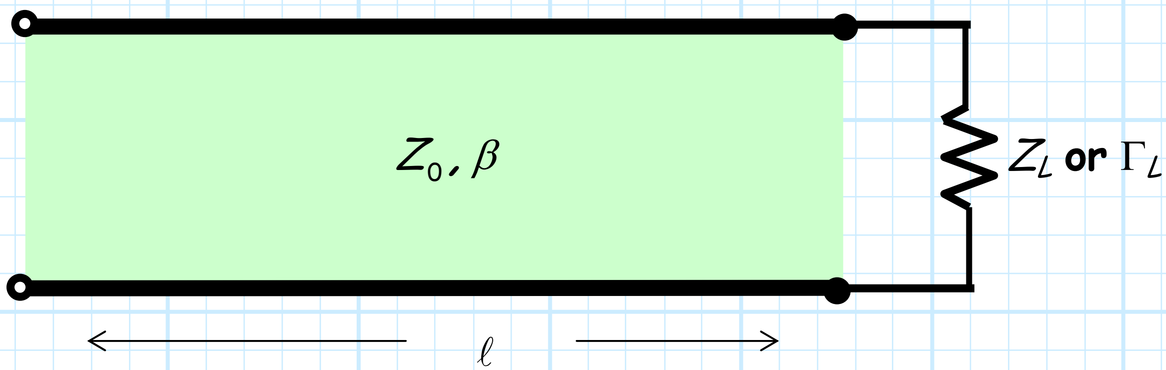
The Reflection Coefficient Transformation

The **load** at the end of some length of a transmission line (with characteristic impedance Z_0) can be specified in terms of its impedance Z_L **or** its reflection coefficient Γ_L.

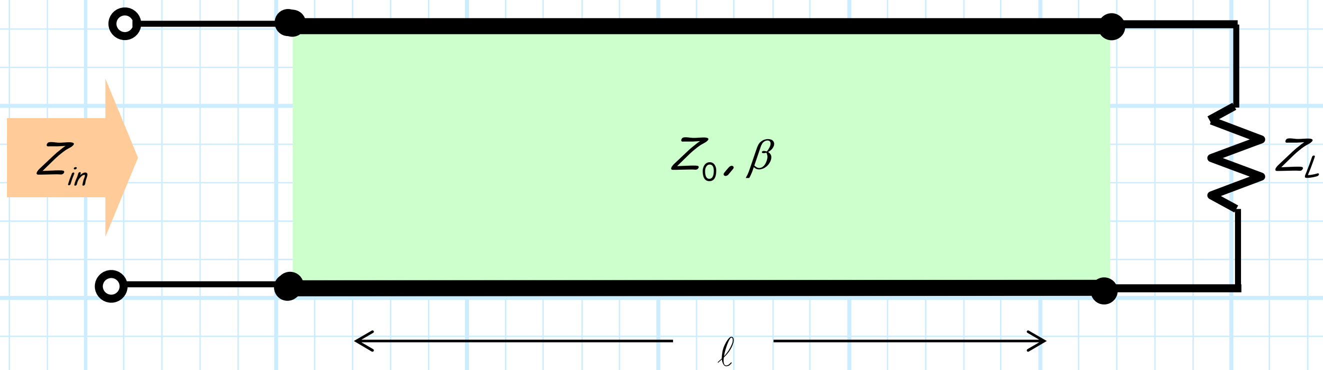


Note **both** values are complex, and **either one** completely specifies the load—if you know **one**, you know the **other**!

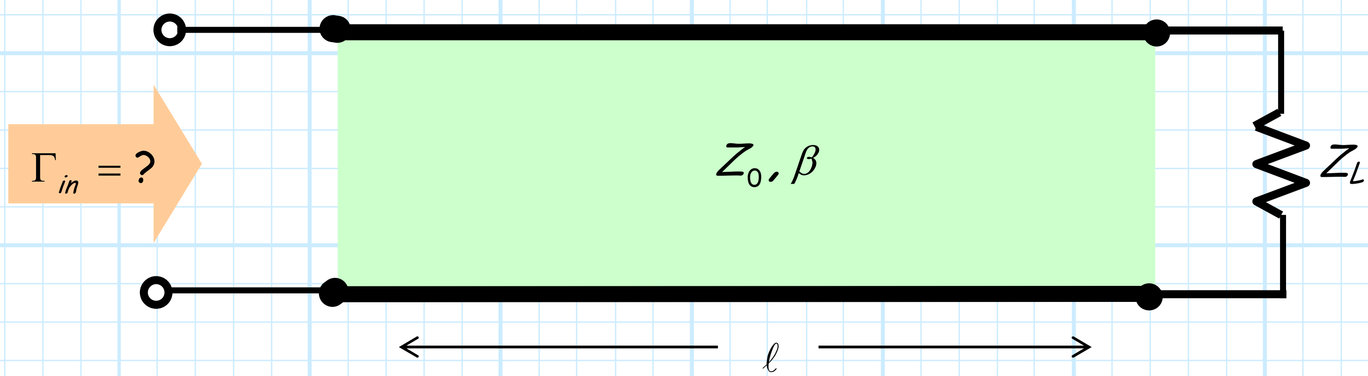
$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad Z_L = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

Is there such a thing as Γ_{in} ?

Recall that we determined how a length of transmission line **transformed** the load **impedance** into an input **impedance** of a (generally) different value:



Q: *Can we likewise express this input impedance in terms of the reflection coefficient (i.e., Γ_{in})? If so, what does Γ_{in} mean?*



The hard way

A: Well, we **could** execute these **three** steps:

1. Convert Γ_L to Z_L:

$$Z_L = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

2. Transform Z_L down the line to Z_{in}:

$$Z_{in} = Z_0\left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell}\right)$$

3. Convert Z_{in} to Γ_{in}:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Q: *Yikes! This is a **ton** of complex arithmetic—isn't there an **easier** way?*

A: Actually, there **is**!

Déjà vu all over again

Slugging through all the algebra, find that the result I really simple:

$$\Gamma_{in} = \Gamma_L e^{-j2\beta\ell}$$

Q: *Hey! This result looks **familiar**.*

*Haven't we seen something like this **before**?*

A: Absolutely!

Recall that we found that the reflection coefficient **function** $\Gamma(z)$ can be expressed as:

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$

Evaluating this function at the **beginning** of the line (i.e., at $z = -\ell$):

$$\begin{aligned} \Gamma(z)\big|_{z=-\ell} &= \Gamma_0 e^{+j2\beta(-\ell)} \\ &= \Gamma_0 e^{-j2\beta\ell} \end{aligned}$$

Just evaluate the reflection coefficient function

But, we recognize that:

$$\Gamma_0 = \Gamma(z=0) = \Gamma_L$$

And so:

$$\Gamma(z=-\ell) = \Gamma_L e^{-j2\beta\ell}$$

Thus, we find that Γ_{in} is simply the value of function $\Gamma(z)$ **evaluated** at the line **input** of $z = -\ell$!

$$\Gamma_{in} = \Gamma(z=-\ell) = \Gamma_L e^{-j2\beta\ell}$$

Makes sense!

After all, the input impedance is **likewise** simply the line impedance evaluated at the line input of $z=-\ell$:

$$Z_{in} = Z(z=-\ell)$$

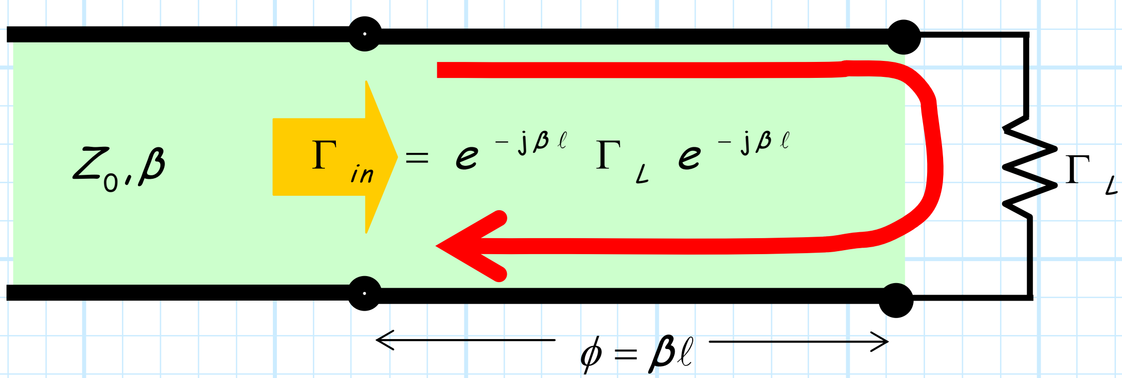
Only the phase changes as we move along the transmission line

It is apparent that from the above expression that the reflection coefficient at the input (i.e., Γ_{in}) is simply related to Γ_L by a **phase shift** of $2\beta\ell$.

In other words, the **magnitude** of Γ_{in} is the **same** as the magnitude of Γ_L!

$$|\Gamma_{in}| = |\Gamma_L| \left| e^{j(\theta_\Gamma - 2\beta\ell)} \right| = |\Gamma_L| (1) = |\Gamma_L|$$

The **phase shift** associated with transforming the load Γ_L down a transmission line can be attributed to the phase shift associated with the wave propagating a length ℓ down the line, reflecting from load Γ_L, and then propagating a length ℓ back up the line:



Three physical events

To **emphasize** this wave interpretation, we begin with the knowledge that:

$$V^{-}(z=-\ell) = V_0^{-}\, e^{-j\beta\ell}$$

In other "words" the minus-wave at $z=-\ell$ is just the minus-wave at $z=0$ (i.e., V_0^{-}), "**shifted**" in phase by $-\beta\ell$.

Now, we also know that the minus-wave and plus-wave at $z=0$ are related by the **reflection coefficient** $\Gamma_0 = \Gamma_L$:

$$V_0^{-} = \Gamma_L\, V_0^{+}$$

Likewise, we know that:

$$V^{+}(z=-\ell) = V_0^{+}\, e^{+j\beta\ell}$$

In other "words" the plus-wave at $z=-\ell$ is just the minus-wave at $z=0$ (i.e., V_0^{-}), **"shifted"** in phase by $+\beta\ell$.

A causal interpretation

Putting these statements together, we find:

$$\begin{aligned} V^{-}(z=-\ell) &= e^{-j\beta\ell}\, V_0^{-} \\ &= e^{-j\beta\ell}\, \Gamma_L\, V_0^{+} \\ &= e^{-j\beta\ell}\, \Gamma_L\, e^{-j\beta\ell}\, V^{-}(z=-\ell) \end{aligned}$$

And from the definition of the **input reflection coefficient** we have thus confirmed:

$$\frac{V^{-}(z=-\ell)}{V^{-}(z=-\ell)} = e^{-j\beta\ell}\, \Gamma_L\, e^{-j\beta\ell} = \Gamma_{in}$$

Note the **"causal"** interpretation of this result: propagate **down** the line, **reflect** off the load, and propagate back **up** the line!