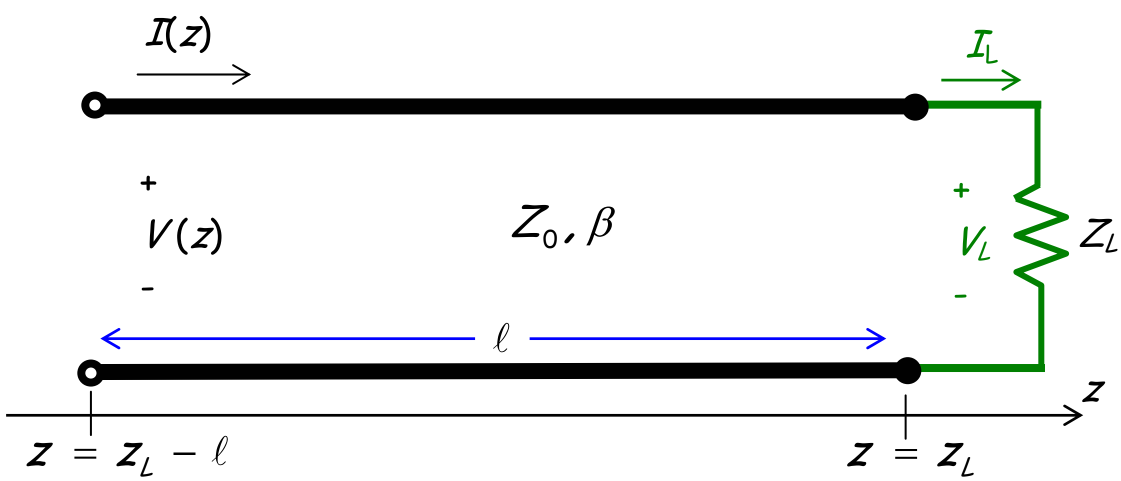
# The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length $\ell$, terminated with a **load** $Z_L$.




**Q:** *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points $z$ where $z_L - \ell \le z \le z_L$ ?)?*

**A:** To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

**1.** To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution**:

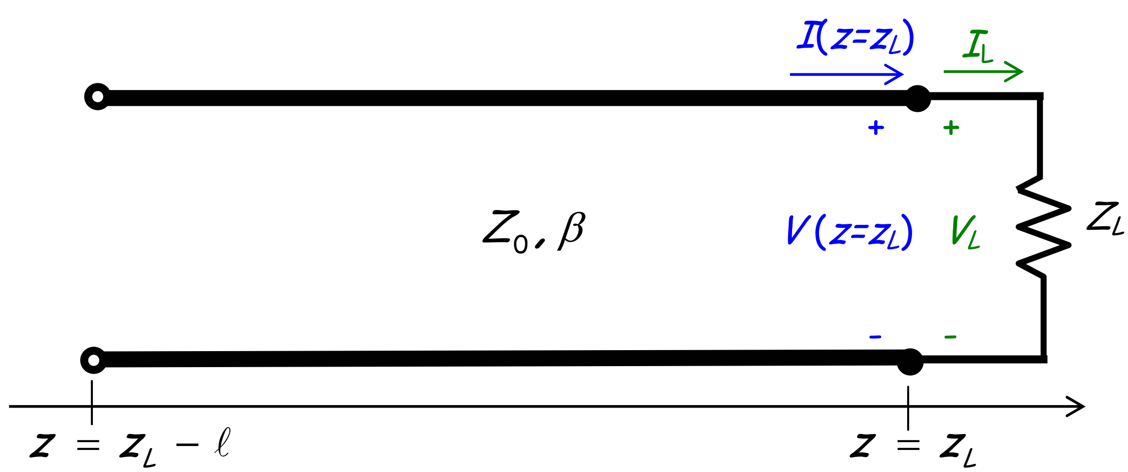
$$\begin{aligned} V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\ &= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L} \end{aligned}$$

$$\begin{aligned} I(z=z_L) &= \frac{V_0^+(z=z_L)}{Z_0} - \frac{V_0^-(z=z_L)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L} \end{aligned}$$

**2.** Likewise, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L I_L$$

**3.** Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and $I_L$, $V_L$ are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!



From KVL and KCL we find these requirements:

$$V(z = z_L) = V_L$$

$$I(z = z_L) = I_L$$

These are the **boundary conditions** for **this** particular problem.

→ **Careful**! Different transmission line problems lead to **different** boundary conditions—**you** must access each problem **individually** and **independently**!

**Combining** these equations and boundary conditions, we find that:

$$V_L = Z_L\, I_L$$

$$V(z = z_L) = Z_L\, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0}\left(V^+(z = z_L) - V^-(z = z_L)\right)$$

Rearranging, we can conclude:

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

**Q:** *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as* ***reflection coefficient*** *$\Gamma(z)$. How does this relate to the expression* ***above****?*

**A:** Recall that $\Gamma(z)$ is a **function** of transmission line position *z*. The value $V^-(z=z_L)/V^+(z=z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z=z_L$ (i.e., evaluated at the **end** of the line):

$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol ($\Gamma_L$) !

$$\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

**Q:** *Wait! We* ***earlier*** *determined that:*

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

*so it would seem that:*

$$\Gamma_L = \Gamma(z=z_L) = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

***Which*** *expression is correct??*

**A:** They **both** are! It is evident that the two expressions:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}$$

are **equal** if:

$$Z(z = z_L) = Z_L$$

And since we know that from **Ohm's Law**:

$$Z_L = \frac{V_L}{I_L}$$

and from **Kirchoff's Laws**:

$$\frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)}$$

and that **line impedance** is:

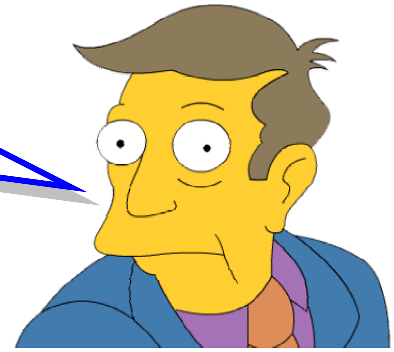
$$\frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L)$$

we find it apparent that the l**ine impedance** at the **end** of the transmission line is **equal** to the l**oad impedance**:

$$Z(z = z_L) = Z_L$$

The above expression is essentially **another** expression of the **boundary condition** applied at the **end** of the transmission line.

**Q:** *I'm confused! Just what are were we trying to accomplish in this handout?*



**A:** We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load $Z_L$!

We can now determine the value of $V_0^-$ in terms of $V_0^+$. Since:

$$\Gamma_L = \frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{V_0^- e^{+j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L}\,\Gamma_L\,V_0^+$$

And therefore we find:

$$V^-(z) = \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z}$$

$$V(z) = V_0^+\left[e^{-j\beta z} + \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

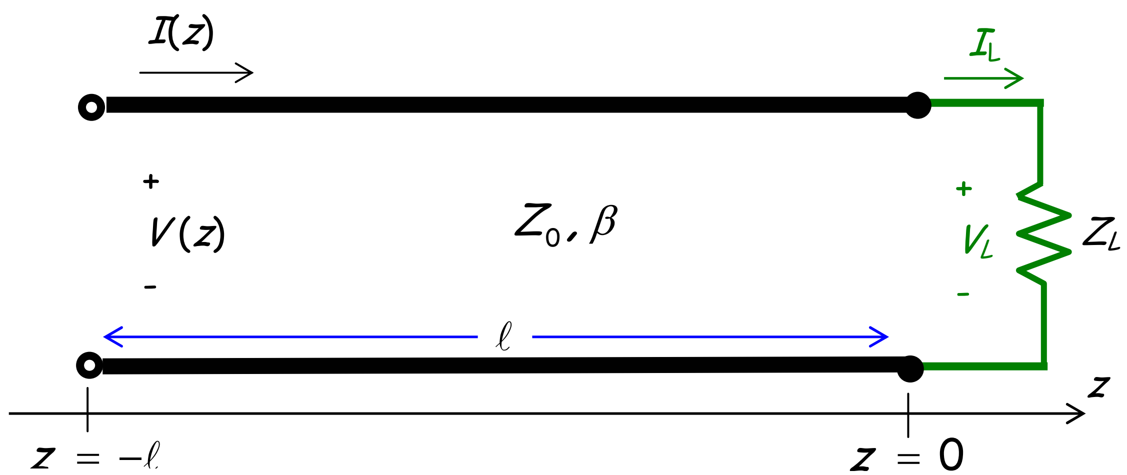
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

## $z_L = 0$

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point $z_L$ a **zero** value (i.e., $z_L = 0$):




**If** the load is located at $z$=0 (i.e., **if** $z_L = 0$), we find that:

$$
\begin{aligned}
V(z=0) &= V^+(z=0) + V^-(z=0) \\
&= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\
&= V_0^+ + V_0^-
\end{aligned}
$$

$$
\begin{aligned}
I(z=0) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\
&= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\
&= \frac{V_0^+ - V_0^-}{Z_0}
\end{aligned}
$$

$$
Z(z=0) = Z_0 \left( \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)
$$

Likewise, it is apparent that **if** $z_L = 0$, $\Gamma_L$ and $\Gamma_0$ are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+ \left[ e^{-j\beta z} + \Gamma_0 \, e^{+j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - \Gamma_0 \, e^{+j\beta z} \right]$$

$$\left[ \text{for } z_L = 0 \right]$$

**Q:** *But, how do we determine $V_0^+$ ??*

**A:** We require a **second** boundary condition to determine $V_0^+$. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !