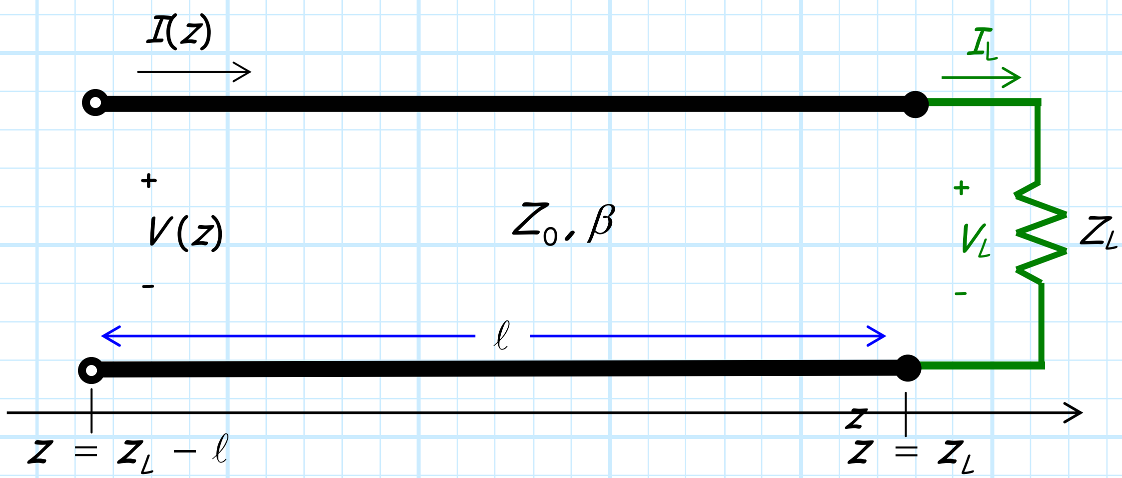
The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.

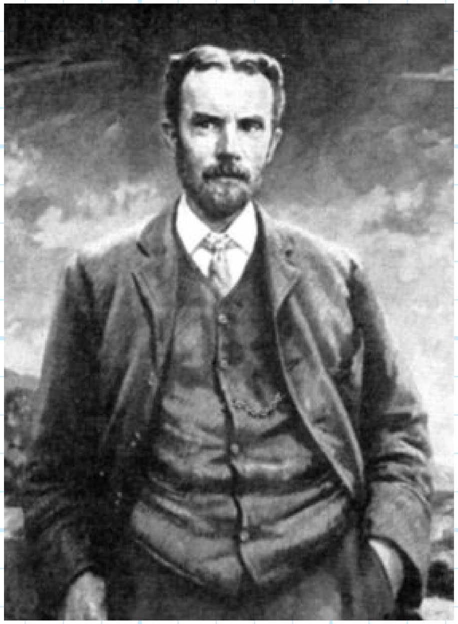


Q: *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $z_L - \ell \leq z \leq z_L$?)?*

A: To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

The First Two Requirements



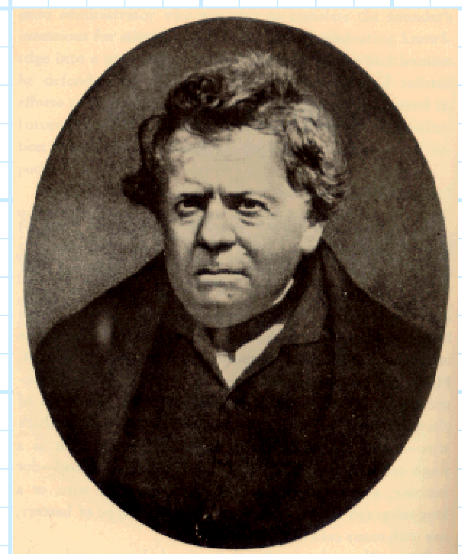
Requirement 1. To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution** (i.e., satisfy the **telegraphers equations**):

$$\begin{aligned} V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\ &= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L} \end{aligned}$$

$$\begin{aligned} I(z=z_L) &= \frac{V^+(z=z_L)}{Z_0} - \frac{V^-(z=z_L)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L} \end{aligned}$$

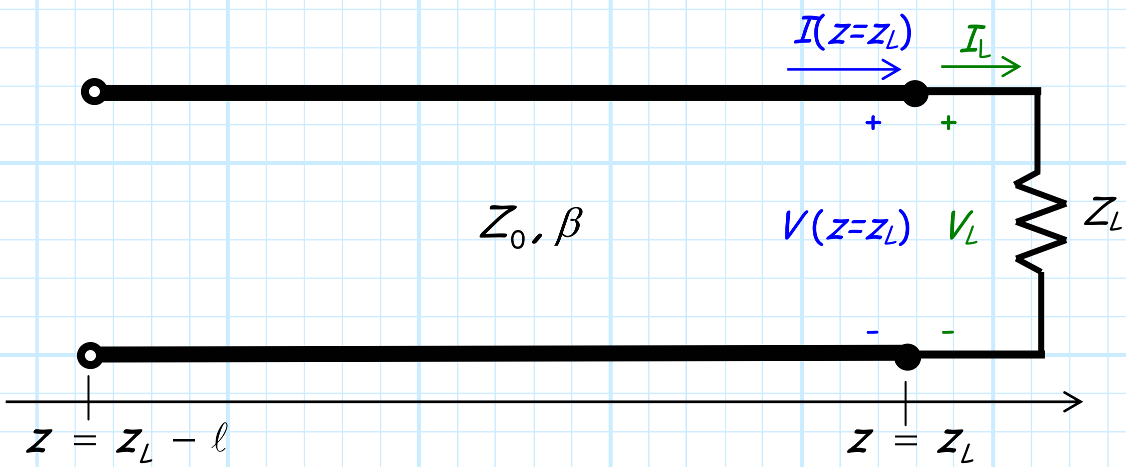
Requirement 2. Likewise, the load voltage and current must be related by **Ohm's law**:

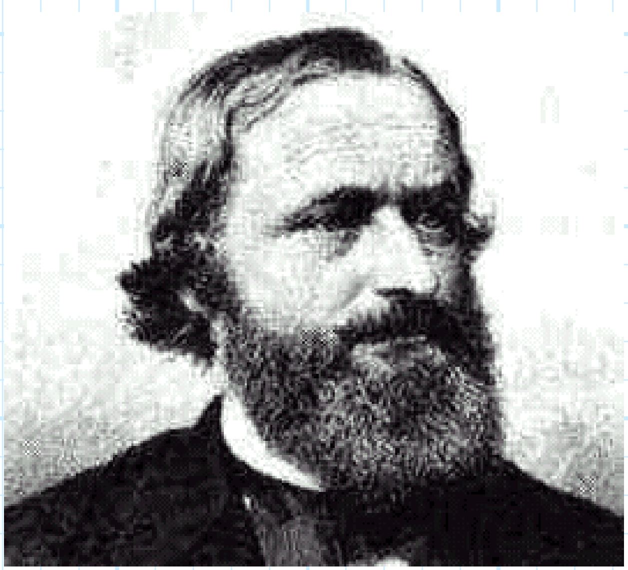
$$V_L = Z_L \, I_L$$



Boundary Conditions !!!!!!

Requirement 3. Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!





From KVL and KCL we find these requirements:

$$V(z=z_L)=V_L$$

$$I(z=z_L)=I_L$$

These are our **boundary conditions!**

A Solution for all Requirements

Combining the mathematical results of these three requirements, we find that:

$$Z(z = z_L) = Z_L$$

In other words, the line impedance at the end of the transmission line (i.e., at $z = z_L$) must be equal to the load impedance attached to that end!

Q: *But the result above is useful for the "**old**" $V(z), I(z), Z(z)$ description of transmission line activity. What does the boundary condition enforce with respect to our **"new" wave viewpoint** (i.e., $V^+(z), V^-(z), \Gamma(z)$??*

A: The three requirements lead us to these relationships:

$$V_L = Z_L \, I_L$$

$$V(z = z_L) = Z_L \, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0} \left(V^+(z = z_L) - V^-(z = z_L) \right)$$

Rearranging, we can conclude:

$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as **reflection coefficient** $\Gamma(z)$. How does this relate to the expression **above**?*

A: Recall that $\Gamma(z)$ is a **function** of transmission line position *z*. The value $V^-(z=z_L)/V^+(z=z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z=z_L$ (i.e., evaluated at the **end** of the line):

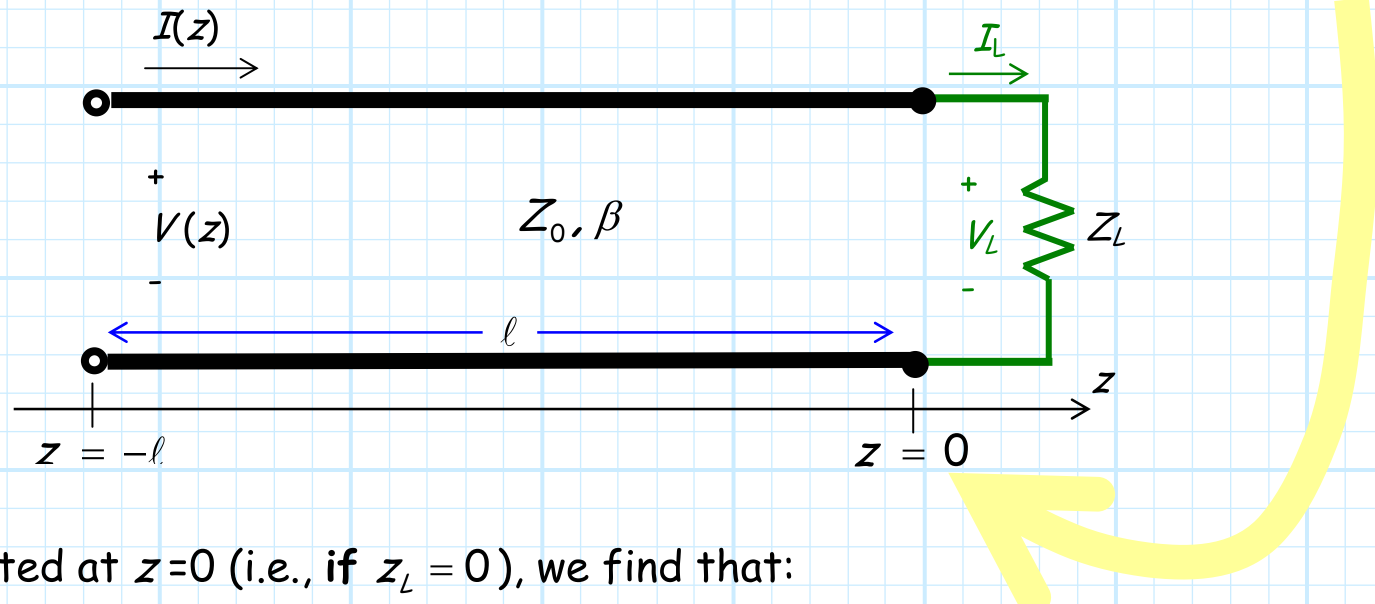
$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_L) !

$$\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

A Useful Simplification

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point z_L a **zero** value (i.e., $z_L = 0$):



If the load is located at z=0 (i.e., **if** $z_L = 0$), we find that:

$$V(z=0) = V^{+}(z=0) + V^{-}(z=0)$$
$$= V_0^{+}\, e^{-j\beta(0)} + V_0^{-}\, e^{+j\beta(0)}$$
$$= V_0^{+} + V_0^{-}$$

$$I(z=0) = \frac{V_0^{+}(z=0)}{Z_0} - \frac{V_0^{-}(z=0)}{Z_0}$$
$$= \frac{V_0^{+}}{Z_0} e^{-j\beta(0)} - \frac{V_0^{-}}{Z_0} e^{+j\beta(0)}$$
$$= \frac{V_0^{+} - V_0^{-}}{Z_0}$$

$$Z(z=0) = Z_0 \left(\frac{V_0^{+} + V_0^{-}}{V_0^{+} - V_0^{-}} \right)$$

Likewise, it is apparent that **if** $z_L = 0$, Γ_L and Γ_0 are the same:

$$\Gamma_L = \Gamma\left(z = z_L\right) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_0\, e^{+j\beta z}\right]$$

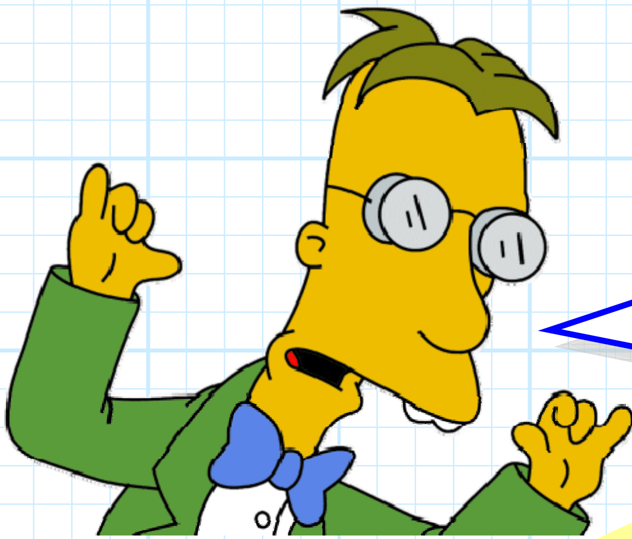
$$\left[\text{for } z_L = 0\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_0\, e^{+j\beta z}\right]$$

$$Z(z) = Z_0\left(\frac{e^{-j\beta z} + \Gamma_0\, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_0\, e^{+j\beta z}}\right)$$

What About V_0^+ ??

Q: *But, how do we determine V_0^+ ??*



A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !