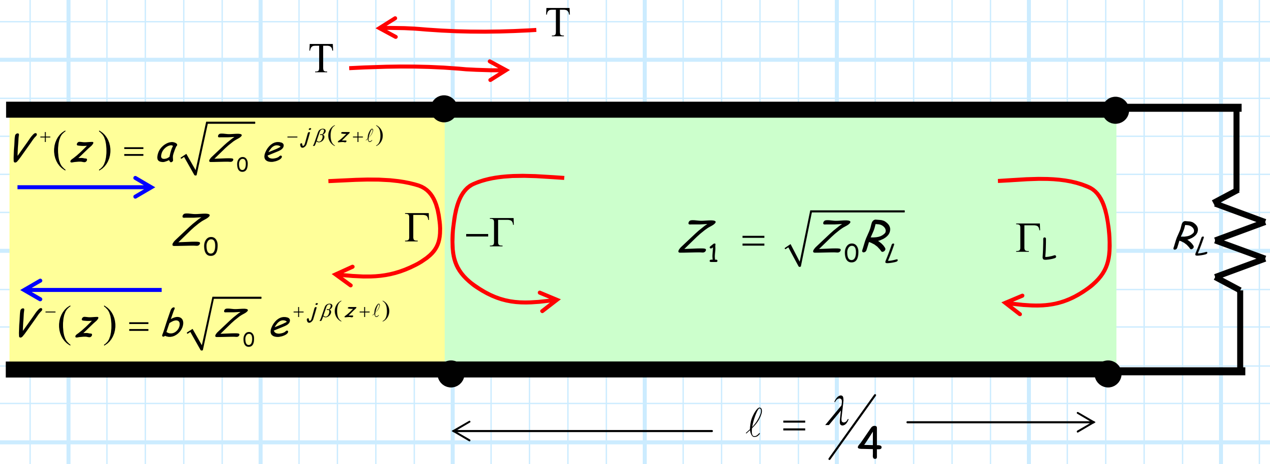
The Theory of Small Reflections

Recall that we analyzed a **quarter-wave** transformer using the multiple reflection view point.

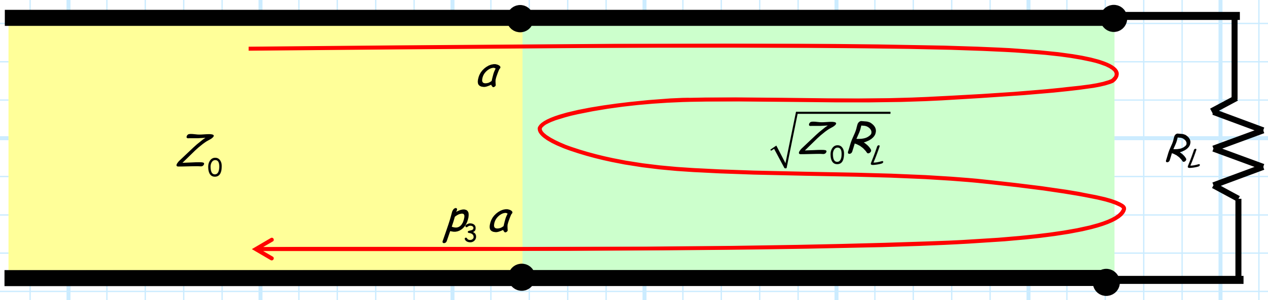


We found that the solution could thus be written as an **infinite** summation of terms (the **propagation series**):

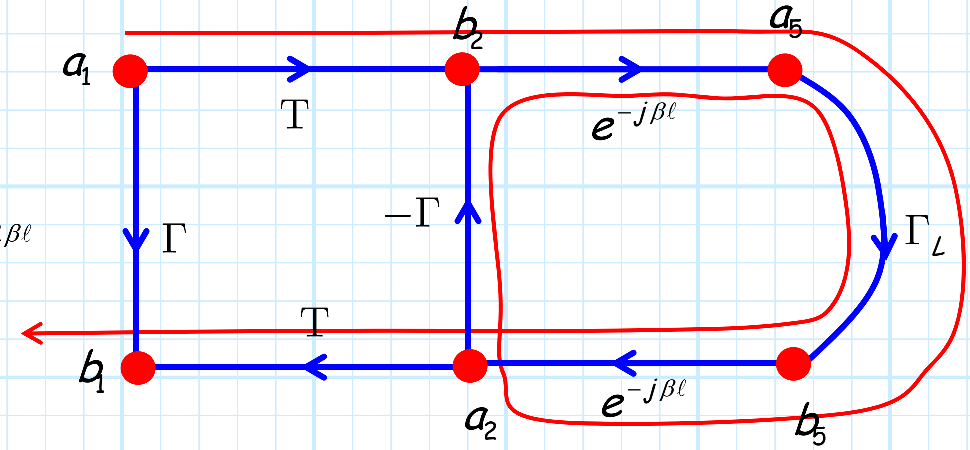
$$b = a\sum_{n=1}^{\infty} p_n$$

where each term had a specific **physical** interpretation, in terms of reflections, transmissions, and propagations.

For example, the **third** term was **path**:



$$p_3 = \mathrm{T}^2 \left(\Gamma_L\right)^2 \Gamma\, e^{-j2\beta\ell}$$



Now let's consider the **magnitude** of this path:

$$\begin{aligned} |p_3| &= |\mathrm{T}|^2 |\Gamma_L|^2 |\Gamma| \left|e^{-j2\beta\ell}\right| \\ &= |\mathrm{T}|^2 |\Gamma_L|^2 |\Gamma| \end{aligned}$$

Recall that $\Gamma = \Gamma_L$ for a **properly designed** quarter-wave transformer :

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L$$

and so:

$$|p_3| = |\mathrm{T}|^2 |\Gamma_L|^2 |\Gamma| = |\mathrm{T}|^2 |\Gamma_L|^3$$

For the case where values R_L and Z_1 are numerically **"close"** in —i.e., when:

$$|R_L - Z_1| \ll |R_L + Z_1|$$

we find that the magnitude of the reflection coefficient will be **very** small:

$$|\Gamma_L| = \left|\frac{R_L - Z_1}{R_L + Z_1}\right| \ll 1.0$$

As a result, the value $|\Gamma_L|^3$ will be **very**, **very**, **very** small.

Moreover, we know (since the connector is **lossless**) that:

$$1 = |\Gamma|^2 + |T|^2 = |\Gamma_L|^2 + |T|^2$$

and so:

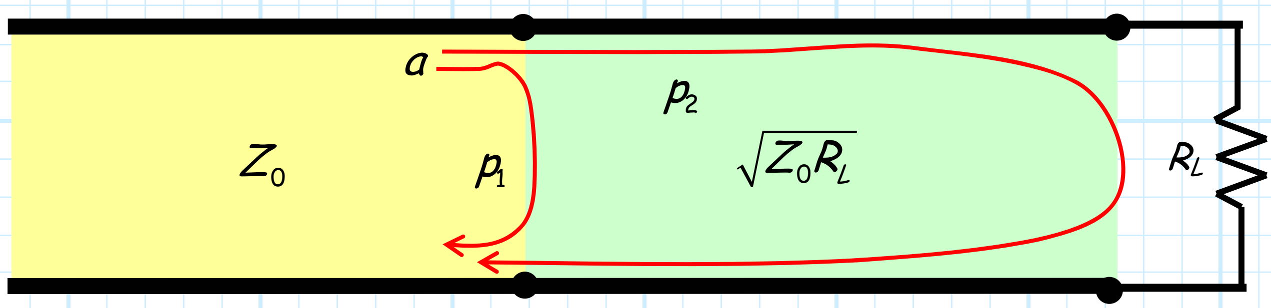
$$|T|^2 = 1 - |\Gamma_L|^2 \approx 1$$

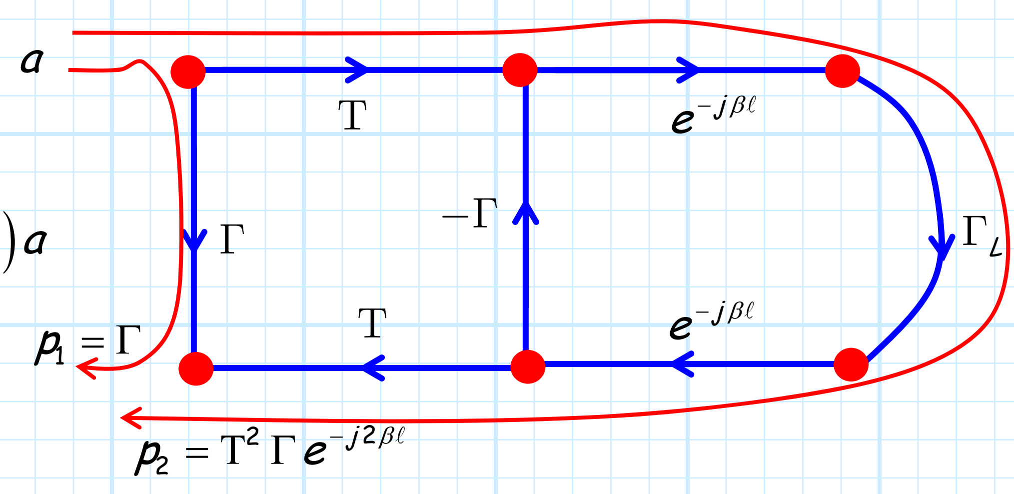
We can thus conclude that the **magnitude** of path p_3 is likewise **very, very, very** small:

$$|p_3| = |T|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1$$

This is a **classic case** where we can approximate the propagation series using only the **forward paths**!!

Recall there are **two** forward paths:





$$b \simeq (p_1 + p_2)\,a$$
$$= \left(\Gamma + T^2\,\Gamma_L\, e^{j2\beta\ell}\right) a$$

Therefore **IF** Z_0 and R_L are very **close** in value, we find that we can **approximate** the reflected wave using only the **direct paths** of the infinite series:

$$\begin{aligned} b &\simeq \left(p_1 + p_2\right) a \\ &= \left(\Gamma + T^2\, \Gamma_L\, e^{j2\beta\ell}\right) a \end{aligned}$$

Therefore:

$$\begin{aligned} V^-(z) &= b\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \\ &\cong \left(\Gamma + T^2\, \Gamma_L\, e^{j2\beta\ell}\right) a\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \end{aligned}$$

Now, if we likewise apply the **approximation** that $|T| \simeq 1.0$, we conclude for this quarter wave transformer (at the design frequency):

$$\begin{aligned} b &\simeq \left(p_1 + p_2\right) a \\ &= \left(\Gamma + \Gamma_L\, e^{j2\beta\ell}\right) a \end{aligned}$$

Therefore:

$$\begin{aligned} V^-(z) &= b\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \\ &\cong \left(\Gamma + \Gamma_L\, e^{j2\beta\ell}\right) a\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \end{aligned}$$

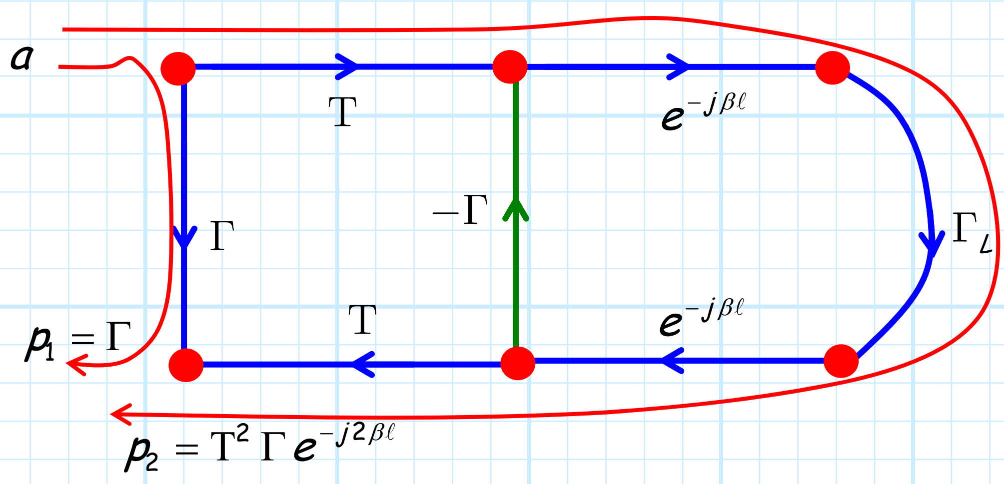
This **approximation**, where we:

1. use only the **direct paths** to calculate the propagation series,

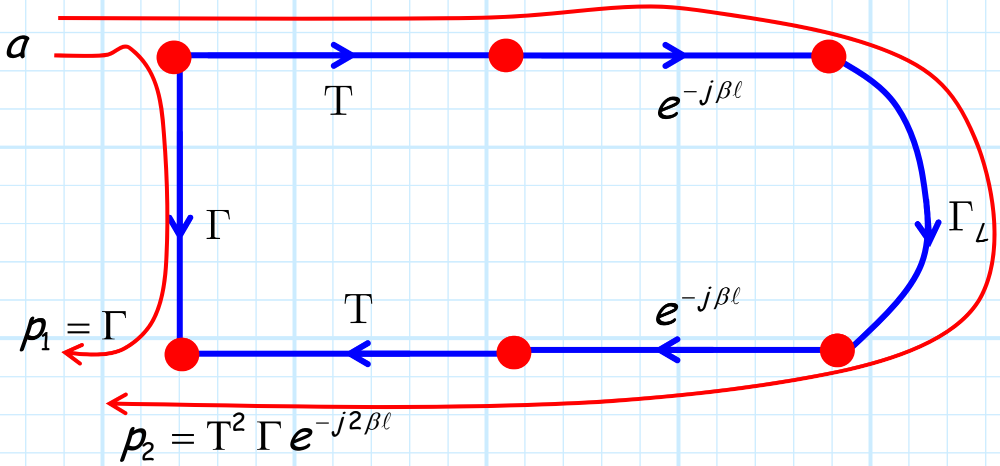
2. approximate the **transmission** coefficients as **one** (i.e., $\mathrm{T} = 1$).

is known as the **Theory of Small Reflections**, and allows us to use the propagation series as an **analysis** tool (we don't have to consider an **infinite** number of terms!).

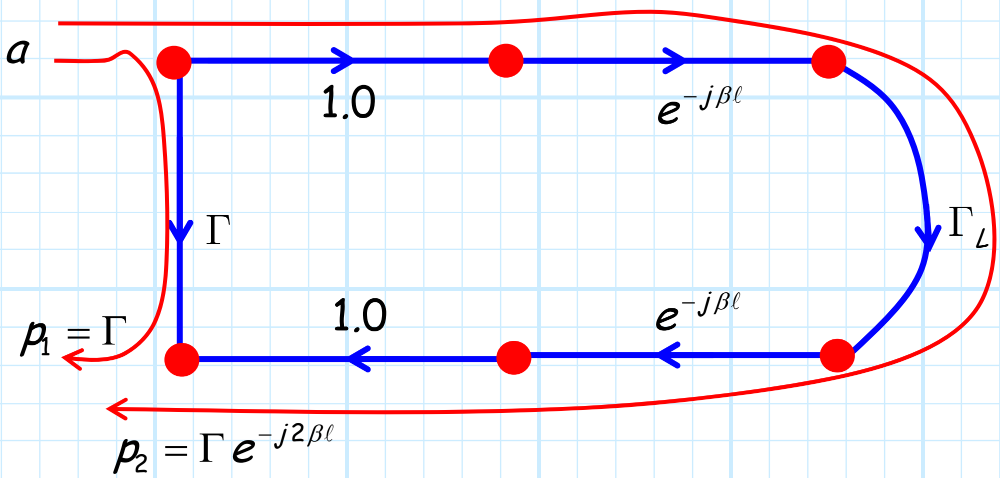
Consider again the quarter-wave matching network *SFG*. Note there is **one branch** ($-\Gamma = S_{22}$ of the connector), that is **not included** in either **direct path**.



With respect to the theory of small reflections (where **only** direct paths are considered), this branch can be **removed** from the *SFG* **without affect**.

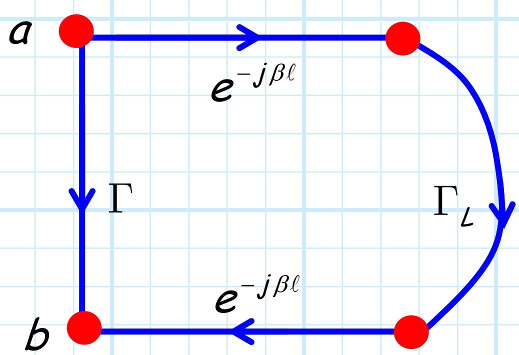


Moreover, the theory of small reflections implements the **approximation** $T = 1$, so that the *SFG* becomes:



Reducing this SFG by combining the 1.0 branch and the $e^{-j\beta\ell}$ branch via the **series rule**, we get the following **approximate** SFG:

$$\Gamma_{in} = \frac{b}{a}$$
$$= \Gamma + \Gamma_L\, e^{j2\beta\ell}$$



The approximate SFG when applying the theory of small reflections!

Note this **approximate** SFG provides **precisely** the results of the theory of small reflections!

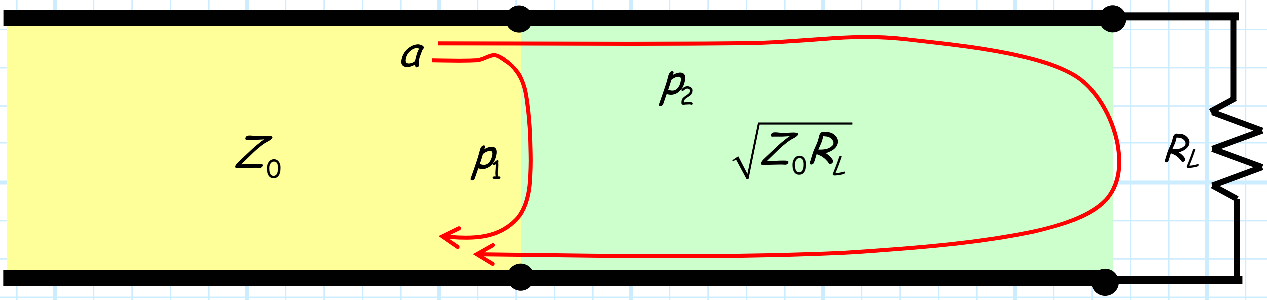
Q: *Why is that?*

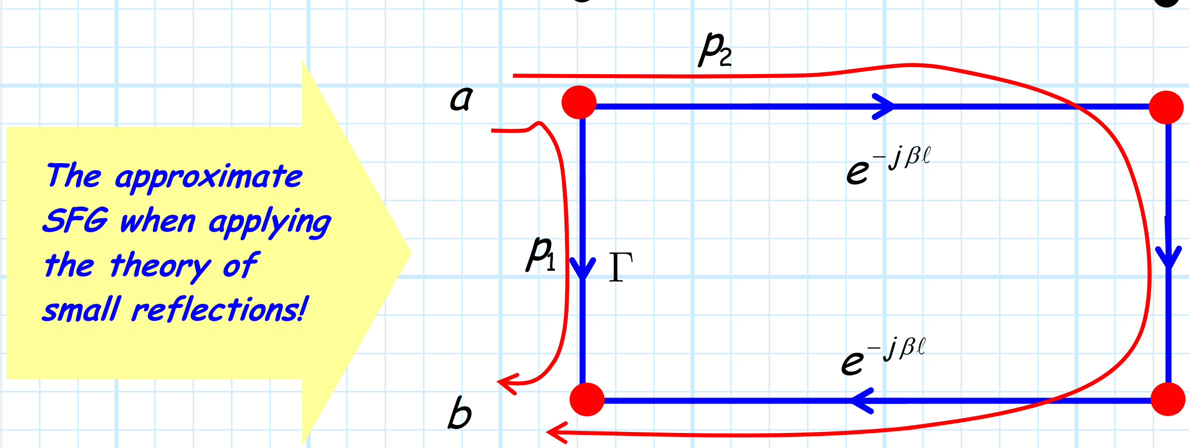
A: The approximate "theory of small reflections SFG" Contains all of the **significant physical propagation mechanisms** of the two *forward paths*, and **only** the two significant propagation mechanisms of the two forward paths.

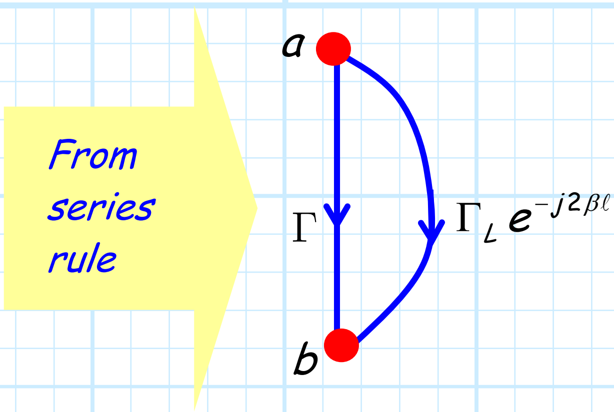
Namely:

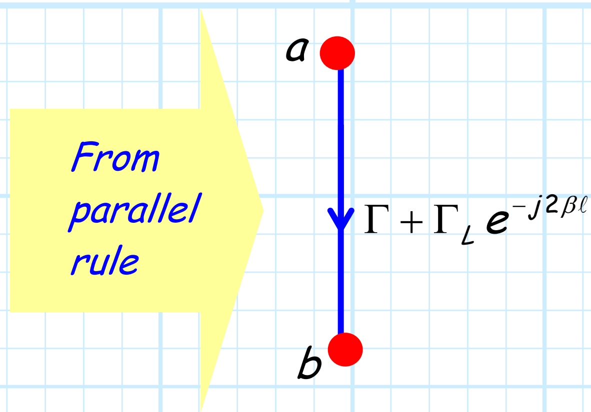
1. The **reflection** at the connector (i.e., Γ).

2. The **propagation down** the quarter-wave transmission line ($e^{-j\beta\ell}$), the **reflection** off the load (Γ_L), and the **propagation back up** the quarter-wave transmission line ($e^{-j\beta\ell}$).









Q: *But wait! The quarter-wave transformer is a **matching** network, therefore $\Gamma_{in} = 0$. The **theory of small reflections**, however, provides the **approximate result**:*

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell}$$

*Is this **approximation** very **accurate**? How **close** is this **approximate** value to the correct answer of $\Gamma_{in} = 0$?*

A: Let's find out!

Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\begin{aligned}\Gamma_{in} &\approx \Gamma + \Gamma_L e^{-j2\beta\ell} \\ &= \Gamma_L \left(1 + e^{-j2\beta\ell}\right)\end{aligned}$$

Likewise, $\ell = \lambda/4$ (but **only** at the design frequency!) so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = 2\pi/\lambda$!

Thus:

$$\begin{aligned}\Gamma_{in} &\approx \Gamma_L\left(1+e^{-j2\beta\ell}\right)\\ &= \Gamma_L\left(1+e^{-j\pi}\right)\\ &= \Gamma_L\left(1-1\right)\\ &= 0 \quad !!!\end{aligned}$$

Q: ***Wow!*** *The theory of small reflections appears to be a* ***perfect*** *approximation—****no error*** *at all!?!*

A: Not so fast.

The **theory of small reflections** most definitely provides an **approximate** solution (e.g., it **ignores** most of the terms of the propagation series, and it **approximates** connector transmission as $T = 1$, when in fact $T \neq 1$).

As a result, the solutions derived using the **theory of small reflections** will—generally speaking—exhibit **some** (hopefully small) **error**.

We just got a bit **"lucky"** for the quarter-wave matching network; the "approximate" result $\Gamma_{in} = 0$ was exact for this one case!

→ The **theory of small reflections** is an **approximate** analysis tool!