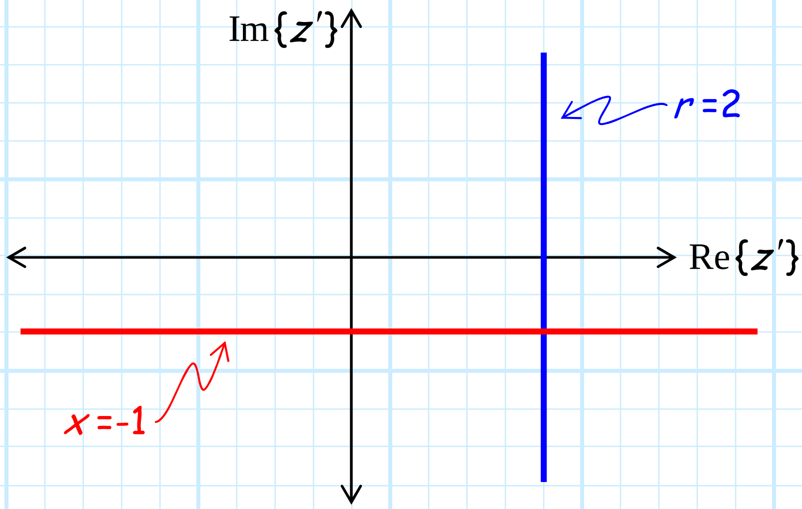
The Smith Chart

Say we wish to map a **line** on the **normalized** complex **impedance** plane **onto** the complex Γ plane.

For example, we could **map** the vertical line r=2 ($\mathrm{Re}\{z'\} = 2$) or the horizontal line x=-1 ($\mathrm{Im}\{z'\} = -1$).



Recall we know how to map the vertical line r=0; it simply maps to the **circle** $|\Gamma| = 1$ on the complex Γ plane.

Likewise, we know how to map the horizontal line x= 0; it simply maps to the **line** $\Gamma_i = 0$ on the complex Γ plane.

But for the examples given above, the mapping is **not** so straight forward. The contours will in general be functions of both Γ_r and Γ_i (e.g., $\Gamma_r^2 + \Gamma_i^2 = 0.5$), and thus the mapping **cannot** be stated with **simple** functions such as $|\Gamma| = 1$ or $\Gamma_i = 0$.

As a matter of fact, a **vertical line** on the normalized **impedance** plane of the form:

$$r = c_r \;,$$

where c_r is some **constant** (e.g. $r = 2$ or $r = 0.5$), is **mapped** onto the complex Γ plane as:

$$\left(\Gamma_r - \frac{c_r}{1+c_r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+c_r}\right)^2$$

Note this equation is of the same form as that of a **circle**:

$$(x - x_c)^2 + (y - y_c)^2 = a^2$$

where:

a = the radius of the circle

$P_c(x = x_c, y = y_c) \;\Rightarrow\;$ point located at the center of the circle

Thus, the vertical **line** $r = c_r$ maps into a **circle** on the complex Γ plane!

By inspection, it is apparent that the **center** of this circle is located at this point on the complex Γ plane:

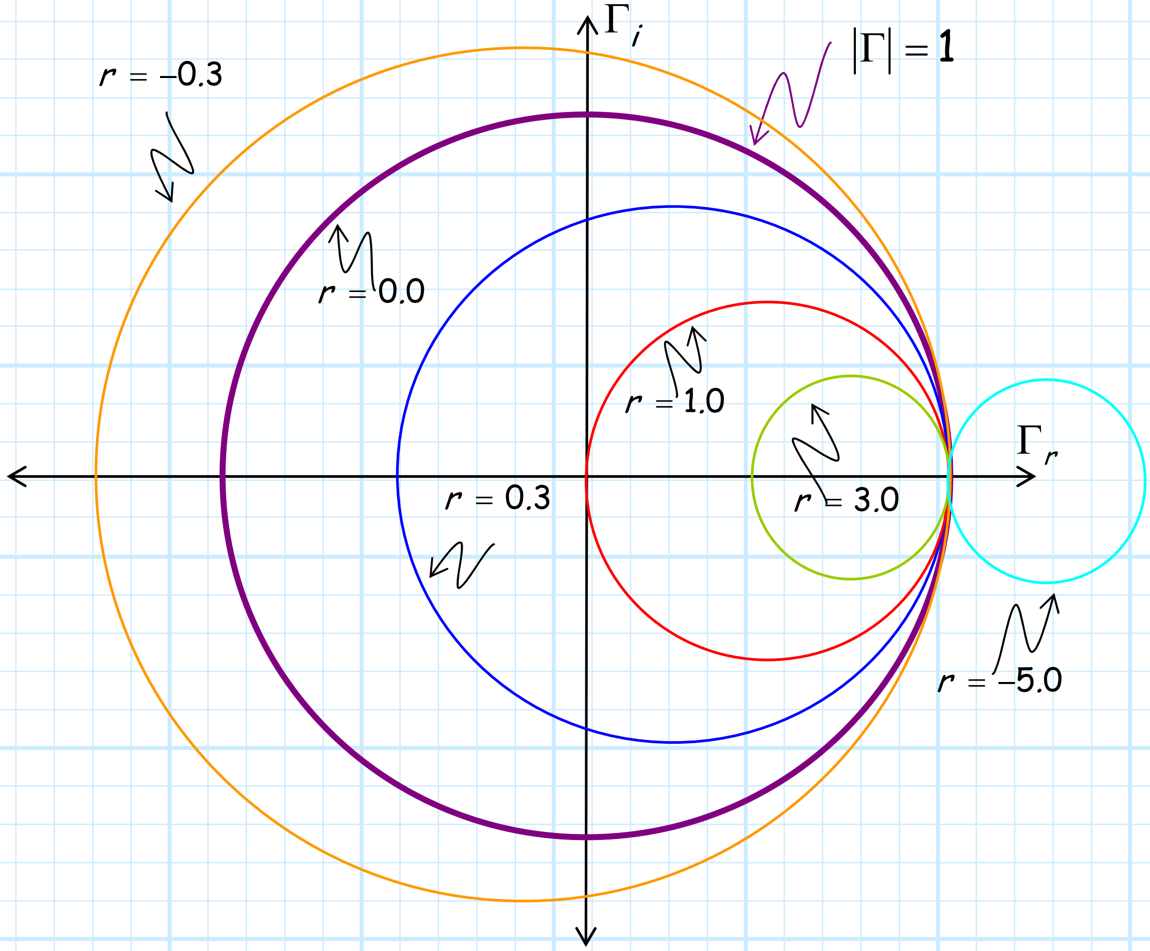
$$P_c\left(\Gamma_r = \frac{c_r}{1+c_r}, \Gamma_i = 0\right)$$

In other words, the center of this circle **always** lies somewhere along the $\Gamma_i = 0$ line.

Likewise, by inspection, we find the **radius** of this circle is:

$$a = \frac{1}{1+c_r}$$

We perform a few of these **mappings** and see where these **circles** lie on the complex Γ plane:



We see that as the constant c_r **increases**, the radius of the circle **decreases**, and its center moves to the **right**.

Note:

1. If $c_r > 0$ then the circle lies entirely **within** the circle $|\Gamma| = 1$.

2. If $c_r < 0$ then the circle lies entirely **outside** the circle $|\Gamma| = 1$.

3. If $c_r = 0$ (i.e., a reactive impedance), the circle lies **on** circle $|\Gamma| = 1$.

4. If $c_r = \infty$, then the **radius** of the circle is **zero**, and its **center** is at the point $\Gamma_r = 1, \Gamma_i = 0$ (i.e., $\Gamma = 1e^{j0}$). In other words, the **entire** vertical line $r = \infty$ on the normalized **impedance** plane is mapped onto just a **single point** on the complex Γ plane!

   But of course, this **makes sense**! If $r = \infty$, the impedance is **infinite** (an open circuit), regardless of what the value of the **reactive** component x is.

Now, let's turn our attention to the mapping of **horizontal lines** in the normalized impedance plane, i.e., lines of the form:

$$x = c_i$$

where c_i is some **constant** (e.g. $x = -2$ or $x = 0.5$).

We can show that this **horizontal** line in the normalized impedance plane is **mapped** onto the **complex Γ plane** as:

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{c_i}\right)^2 = \frac{1}{c_i^2}$$

Note this equation is **also** that of a **circle**! Thus, the horizontal line $x = c_i$ maps into a circle on the complex Γ plane!

By inspection, we find that the **center** of this circle lies at the point:

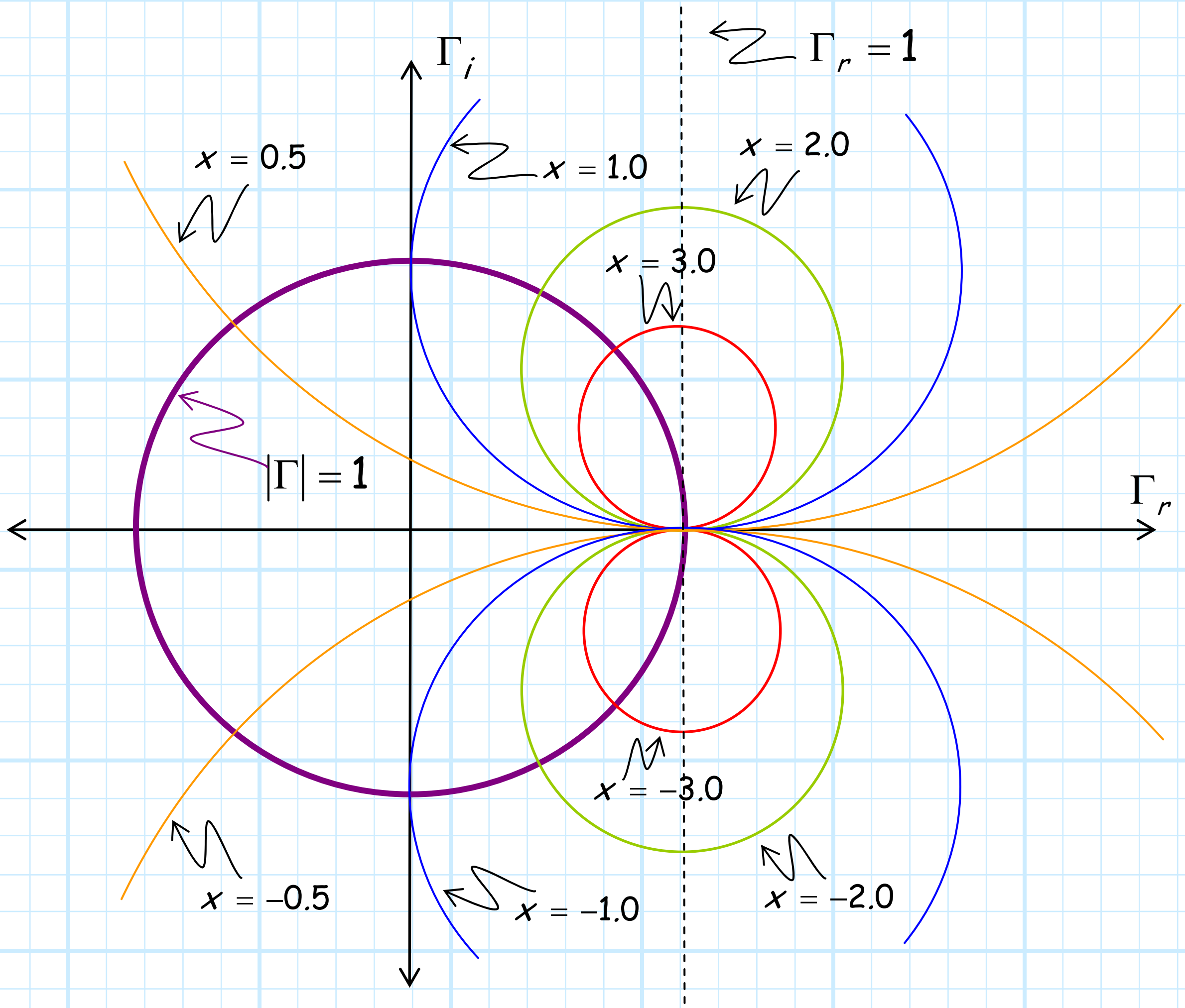
$$P_c\left(\Gamma_r = 1, \Gamma_i = \frac{1}{c_i}\right)$$

in other words, the center of this circle **always** lies somewhere along the vertical $\Gamma_r = 1$ line.

Likewise, by inspection, the **radius** of this circle is:

$$a = \frac{1}{|c_i|}$$

We perform a few of these **mappings** and see where these circles lie on the complex Γ plane:



We see that as the **magnitude** of constant c_i **increases**, the radius of the circle **decreases**, and its **center** moves toward the point $(\Gamma_r = 1, \Gamma_i = 0)$.

Note:

1. If $c_i > 0$ (i.e., reactance is **inductive**) then the circle lies entirely in the **upper half** of the complex Γ plane (i.e., where $\Gamma_i > 0$)—the upper half-plane is known as the **inductive** region.

2. If $c_i < 0$ (i.e., reactance is **capacitive**) then the circle lies entirely in the **lower half** of the complex Γ plane (i.e., where $\Gamma_i < 0$)—the lower half-plane is known as the **capacitive** region.

3. If $c_i = 0$ (i.e., a **purely resistive** impedance), the circle has an infinite radius, such that it lies **entirely** on the line $\Gamma_i = 0$.

4. If $c_i = \pm\infty$, then the **radius** of the circle is **zero**, and its **center** is at the point $\Gamma_r = 1, \Gamma_i = 0$ (i.e., $\Gamma = 1e^{j0}$). In other words, the **entire** vertical line $x = \infty$ or $x = -\infty$ on the normalized impedance plane is mapped onto just a **single point** on the complex Γ plane!

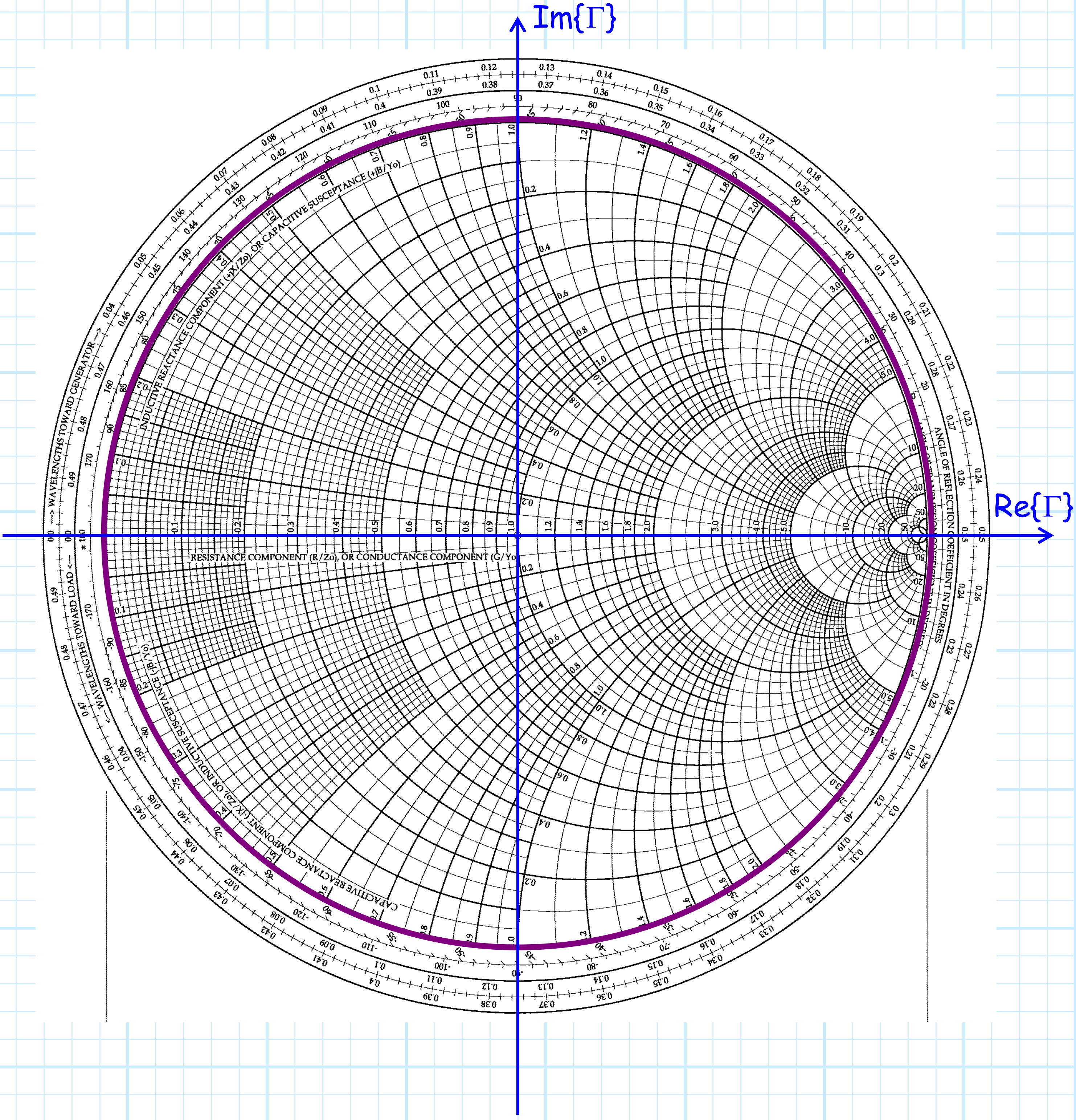
But of course, this **makes sense**! If $x = \infty$, the impedance is **infinite** (an **open** circuit), **regardless** of what the value of the resistive component r is.

5. Note also that **much** of the circle formed by mapping $x = c_i$ onto the complex Γ plane lies **outside** the circle $|\Gamma| = 1$.

This **makes sense**! The portions of the circles laying **outside** $|\Gamma| = 1$ circle correspond to impedances where the **real** (resistive) part is **negative** (i.e., $r < 0$).

Thus, we typically can completely **ignore** the portions of the circles that lie **outside** the $|\Gamma| = 1$ circle !

Mapping **many** lines of the form $r = c_r$ and $x = c_i$ onto circles on the complex Γ plane results in tool called the **Smith Chart**.

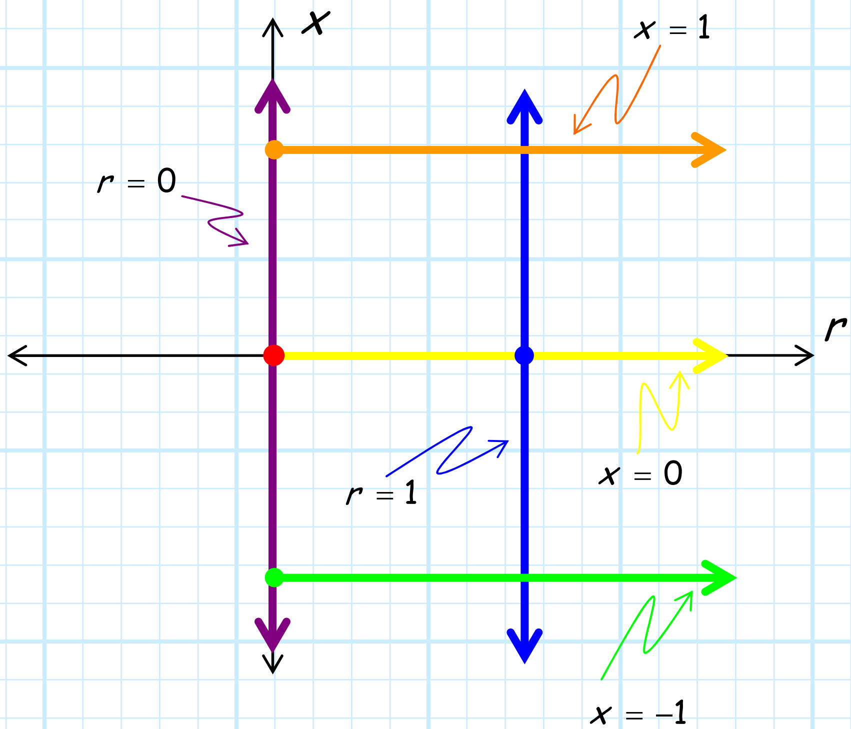


Note the Smith Chart is simply the vertical lines $r = c_r$ and horizontal lines $x = c_i$ of the normalized **impedance** plane, **mapped** onto the two types of **circles** on the complex Γ plane.

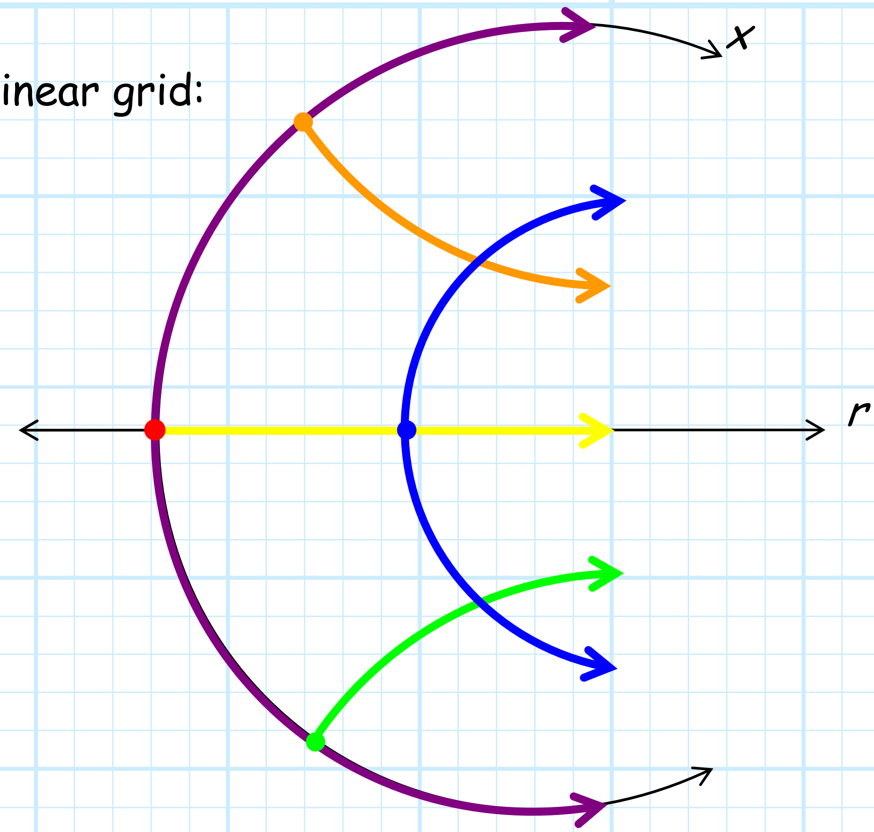
Note for the normalized **impedance** plane, a vertical line $r = c_r$ and a horizontal line $x = c_i$ are always **perpendicular** to each other when they intersect. We say these lines form a **rectilinear grid**.

However, a similar thing is true for the **Smith Chart**! When a mapped circle $r = c_r$ **intersects** a mapped circle $x = c_i$, the two circles are **perpendicular** at that intersection point. We say these circles form a **curvilinear grid**.
In fact, the Smith Chart is formed by **distorting** the **rectilinear** grid of the normalized impedance plane into the **curvilinear** grid of the Smith Chart!

I.E.,:



Distorting this rectilinear grid:



And then **distorting** some more—we have the **curvilinear** grid of the Smith Chart!