The Theory of Small Reflections

Recall that we analyzed a \textit{quarter-wave} transformer using the multiple reflection viewpoint.

\[ V' (z) = a \sqrt{Z_0} e^{-j \beta (z + \ell)} \]
\[ V^- (z) = b \sqrt{Z_0} e^{+j \beta (z + \ell)} \]

We found that the solution could thus be written as an \textit{infinite} summation of terms (the \textit{propagation series}):

\[ b = a \sum_{n=1}^{\infty} p_n \]

where each term had a specific \textit{physical} interpretation, in terms of reflections, transmissions, and propagations.

For example, the \textit{third} term was path:
Now let's consider the magnitude of this path:

\[ |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| |e^{-j2\beta_l}| \]

\[ = |T|^2 |\Gamma_L|^2 |\Gamma| \]

Recall that \( \Gamma = \Gamma_L \) for a properly designed quarter-wave transformer:

\[ \Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \]

and so:

\[ |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| = |T|^2 |\Gamma_L|^3 \]

For the case where values \( R_L \) and \( Z_1 \) are numerically "close" — i.e., when:

\[ |R_L - Z_1| \ll |R_L + Z_1| \]

we find that the magnitude of the reflection coefficient will be very small:

\[ |\Gamma_L| = \left| \frac{R_L - Z_1}{R_L + Z_1} \right| \ll 1.0 \]

As a result, the value \( |\Gamma_L|^3 \) will be very, very, very small.
Moreover, we know (since the connector is lossless) that:

\[ 1 = |\Gamma|^2 + |T|^2 = |\Gamma_L|^2 + |T|^2 \]

and so:

\[ |T|^2 = 1 - |\Gamma_L|^2 \approx 1 \]

We can thus conclude that the magnitude of path \( p_3 \) is likewise very, very, very small:

\[ |p_3| = |T|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1 \]

This is a classic case where we can approximate the propagation series using only the forward paths!!

Recall there are two forward paths:
Therefore \textbf{IF} \( Z_0 \) and \( R_L \) are very close in value, we find that we can \textbf{approximate} the reflected wave using only the \textbf{direct paths} of the infinite series:

\[
b = (p_1 + p_2) a = (\Gamma + \Gamma^2 \Gamma_L e^{j2\beta t}) a
\]

Therefore:

\[
V^-(z) = b \sqrt{Z_0} e^{+j\beta(z+t)}
\]

\[
\approx (\Gamma + \Gamma^2 \Gamma_L e^{j2\beta t}) a \sqrt{Z_0} e^{+j\beta(z+t)}
\]

Now, if we likewise apply the \textbf{approximation} that \(|\Gamma| = 1.0\), we conclude for this quarter wave transformer (at the design frequency):

\[
b = (p_1 + p_2) a = (\Gamma + \Gamma_L e^{j2\beta t}) a
\]

Therefore:

\[
V^-(z) = b \sqrt{Z_0} e^{+j\beta(z+t)}
\]

\[
\approx (\Gamma + \Gamma_L e^{j2\beta t}) a \sqrt{Z_0} e^{+j\beta(z+t)}
\]
This \textbf{approximation}, where we:

1. use only the \textbf{direct paths} to calculate the propagation series,

2. approximate the \textbf{transmission} coefficients as \textbf{one} (i.e., \( T = 1 \)).

is known as the \textbf{Theory of Small Reflections}, and allows us to use the propagation series as an \textbf{analysis} tool (we don't have to consider an \textbf{infinite} number of terms!).

Consider again the quarter-wave matching network \textit{SFG}. Note there is \textbf{one branch} (\(-\Gamma = S_{22}\) of the connector), that is \textbf{not} included in either \textbf{direct path}.

\[
p_1 = \Gamma \\
p_2 = T^2 \Gamma e^{-j2\beta L}
\]
With respect to the theory of small reflections (where only direct paths are considered), this branch can be **removed** from the **SFG** without affect.

Moreover, the theory of small reflections implements the **approximation** $\tau = 1$, so that the **SFG** becomes:

Reducing this **SFG** by combining the 1.0 branch and the $e^{-j\beta l}$ branch via the **series rule**, we get the following **approximate** **SFG**:

$$\Gamma_m = \frac{b}{a} = \Gamma + \Gamma_L e^{j2\beta l}$$

**The approximate SFG when applying the theory of small reflections!**
Note this approximate SFG provides precisely the results of the theory of small reflections!

Q: Why is that?

A: The approximate “theory of small reflections SFG” contains all of the significant physical propagation mechanisms of the two forward paths, and only the two significant propagation mechanisms of the two forward paths.

Namely:

1. The reflection at the connector (i.e., $\Gamma$).

2. The propagation down the quarter-wave transmission line ($e^{-j\beta t}$), the reflection off the load ($\Gamma_L$), and the propagation back up the quarter-wave transmission line ($e^{-j\beta t}$).
Q: But wait! The quarter-wave transformer is a matching network, therefore $\Gamma_{in} = 0$. The theory of small reflections, however, provides the approximate result:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell}$$

Is this approximation very accurate? How close is this approximate value to the correct answer of $\Gamma_{in} = 0$?

A: Let's find out!

Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell} = \Gamma_L (1 + e^{-j2\beta\ell})$$

Likewise, $\ell = \frac{\lambda}{4}$ (but only at the design frequency!) so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where you of course recall that $\beta = \frac{2\pi}{\lambda}$!
Thus:

\[
\Gamma_{in} \approx \Gamma_L \left( 1 + e^{-j2\beta L} \right) \\
= \Gamma_L \left( 1 + e^{-j\pi} \right) \\
= \Gamma_L (1 - 1) \\
= 0
\]

Q: Wow! The theory of small reflections appears to be a perfect approximation—no error at all?!

A: Not so fast.

The theory of small reflections most definitely provides an approximate solution (e.g., it ignores most of the terms of the propagation series, and it approximates connector transmission as \( T = 1 \), when in fact \( T \neq 1 \)).

As a result, the solutions derived using the theory of small reflections will—generally speaking—exhibit some (hopefully small) error.

We just got a bit “lucky” for the quarter-wave matching network; the “approximate” result \( \Gamma_{in} = 0 \) was exact for this one case!

\( \rightarrow \) The theory of small reflections is an approximate analysis tool!