The Transmission Line Wave Equations

So let's assume that $v(z,t)$ and $i(z,t)$ each have the **time-harmonic** form:

$$v(z,t) = \text{Re}\left\{V(z)\, e^{j\omega t}\right\} \qquad \text{and} \qquad i(z,t) = \text{Re}\left\{I(z)\, e^{j\omega t}\right\}$$

The **time**-derivative of these **eigen** functions are easily determined. E.G., :

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega\, V(z)\, e^{j\omega t}\right\}$$

From this we can show that the **telegrapher equations** relate $I(z)$ and $V(z)$ as:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z) \qquad \frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

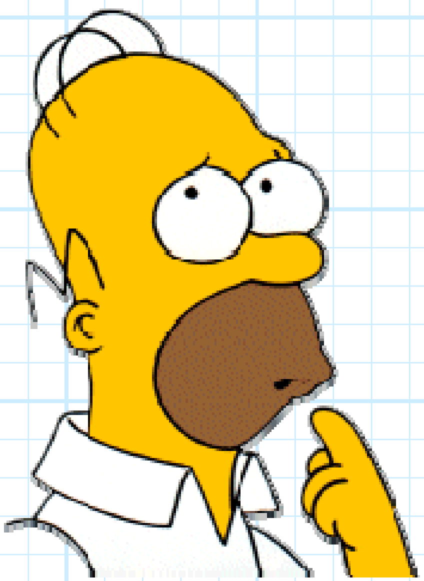
These are the **complex form** of the **telegrapher equations**.

What's your quest?

Note that these complex differential equations are **not** a function of **time** t!

$$\frac{\partial V(z)}{\partial z} = -(R + jwL)\, I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + jwC)\, V(z)$$



* The functions $I(z)$ and $V(z)$ are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position z.

* **Remember**, not just **any** function $I(z)$ and $V(z)$ can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.

Our quest, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

The Transmission Line Wave Equations

Q: *So, what functions $I(z)$ and $V(z)$* ***do*** *satisfy both telegrapher's equations??*

A: The complex telegrapher's equations are a pair of **coupled** differential equations.

With a little mathematical elbow grease, we can **decouple** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z)$$

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)$$

where

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

These equations are known as the transmission line **wave equations**. Since they each involve only **one** unknown function they are **easily** solved!

The (one and only) solution to the Wave Equations

The **solutions** to these wave equations are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

The wave interpretation

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z) \qquad I(z) = I^+(z) + I^-(z)$$

where:

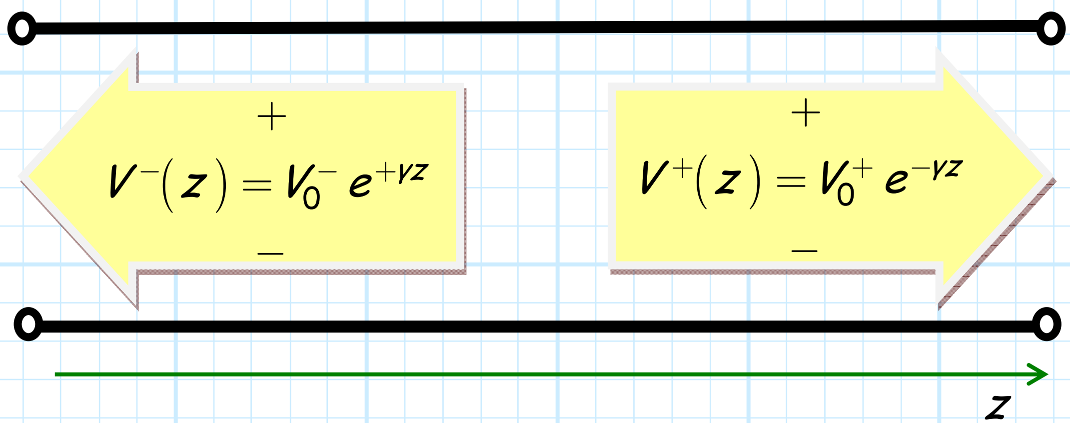
$$V^+(z) \doteq V_0^+ \, e^{-\gamma z} \qquad V^-(z) \doteq V_0^- \, e^{+\gamma z}$$

$$I^+(z) \doteq I_0^+ \, e^{-\gamma z} \qquad I^-(z) \doteq I_0^- \, e^{+\gamma z}$$

Q: *Just what do the two functions $V^+(z)$ and $V^-(z)$ **tell** us? Do they have any **physical meaning**?*

A: An **incredibly important** physical meaning!

Function $V^+(z)$ describes a **wave propagating** in the direction of **increasing** z, and $V^-(z)$ describes a **wave** in the opposite direction.



Complex amplitudes

Q: *So just what **are** the complex values $V_0^+, V_0^-, I_0^+, I_0^-$?*

A: They are called the **complex amplitudes** of each propagating **wave**.

Q: *Do they have any **physical meaning**?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point where $z=0$. We find that the **complex value** of the wave at that point is:

$$\begin{aligned} V^+(z=0) &= V_0^+ e^{-\gamma(z=0)} \\ &= V_0^+ e^{-(0)} \\ &= V_0^+(1) \\ &= V_0^+ \end{aligned}$$

likewise:

$$V_0^- = V^-(z=0)$$

$$I_0^+ = I^+(z=0)$$

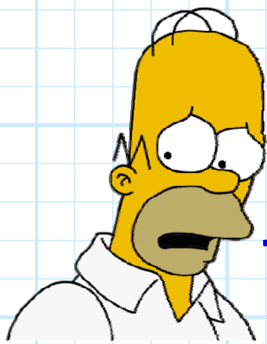
$$I_0^- = I^-(z=0)$$

So, the complex **wave amplitude** V_0^+ is simply the **complex** value of the wave function $V^+(z=0)$ at the point $z=0$ on the transmission line (that's what the **subscript** $_0$ means—the value at $z=0$)!

Determining the 4 complex wave amplitudes

Again, the **four** complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at **any and all** points on the transmission line!

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z), I^+(z), V^-(z), I^-(z)$.



Q: *But what* ***determines*** *these wave functions? How do we* ***find*** *the values of constants* V_0^+, I_0^+, V_0^-, I_0^-*?*

A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active **sources** and/or passive **loads**)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!