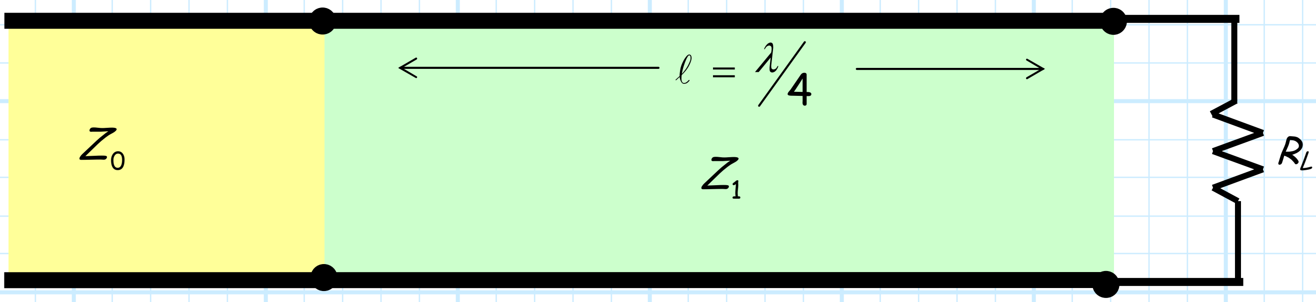
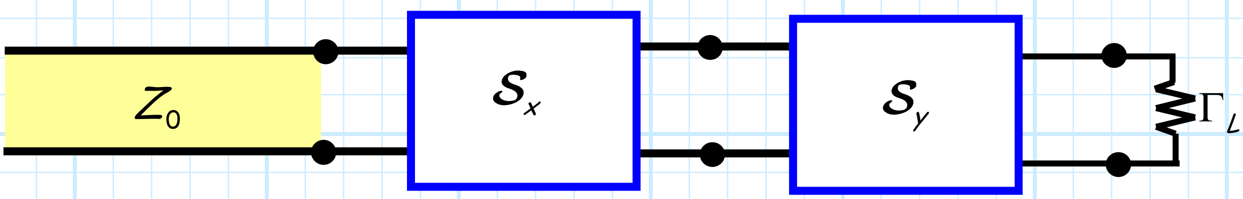
The Signal Flow Graph of a Quarter-Wave Transformer

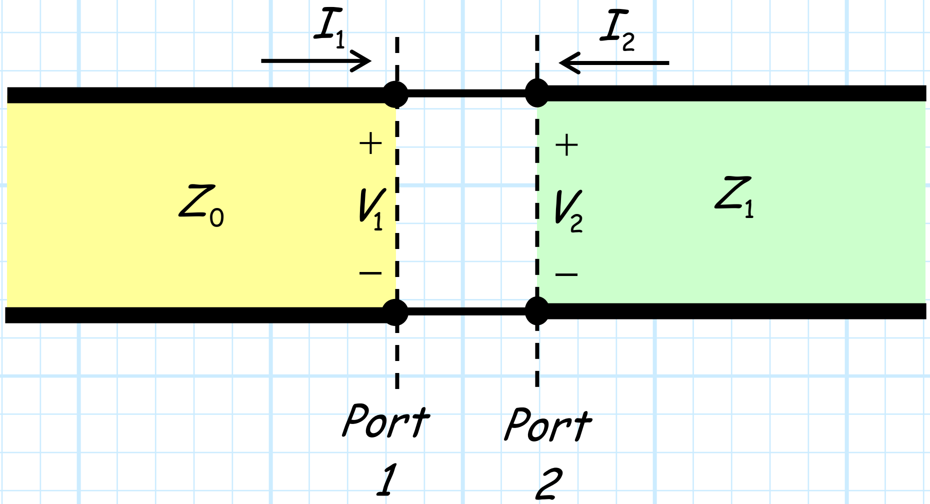
A quarter wave transformer can be thought of as a cascaded series of two two-port devices, terminated with a load R_L:





Q: ***Two*** *two-port devices? It appears to me that a quarter-wave transformer is* ***not*** *that complex. What* ***are*** *the two two-port devices?*

A: The **first** is a "**connector**". Note a connector is the interface between one transmission line (characteristic impedance Z_0) to a second transmission line (characteristic impedance Z_1).



Recall that we **earlier** determined the scattering matrix of this two-port device:

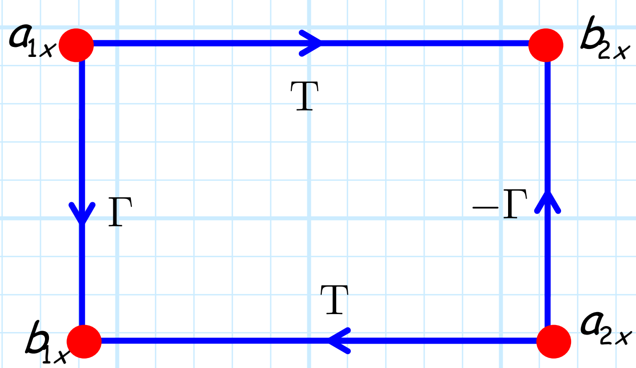
$$\mathcal{S}_x = \begin{bmatrix} \dfrac{Z_1 - Z_0}{Z_1 + Z_0} & \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} \\ \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} & \dfrac{Z_0 - Z_1}{Z_0 + Z_1} \end{bmatrix}$$

This result can be more **compactly** stated as:

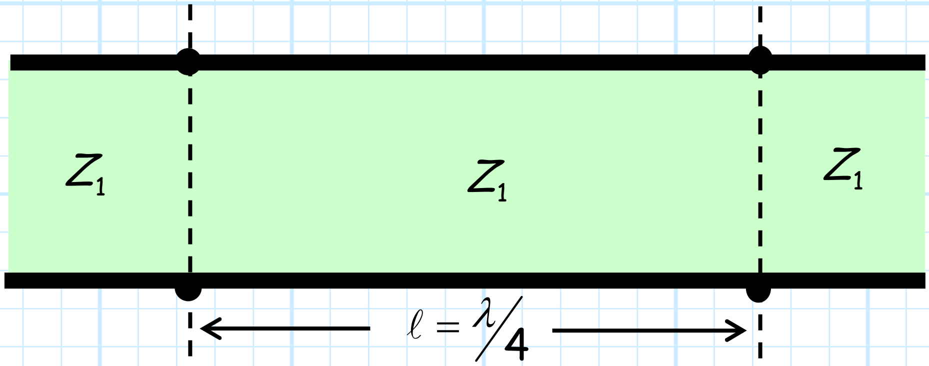
$$\mathcal{S} = \begin{bmatrix} \Gamma & \mathrm{T} \\ \mathrm{T} & -\Gamma \end{bmatrix}$$

where $\Gamma \doteq \dfrac{Z_1 - Z_0}{Z_1 + Z_0}$ and $\mathrm{T} \doteq \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1}$

The signal flow graph of this device is therefore:

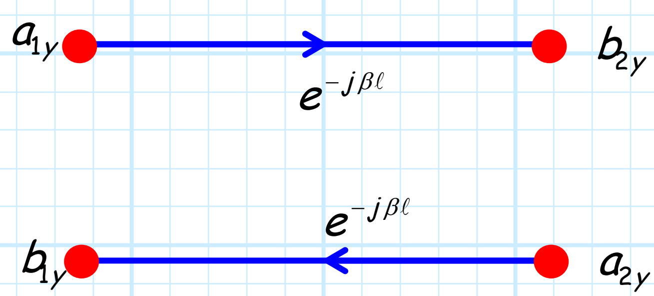


Now, the second two-port device is a quarter wavelength of **transmission line.**

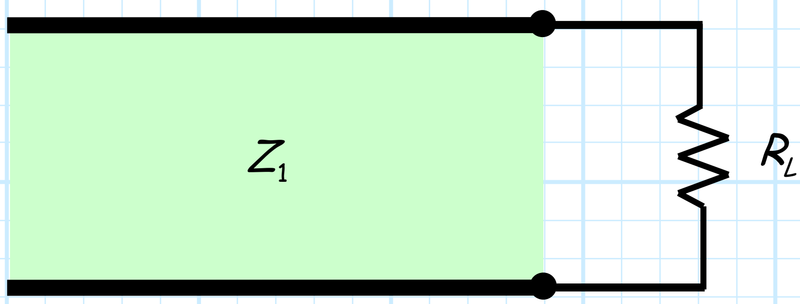


We know that it has the scattering matrix:

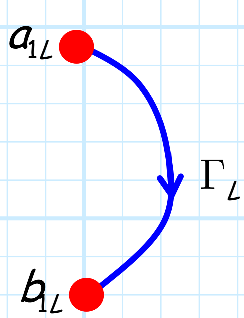
$$\mathcal{S}_y = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}$$



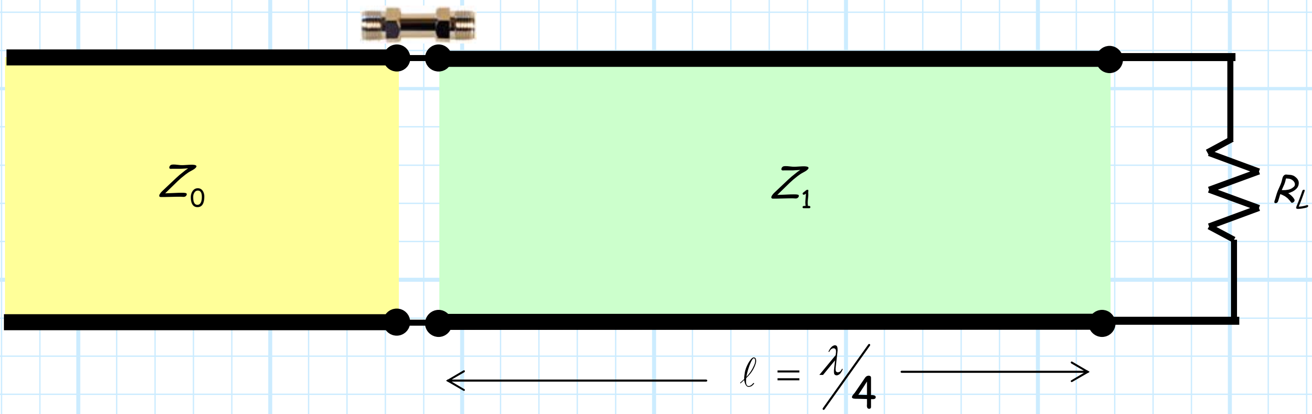
Finally, a **load** has a "scattering matrix" of:



$$\mathcal{S} = \left[\frac{R_L - Z_1}{R_L + Z_1}\right] = \Gamma_L$$



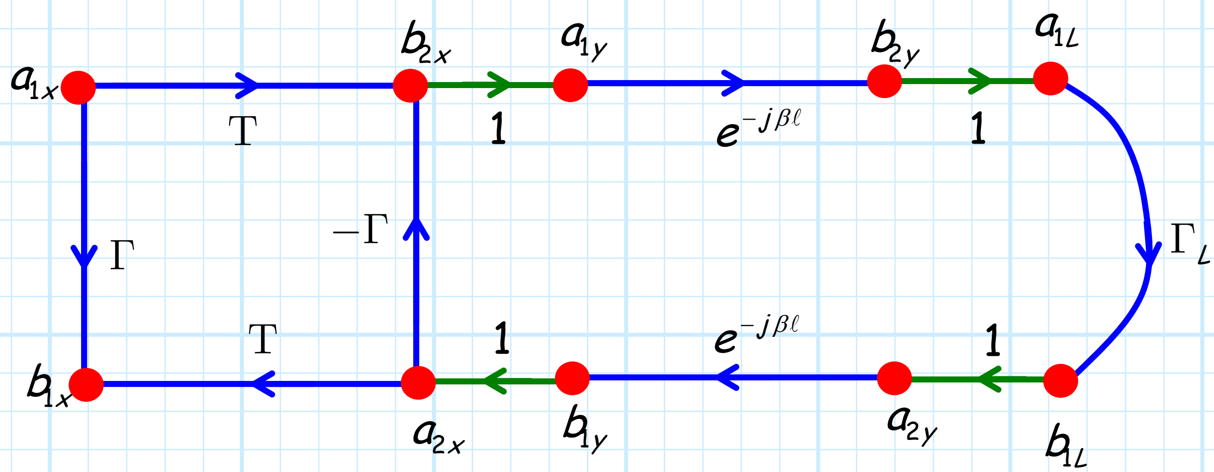
Of course, if we connect the ideal connector to a quarter wavelength of transmission line, and terminate the whole thing with load R_L, we have formed a **quarter wave matching network**!



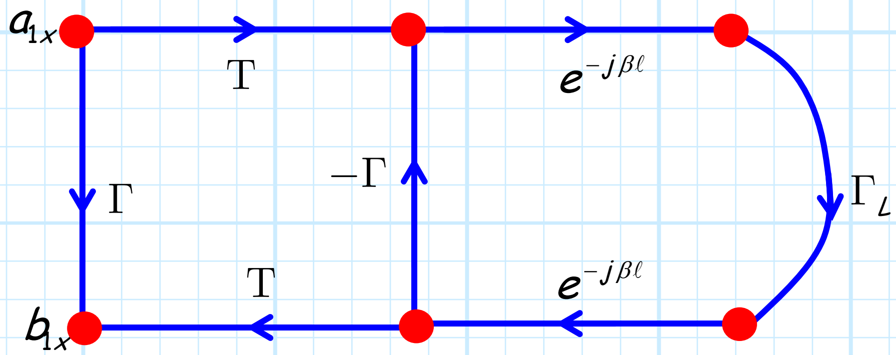
The boundary conditions associated with these connections are likewise:

$$a_{1y} = b_{2x} \qquad a_{2x} = b_{1y} \qquad a_{1L} = b_{2y} \qquad a_{2y} = b_{1L}$$

We can thus put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:



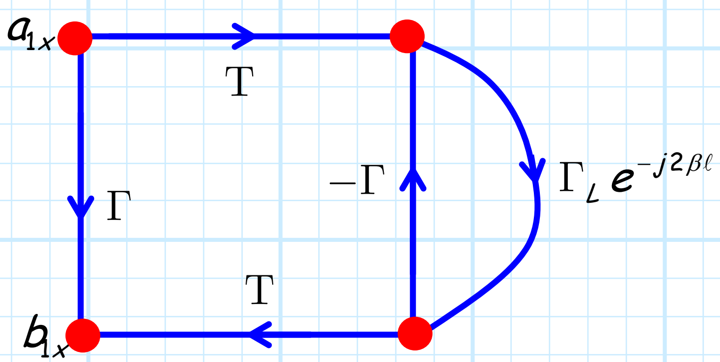
And simplifying:



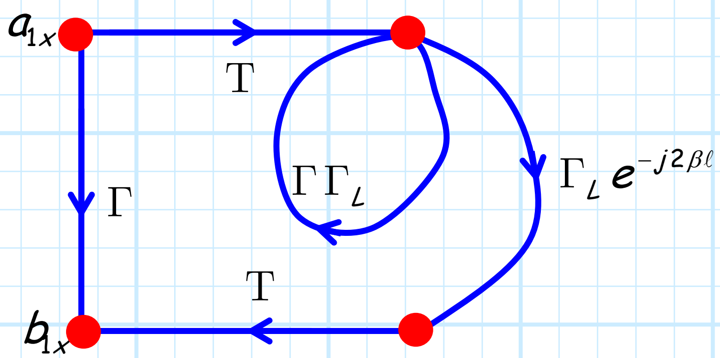
Now, let's see if we can **reduce** this graph to determine:

$$\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}}$$

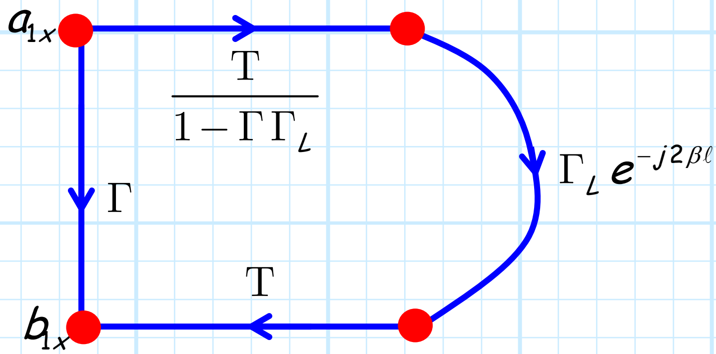
From the **series** rule:



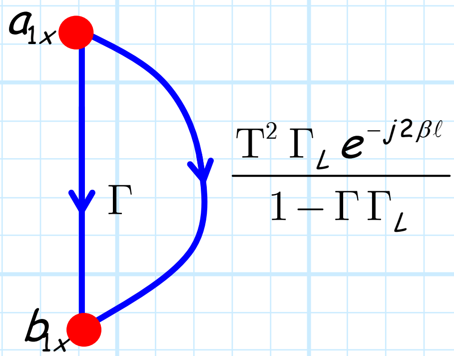
From the **splitting** rule:



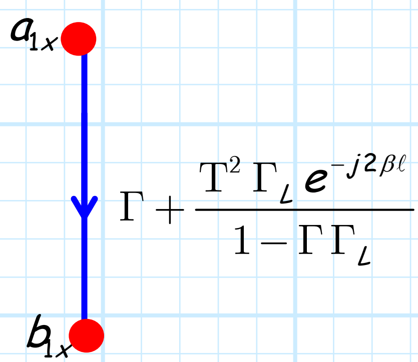
From the **self-loop** rule:



Again with the **series** rule:



And finally with the **parallel** rule:



So that:

$$\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta\ell}}{1-\Gamma\Gamma_L}$$

Q: *Hey wait! If the quarter-wave transformer is a **matching network**, shouldn't* $\Gamma_{in} = 0$*??*

A: Who says it isn't! Consider now **three important facts**.

For a **quarter wave transformer**, we set Z_1 such that:

$$Z_1^2 = Z_0 R_L \qquad \Rightarrow \qquad Z_0 = Z_1^2 / R_L$$

Inserting this into the scattering parameter S_{11} of the connector, we find:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - {Z_1^2}/{R_L}}{Z_1 + {Z_1^2}/{R_L}} = \frac{R_L - Z_1}{R_L + Z_1}$$

Look at this result! For the quarter-wave transformer, the **connector** S_{11} value (i.e., Γ) is the **same** as the **load** reflection coefficient Γ_L:

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \qquad \leftarrow \textbf{Fact 1}$$

Since the connector is **lossless** (unitary scattering matrix!), we can conclude (and likewise show) that:

$$1 = |S_{11}|^2 + |S_{21}|^2 = |\Gamma|^2 + |T|^2$$

Since Z_0, Z_1, and R_L are all real, the values Γ and T are also **real valued**. As a result, $|\Gamma|^2 = \Gamma^2$ and $|T|^2 = T^2$, and we can likewise conclude:

$$\Gamma^2 + T^2 = 1 \qquad \leftarrow \textbf{Fact 2}$$

Likewise, the Z1 transmission line has $\ell = \lambda/4$, so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = 2\pi/\lambda$! Thus:

$$e^{-j2\beta\ell} = e^{-j\pi} = -1 \qquad \leftarrow \textbf{Fact 3}$$

As a result:

$$\Gamma_{in} = \Gamma + \frac{T^2\,\Gamma_L\, e^{-j2\beta\ell}}{1-\Gamma\,\Gamma_L} = \Gamma - \frac{T^2\,\Gamma_L}{1-\Gamma\,\Gamma_L}$$

And using the **newly discovered** fact that (for a correctly designed transformer) $\Gamma_L = \Gamma$:

$$\Gamma_{in} = \Gamma - \frac{T^2\,\Gamma_L}{1-\Gamma\,\Gamma_L} = \Gamma - \frac{T^2\,\Gamma}{1-\Gamma^2}$$

And also are **recent** discovery that $T^2 = 1 - \Gamma^2$:

$$\Gamma_{in} = \Gamma - \frac{T^2\,\Gamma}{1-\Gamma^2} = \Gamma - \frac{T^2\,\Gamma}{T^2} = 0$$

A **perfect match**! The quarter-wave transformer does indeed work!