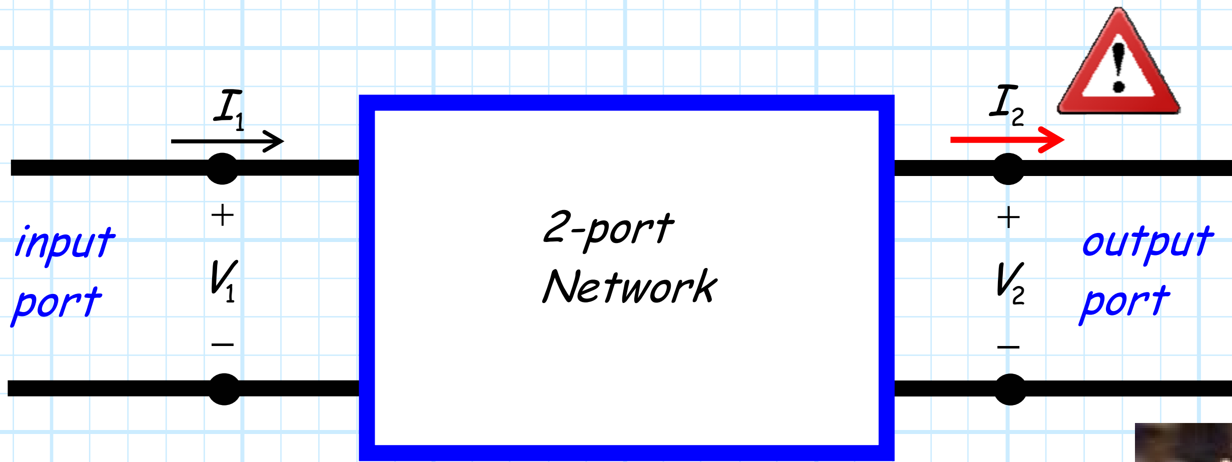
The Transmission Matrix

If a network has **two** ports, then we can **alternatively** define the voltages and currents at each port as:



Q: Say, can *we somehow relate the two* ***input*** *parameters* (I_1, V_1) *to the two* ***output*** *parameters* (I_2, V_2)*?*

A: Yes we can! We can relate them with **four** parameters A, B, C, D:

$$V_1 = A\,V_2 + B\,I_2$$

$$I_1 = C\,V_2 + D\,I_2$$

Or, using linear algebra:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

The matrix is defined as the **transmission matrix** $\mathcal{T}$, otherwise known as the **ABCD matrix** :

$$\mathcal{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Q: *Great. But what exactly* ***are*** *the values A, B, C, D, and how do we determine them?*

A: Similar to the impedance and admittance matrices, we determine the elements of the transmission matrix using **shorts** and **opens**.

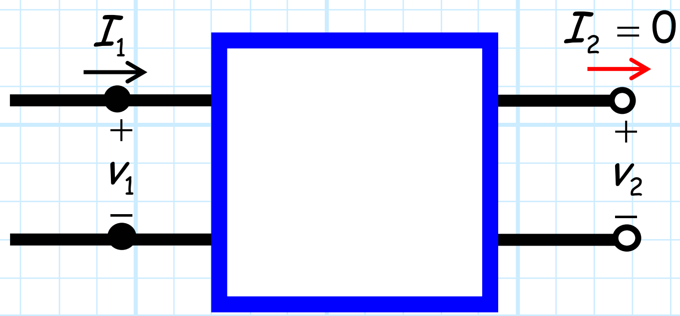
A

Note when $I_2 = 0$ then:

$$V_1 = A\,V_2$$

Therefore, to find value A, place an **open** on port 2, and then determine voltage V_1 in terms of V_2. We find that:

$$A = \frac{V_1}{V_2} \quad \text{(port 2 **open**)}$$



Note the parameter A is unitless (i.e., it is a coefficient).

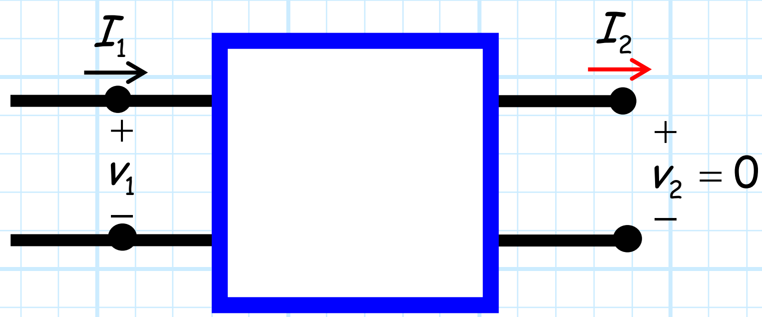
B

Note when $V_2 = 0$ then:

$$V_1 = B\, I_2$$

Therefore, to find value B, place a **short** on port 2, and then determine voltage V_1 in terms of I_2. We find that:

$$B = \frac{V_1}{I_2} \quad \text{(port 2 \textbf{short})}$$



Note parameter B has units of **impedance** (i.e., Ohms).

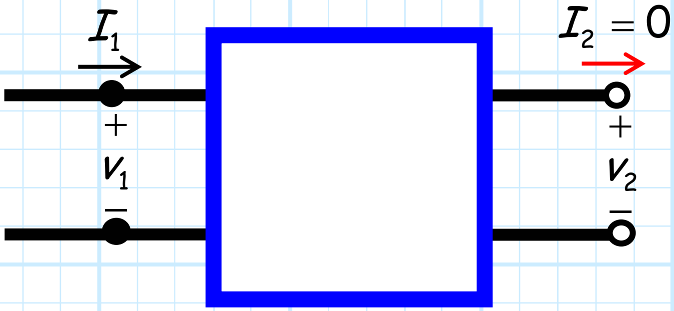
C

Note when $I_2 = 0$ then:

$$I_1 = C\, V_2$$

Therefore, to find value C, place an **open** on port 2, and then determine current I_1 in terms of voltage V_2. We find that:

$$C = \frac{I_1}{V_2} \quad \text{(port 2 } \textbf{open}\text{)}$$



Note parameter C has units of **admittance** (i.e., mhos).

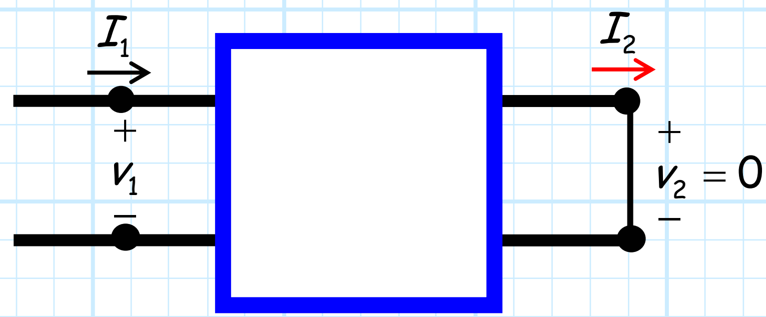
D

Note when $V_2 = 0$ then:

$$I_1 = D\, I_2$$

Therefore, to find value D, place a **short** on port 2, and then determine current I_1 in terms of current I_2. We find that:

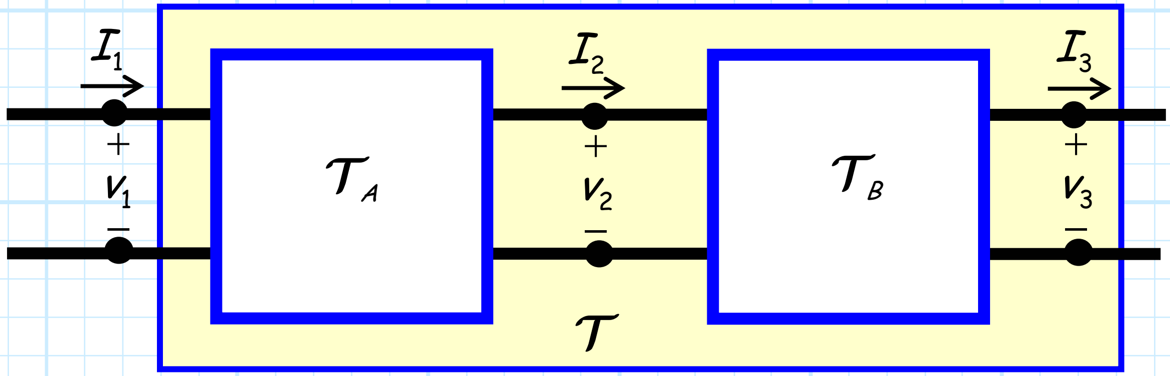
$$D = \frac{I_1}{I_2} \quad \text{(port 2 **short**)}$$



Note parameter D is unitless (another coefficient!).

Q: *For cryin' out loud! We already have the impedance matrix, the scattering matrix, **and** the admittance matrix. **Why** do we need the transmission matrix **also**? Is it somehow **uniquely** useful?*

A: Consider the case where a 2-port network is created by connecting (i.e., cascading) **two** networks:



Note

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathcal{T}_A \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

and

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \mathcal{T}_B \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

also

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathcal{T} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

Combining the first two equations:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathcal{T}_A \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \mathcal{T}_A \, \mathcal{T}_B \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

and comparing to the third:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathcal{T}_A \, \mathcal{T}_B \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \mathcal{T} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

we conclude that:

$$\mathcal{T} = \mathcal{T}_A \mathcal{T}_B$$

Likewise, for N cascaded networks, the **total** transmission matrix $\mathcal{T}$ can be determined as the product of all N networks!

$$\mathcal{T} = \mathcal{T}_1 \, \mathcal{T}_2 \, \mathcal{T}_3 \, \mathcal{T}_4 \cdots \mathcal{T}_4 = \prod_{n=1}^{N} \mathcal{T}_n$$

Note this result is **only** true for the **transmission** matrix $\mathcal{T}$. **No** equivalent result exists for $\mathcal{S}, \mathcal{Z}, \mathcal{Y}$!

Thus, the transmission matrix can greatly **simplify** the analysis of **complex** networks constructed from **two**-port devices. We find that the $\mathcal{T}$ matrix is particularly useful when creating design software for **CAD** applications.