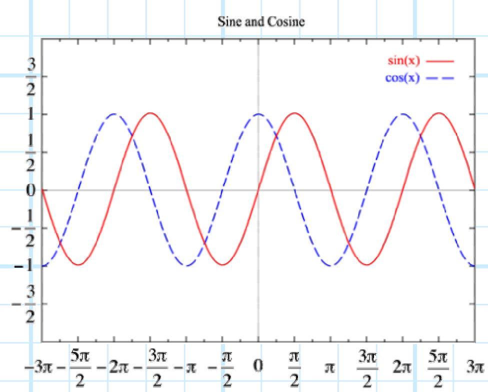
Time-Harmonic Solutions for Linear Circuits

There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations! However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).

Q: *Why on earth would we assume a **sinusoidal** function of time? Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!



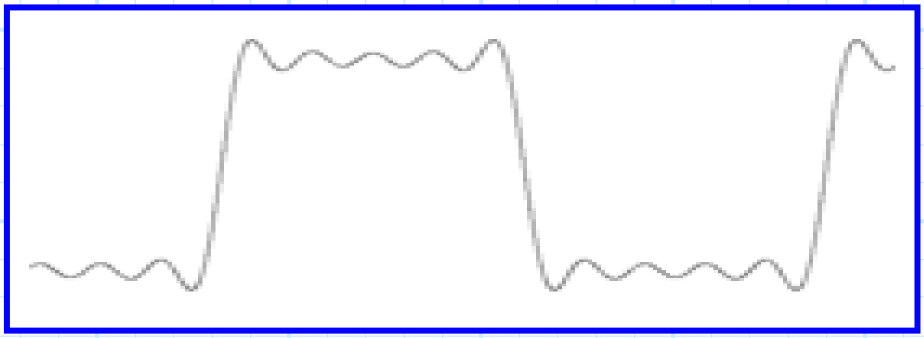
Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

Q: *???*

A: If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So what? Isn't that **obvious**?*

A: Not at all! If you were to excite a linear circuit with a **square wave**, or **triangle wave**, or **sawtooth**, you would find that—generally speaking—**nowhere** else in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth. The linear circuit will effectively **distort** the input signal into something **else**!



Q: *Into what function will the input signal be distorted?*

A: It depends—both on the original form of the **input signal**, and the parameters of the **linear circuit**. At **different** points within the circuit we will discover **different** functions of time—**unless**, of course, we use a **sinusoidal** input. Again, for a sinusoidal excitation, we find at **every** point within circuit an **undistorted** sinusoidal function!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

A: Not quite **exactly** the same. Although at every point within the circuit the voltage will be precisely sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be different at each and every point within the circuit.

Thus, the voltage along a transmission line—when excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos\big(\omega t + \varphi(z)\big)$$

Thus, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

Now, consider Euler's equation, which states:

$$e^{j\psi} = \cos\psi + j\sin\psi$$

Thus, it is apparent that:

$$Re\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$\begin{aligned} v(z,t) &= v(z)\cos\big(\omega t + \varphi(z)\big) \\ &= Re\left\{v(z)\, e^{j(\omega t + \varphi(z))}\right\} \\ &= Re\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\} \end{aligned}$$

Thus, we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

$$V(z) = v(z)\, e^{-j\varphi(z)}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = |V(z)|$$

and the phase of the complex function is the relative phase of the sinusoid :

$$\varphi(z) = arg\{V(z)\}$$

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as qt the excitation source) then this must be time function at **all** transmission line locations z!

The only **unknown** is the **complex** function $V(z)$. Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = Re\{V(z)\, e^{j\omega t}\}$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$.