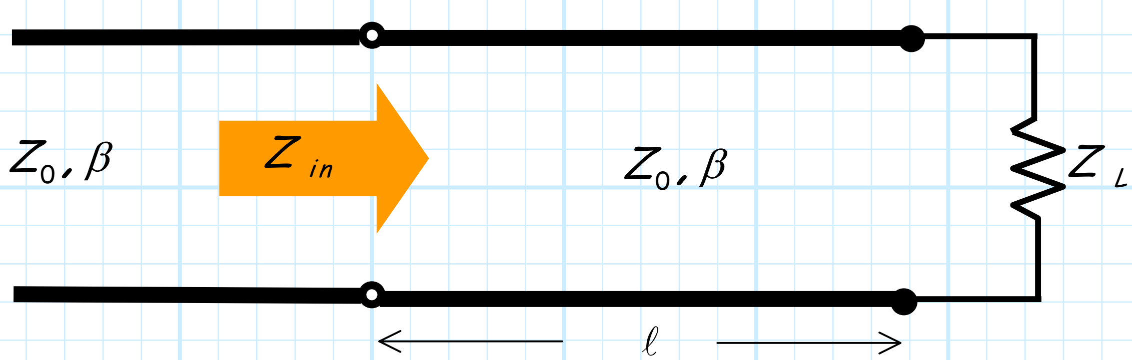
The Reflection Coefficient Transformation

The **load** at the end of some length of a transmission line (with characteristic impedance Z_0) can be specified in terms of its impedance Z_L **or** its reflection coefficient Γ_L.

Note **both** values are complex, and **either one** completely specifies the load—if you know **one**, you know the **other**!

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad Z_L = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right)$$

Recall that we determined how a length of transmission line **transformed** the load **impedance** into an input **impedance** of a (generally) different value:

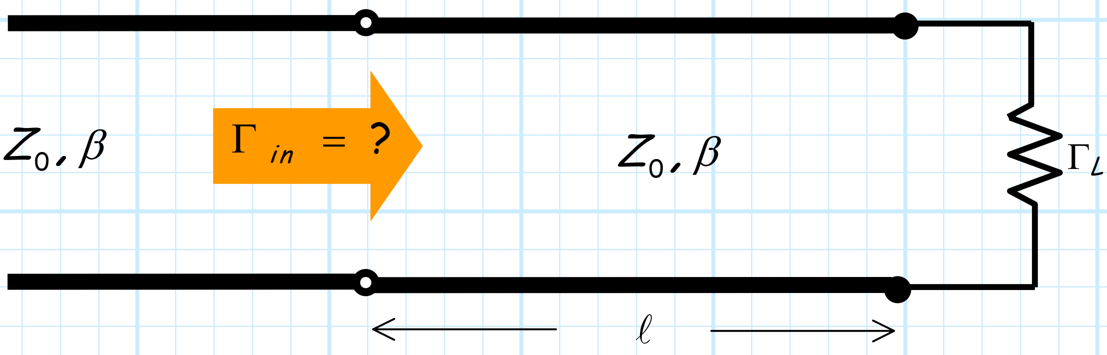


where:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan \beta\ell}{Z_0 + j\, Z_L \tan \beta\ell} \right)$$

Q: *Say we know the load in terms of its **reflection coefficient**. How can we express the **input** impedance in terms its **reflection coefficient** (call this Γ_{in})?*



A: Well, we **could** execute these **three** steps:

1. Convert Γ_L to Z_L:

$$Z_L = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right)$$

2. Transform Z_L down the line to Z_{in}:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right)$$

3. Convert Z_{in} to Γ_{in}:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Q: *Yikes! This is a **ton** of complex arithmetic—isn't there an **easier** way?*

A: Actually, there **is**!

Recall in an **earlier handout** that the input impedance of a transmission line length ℓ, terminated with a load Γ_L, is:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0 \left(\frac{e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}} \right)$$

Note this **directly** relates Γ_L to Z_{in} (steps 1 and 2 combined!).

If we directly **insert** this equation into:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

we get an equation **directly** relating Γ_L to Γ_{in} :

$$\Gamma_{in} = \frac{Z_0\left(e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right) - \left(e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right)}{Z_0\left(e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right) + \left(e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right)}$$

$$= \frac{2\Gamma_L e^{-j\beta\ell}}{2e^{+j\beta\ell}}$$

$$= \Gamma_L e^{-j\beta\ell} e^{-j\beta\ell}$$

$$= \Gamma_L e^{-j2\beta\ell}$$

Q: *Hey! This result looks* ***familiar****. Haven't we seen something like this* ***before****?*

A: Absolutely! Recall that we found that the reflection coefficient **function** $\Gamma(z)$ can be expressed as:

$$\Gamma(z) = \Gamma_0 e^{2\gamma z}$$

Now, for a **lossless** line, we know that $\gamma = j\beta$, so that:

$$\Gamma(z) = \Gamma_0 e^{j2\beta z}$$

Evaluating this function at the **beginning** of the line (i.e., at $z = z_L - \ell$):

$$\Gamma(z = z_L - \ell) = \Gamma_0 e^{j2\beta(z_L - \ell)}$$

$$= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta\ell}$$

But, we recognize that:

$$\Gamma_0 e^{j2\beta z_L} = \Gamma(z = z_L) = \Gamma_L$$

And so:

$$\begin{aligned}\Gamma(z = z_L - \ell) &= \Gamma_0\, e^{j2\beta z_L}\, e^{-j2\beta\ell} \\ &= \Gamma_L\, e^{-j2\beta\ell}\end{aligned}$$

Thus, we find that Γ_{in} is simply the value of function $\Gamma(z)$ **evaluated** at the line **input** of $z = z_L - \ell$!

$$\Gamma_{in} = \Gamma(z = z_L - \ell) = \Gamma_L\, e^{-j2\beta\ell}$$

Makes sense! After all, the input impedance is **likewise** simply the line impedance evaluated at the line input of $z = z_L - \ell$:

$$Z_{in} = Z(z = z_L - \ell)$$

It is apparent that from the above expression that the reflection coefficient at the input is simply related to Γ_L by a **phase shift** of $2\beta\ell$.

In other words, the **magnitude** of Γ_{in} is the **same** as the magnitude of Γ_L!

$$\begin{aligned}|\Gamma_{in}| &= |\Gamma_L|\left|e^{j(\theta_\Gamma - 2\beta\ell)}\right| \\ &= |\Gamma_L|\,(1) \\ &= |\Gamma_L|\end{aligned}$$

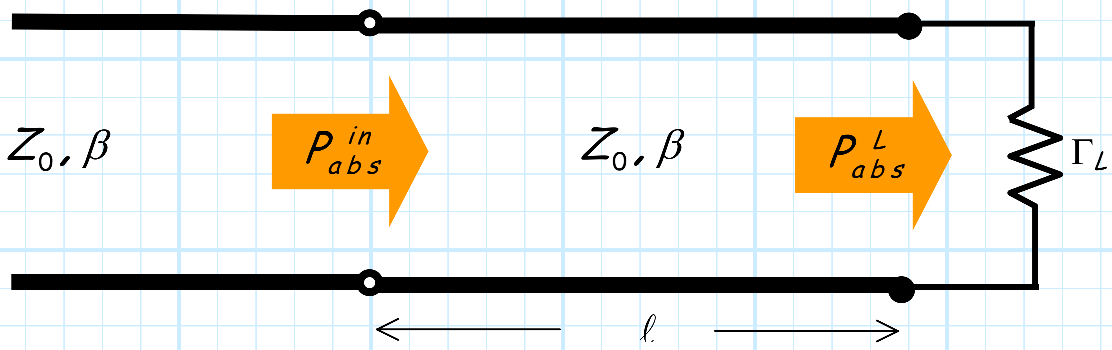
If we **think** about this, it makes **perfect sense**!

Recall that the power **absorbed** by the load Γ_{in} would be:

$$P_{abs}^{in} = \frac{|V_0^+|^2}{2\,Z_0}\left(1-|\Gamma_{in}|^2\right)$$

while that absorbed by the **load** Γ_L is:

$$P_{abs}^{L} = \frac{|V_0^+|^2}{2\,Z_0}\left(1-|\Gamma_{L}|^2\right)$$



Recall, however, that a lossless transmission line can absorb **no** power! By adding a length of transmission line to load Γ_L, we have added only **reactance**. Therefore, the power absorbed by load Γ_{in} is **equal** to the power absorbed by Γ_L:

$$P_{abs}^{in} = P_{abs}^{L}$$

$$\frac{|V_0^+|^2}{2\,Z_0}\left(1-|\Gamma_{in}|^2\right) = \frac{|V_0^+|^2}{2\,Z_0}\left(1-|\Gamma_{L}|^2\right)$$

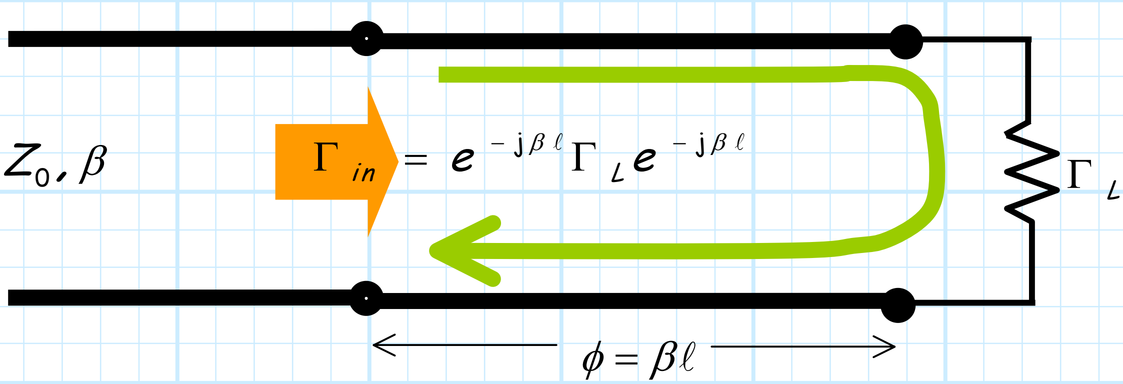
$$1-|\Gamma_{in}|^2 = 1-|\Gamma_{L}|^2$$

Thus, we can conclude from **conservation of energy** that:

$$\left|\Gamma_{in}\right| = \left|\Gamma_L\right|$$

Which of course is **exactly** the result we just found!

Finally, the **phase shift** associated with transforming the load Γ_L down a transmission line can be attributed to the phase shift associated with the wave propagating a length ℓ down the line, reflecting from load Γ_L, and then propagating a length ℓ back up the line:



To **emphasize** this wave interpretation, we recall that by definition, we can write Γ_{in} as:

$$\Gamma_{in} = \Gamma\left(z = z_L - \ell\right) = \frac{V^-\left(z = z_L - \ell\right)}{V^+\left(z = z_L - \ell\right)}$$

Therefore:

$$\begin{aligned} V^-\left(z = z_L - \ell\right) &= \Gamma_{in}\, V^+\left(z = z_L - \ell\right) \\ &= e^{-j\beta\ell}\,\Gamma_L\, e^{-j\beta\ell}\, V^+\left(z = z_L - \ell\right) \end{aligned}$$