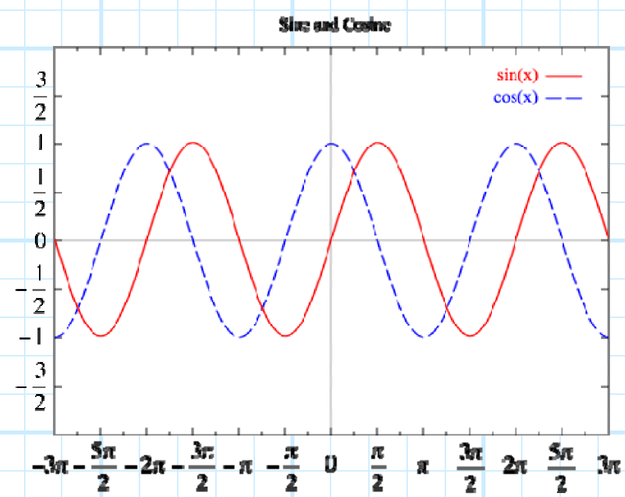
# Time-Harmonic Solutions for Linear Systems

There are an **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations!

However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** $\omega$ (e.g., $\cos \omega t$).




**Q:** *Why on earth would we assume a **sinusoidal** function of time? Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

**A:** Sinusoids have a **very** special property! Sinusoidal time functions—and **only** these functions—are the **eigen functions** of **linear, time-invariant** systems.

→ If a sinusoidal voltage source with frequency $\omega$ is used to excite a linear, time-invariant circuit (e.g., a transmission line), then the voltage at each and **every** point within the circuit will likewise vary sinusoidally—at the same frequency $\omega$!

# Eigen Functions and Transmission Lines

**Q:** *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

**A:** Not quite **exactly** the same. Although at every point within the circuit the voltage will be **precisely** sinusoidal (with frequency $\omega$), the **magnitude** and **relative phase** of the sinusoid will generally be **different** at each and every point within the circuit.

Thus, the voltage along a transmission line—**when** excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, at some arbitrary location $z$ along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

For a given frequency $\omega$, the **two functions** $v(z)$ and $\varphi(z)$ (functions of position $z$ only!) **completely** describe the oscillating voltage at each and **every** point along the transmission line.

# A Complex Representation of $v(z,t)$

**Q:** *I just thought of something! Our sinusoidal oscillations are described by a* ***magnitude*** *($V(z)$) and a* ***phase*** *($\varphi(z)$)—but a* ***complex*** *value is also defined by its* ***magnitude*** *and* ***phase*** *(i.e., $c = |c|\, e^{j\varphi_c}$). Is there a connection between our oscillations and a* ***complex*** *value?*

**A:** Absolutely! A connection made by **Euler's Identity**

$$e^{j\psi} = \cos\psi + j\sin\psi$$

From this it is apparent that:

$$Re\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$\begin{aligned} v(z,t) &= V(z)\cos\left(\omega t + \varphi(z)\right) \\ &= Re\left\{V(z)\, e^{j\left(\omega t + \varphi(z)\right)}\right\} \\ &= Re\left\{V(z)\, e^{+j\varphi(z)}\, e^{j\omega t}\right\} \end{aligned}$$

# The Complex Function V(z)

It is apparent that we can specify the time-harmonic voltage at each an every location $z$ along a transmission line with the **complex** function $V(z)$:

$$V(z) = v(z)\, e^{-j\varphi(z)}$$

So that:

$$\begin{aligned} v(z,t) &= v(z) \cos\left(\omega t + \varphi(z)\right) \\ &= Re\left\{ v(z)\, e^{+j\varphi(z)} e^{j\omega t} \right\} \\ &= Re\left\{ V(z)\, e^{j\omega t} \right\} \end{aligned}$$

where the **magnitude** of the **complex function** is the **magnitude** of the sinusoid:

$$v(z) = \left| V(z) \right|$$

and the **phase** of the **complex function** is the relative **phase** of the sinusoid :

$$\varphi(z) = arg\left\{ V(z) \right\}$$

# The Complex Function $V(z)$ and You

Microwave engineers almost **always** describe the activity of a transmission line (if excited by time harmonic sources) in terms of **complex functions of position** $z$—and **only** in terms of complex functions of position $z$ !!

As a result, it is **unfathomably important** that you understand what these complex functions **mean**. You **must understand** what these complex functions are telling you about the currents, voltages, etc. along a transmission line.

Perhaps it's helpful to think about these functions as sort of a **compression algorithm**, with the important information **"embedded"** in the complex values. To recover the information, we simply take the **magnitude** and **phase** of these complex values.

$$v(z) = |V(z)| \qquad \varphi(z) = arg\{V(z)\}$$

See if **you** can determine what these complex values tell you about the **voltage** at different points along a transmission line:

$$V(z=0)=3 \qquad V(z=1)=j \qquad V(z=2)=e^{j\pi/4}$$

$$V(z=3)=-2 \qquad V(z=4)=\sqrt{2}+j\sqrt{2} \qquad V(z=5)=3\,e^{-j\pi/4}$$

# Why we Love Eigen Functions

Note that the **complex function** $V(z)$ is a function of position $z$ **only**!

**Q:** *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

**A:** There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as the excitation source) then this must be time function at **all** transmission line locations $z$!

The only **unknown** is the **complex function** $V(z)$. Once we determine $V(z)$, we can always (**if** we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = Re\{V(z)e^{j\omega t}\} = v(z)\cos(\omega t + \varphi(z))$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$!!!