Time-Harmonic Solutions for Transmission Lines

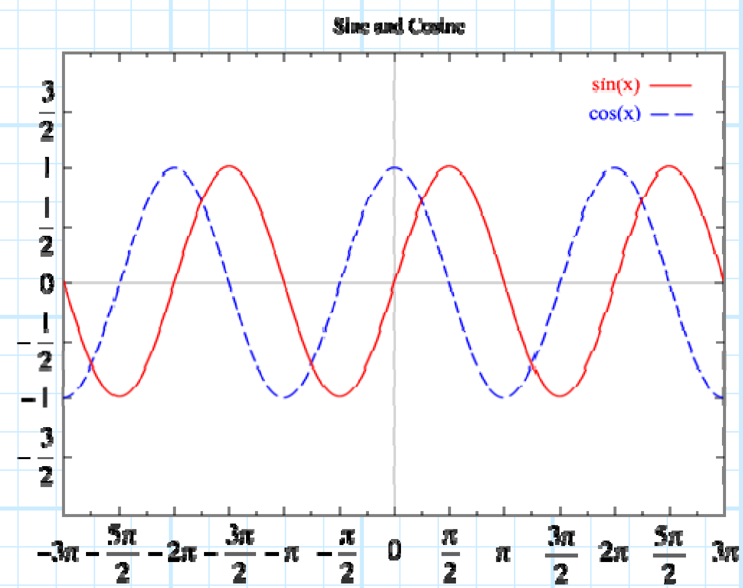
There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations!

However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).

Q: *Why on earth would we assume a* ***sinusoidal*** *function of time?*

Why not a ***square wave****, or* ***triangle wave****, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!



Eigen Functions

Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

→ If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

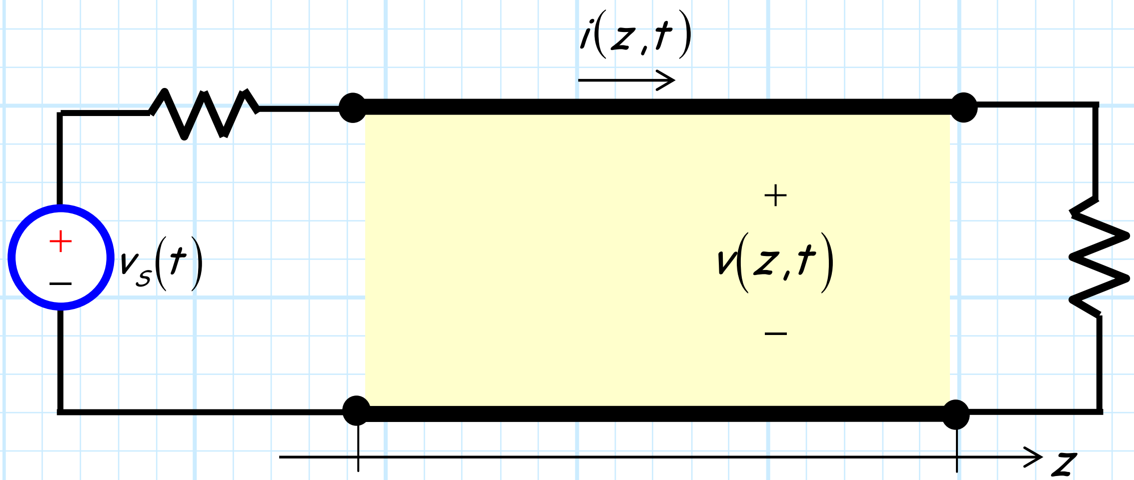
A: Not quite **exactly** the same.

Although at every point within the circuit the voltage will be **precisely** sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be **different** at each and every point within the circuit.

Eigen Functions and Transmission Lines

Thus, the voltage along a transmission line—**when** excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos(\omega t + \varphi(z))$$



In other words, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

For a given frequency ω, the **two functions** $v(z)$ and $\varphi(z)$ (functions of position z only!) **completely** describe the oscillating voltage at each and **every** point along the transmission line.

A Complex Representation of $v(z,t)$

Q: *I just thought of something!*

*Our sinusoidal oscillations are described by a magnitude ($v(z)$) and a phase ($\varphi(z)$)—but a complex value is **also** defined by its magnitude and phase (i.e., $c = |c| e^{j\varphi_c}$).*

*Is there a **connection** between our oscillations and a **complex value**?*

A: Absolutely! A connection made by **Euler's Identity**

$$e^{j\psi} = \cos\psi + j\sin\psi$$

From this it is apparent that:

$$\mathrm{Re}\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the real **voltage on a transmission line** can be expressed as:

$$v(z,t) = v(z)\cos(\omega t + \varphi(z)) = \mathrm{Re}\left\{v(z)\, e^{j(\omega t + \varphi(z))}\right\} = \mathrm{Re}\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\}$$

The Complex Function $V(z)$

It is apparent that we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

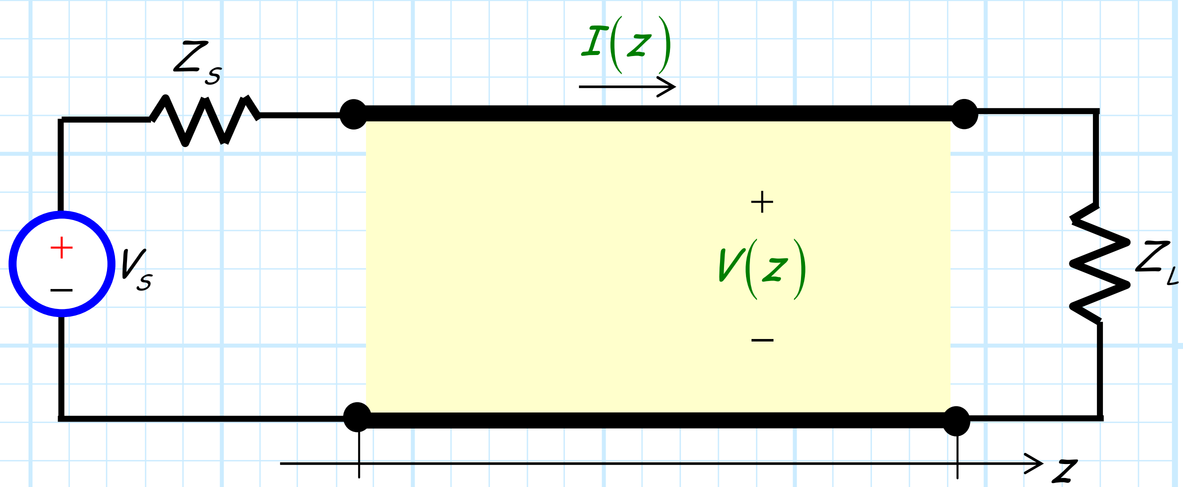
$$V(z) = v(z)\, e^{-j\varphi(z)}$$

So that:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right) = \mathrm{Re}\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\} = \mathrm{Re}\left\{V(z)\, e^{j\omega t}\right\}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = \left|V(z)\right|$$



and the **phase** of the complex function is the **relative phase** of the sinusoid :

$$\varphi(z) = \arg\left\{V(z)\right\}$$

All we need to determine is *V(z)*

Note then that only **unknown** is the **complex** function $V(z)$.

Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} = v(z)\cos(\omega t + \varphi(z))$$

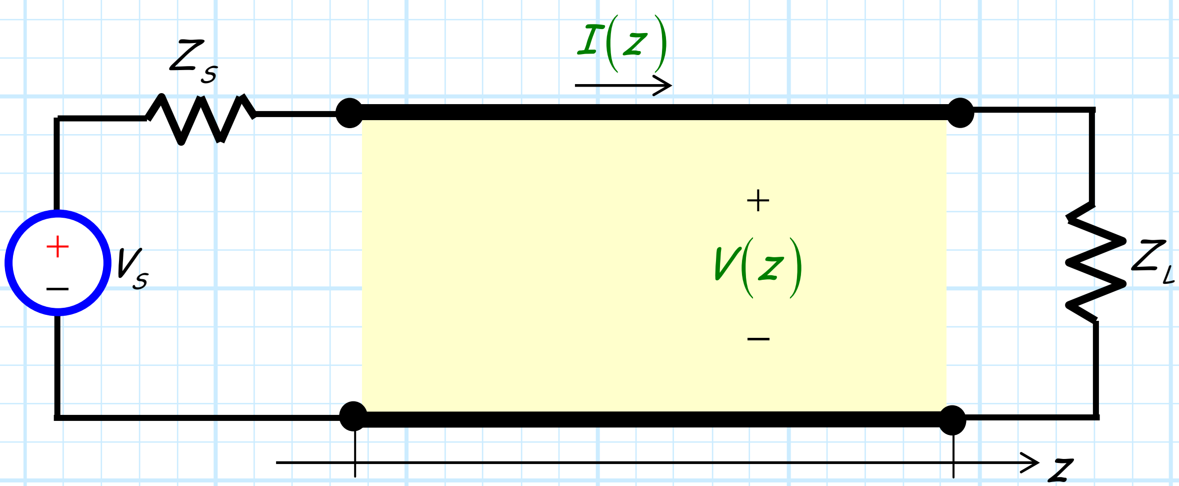
Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$!

Make this make sense to you

Microwave engineers almost **always** describe the activity of a transmission line (if excited by time harmonic sources) in terms of **complex functions of position** z — and **only** in terms of complex functions of position z !!

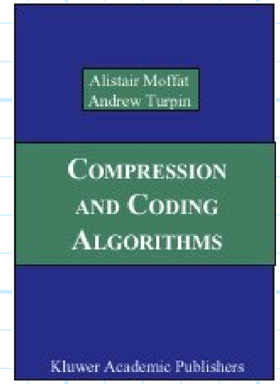
As a result, it is **unfathomably important** that you understand what these complex functions **mean**.

You **must understand** what these complex functions are telling you about the currents, voltages, etc. along a transmission line.



The Complex Function *V(z)* and You

Perhaps it's helpful to think about these functions as sort of a **compression algorithm**, with the important information **"embedded"** in the complex values.



To recover the information, we simply take the magnitude and phase of these complex values.

$$V(z) \rightarrow v(z) = |V(z)|$$

$$V(z) \rightarrow \varphi(z) = \arg\{V(z)\}$$

$$v(z,t) = v(z)\cos(\omega t + \varphi(z))$$

Note that the complex function $V(z)$ is a function of position z **only**!

Why we Love our Eigen Functions

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as the excitation source) then this must be time function at **all** transmission line locations z.

The only **unknown** is the **complex** function $V(z)$!

Once we determine $V(z)$, we can always (**if** we so desire) "recover" the **real** function $v(z,t)$ as:

$$\operatorname{Re}\left\{V(z)e^{j\omega t}\right\} = v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$!!!

Quiz !!

See if **you** can determine what these complex values tell you about the **voltage** at different points z along a transmission line:

$V(z=0)=3$ → $v(z=0,t)=\quad\cos(\omega t+\quad)$

$V(z=1)=j$ → $v(z=1,t)=\quad\cos(\omega t+\quad)$

$V(z=2)=e^{j\pi/4}$ → $v(z=2,t)=\quad\cos(\omega t+\quad)$

$V(z=3)=-2$ → $v(z=3,t)=\quad\cos(\omega t+\quad)$

$V(z=4)=\sqrt{2}+j\sqrt{2}$ → $v(z=4,t)=\quad\cos(\omega t+\quad)$

$V(z=5)=3e^{-j\pi/4}$ → $v(z=5,t)=\quad\cos(\omega t+\quad)$