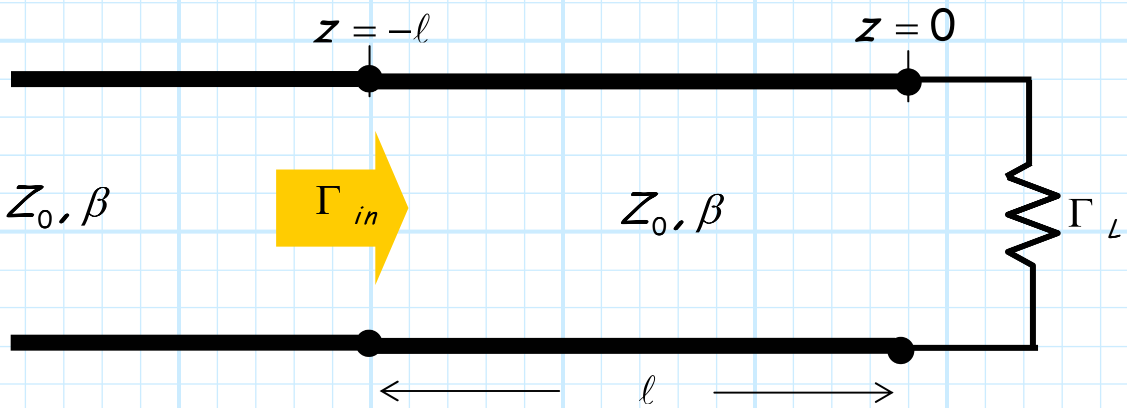
# Transformations on the Complex Γ Plane

The usefulness of the complex $\Gamma$ plane is apparent when we consider again the **terminated, lossless transmission line**:




Recall that the reflection coefficient function for **any** location $z$ along the transmission line can be expressed as (since $z_L = 0$):

$$\begin{aligned}\Gamma(z) &= \Gamma_L\, e^{j2\beta z} \\ &= |\Gamma_L|\, e^{j(\theta_\Gamma + 2\beta z)}\end{aligned}$$

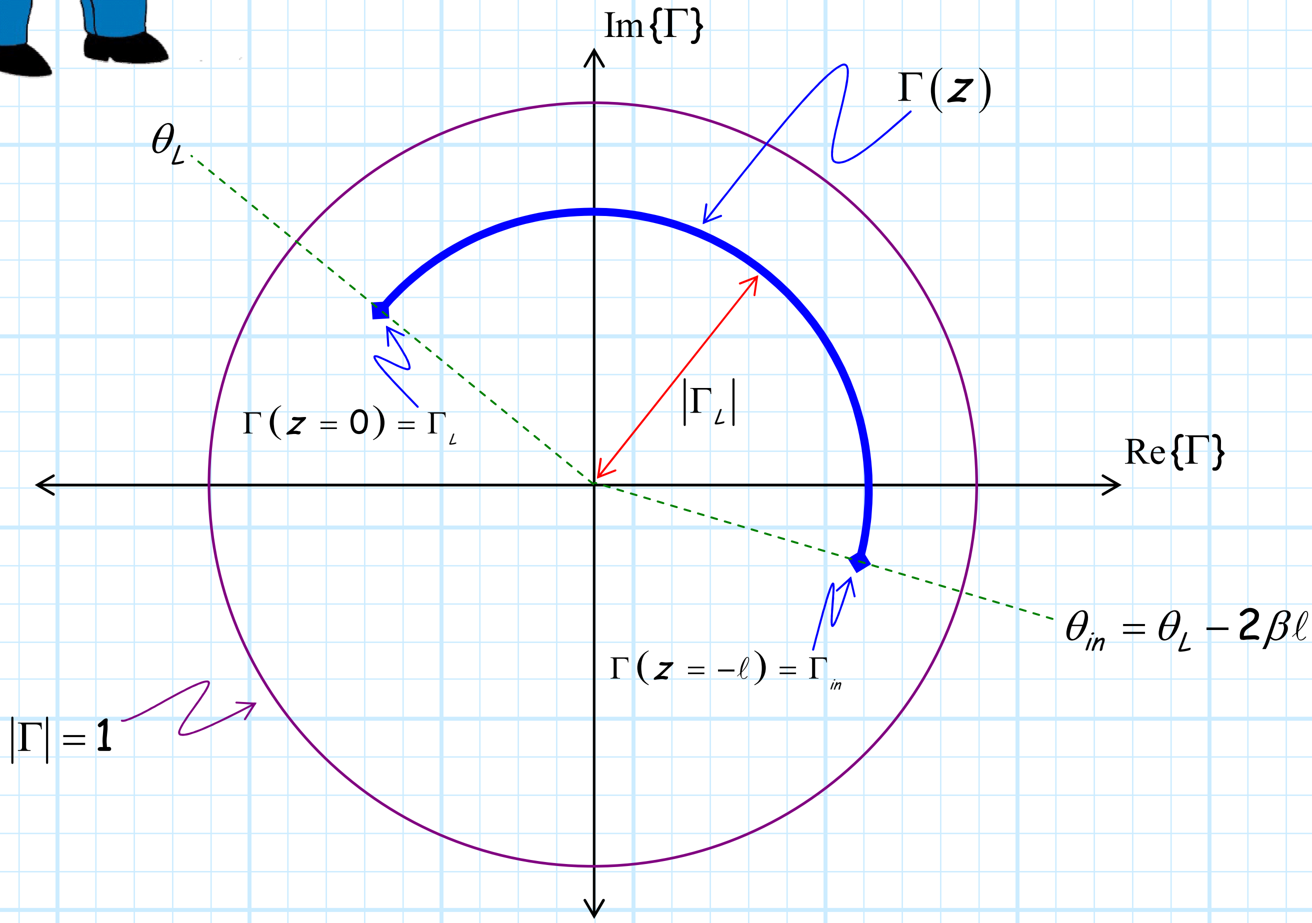
And thus, as we would **expect**:

$$\Gamma(z=0) = \Gamma_L \qquad \text{and} \qquad \Gamma(z=-\ell) = \Gamma_L e^{-j2\beta\ell} = \Gamma_{in}$$

Recall this result "says" that adding a transmission line of length $\ell$ to a load results in a **phase shift** in $\theta_\Gamma$ by $-2\beta\ell$ radians, while the **magnitude** $|\Gamma|$ remains **unchanged**.

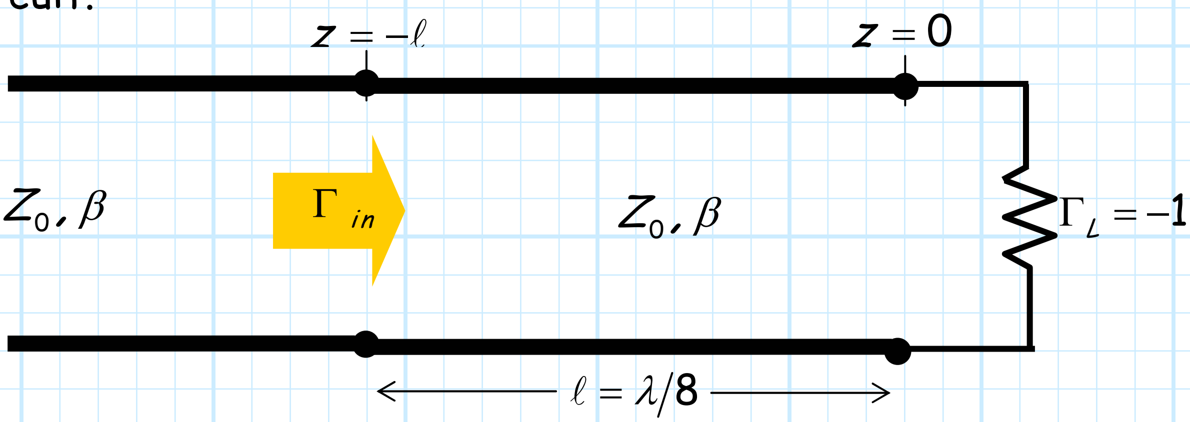
**Q:** *Magnitude $|\Gamma|$ and phase $\theta_\Gamma$ --aren't those the values used when **plotting** on the complex $\Gamma$ plane?*

**A: Precisely**! In fact, **plotting** the transformation of $\Gamma_L$ to $\Gamma_{in}$ along a transmission line length $\ell$ has an interesting **graphical** interpretation. Let's **parametrically** plot $\Gamma(z)$ from $z = z_L$ (i.e., $z = 0$) to $z = z_L - \ell$ (i.e., $z = -\ell$):

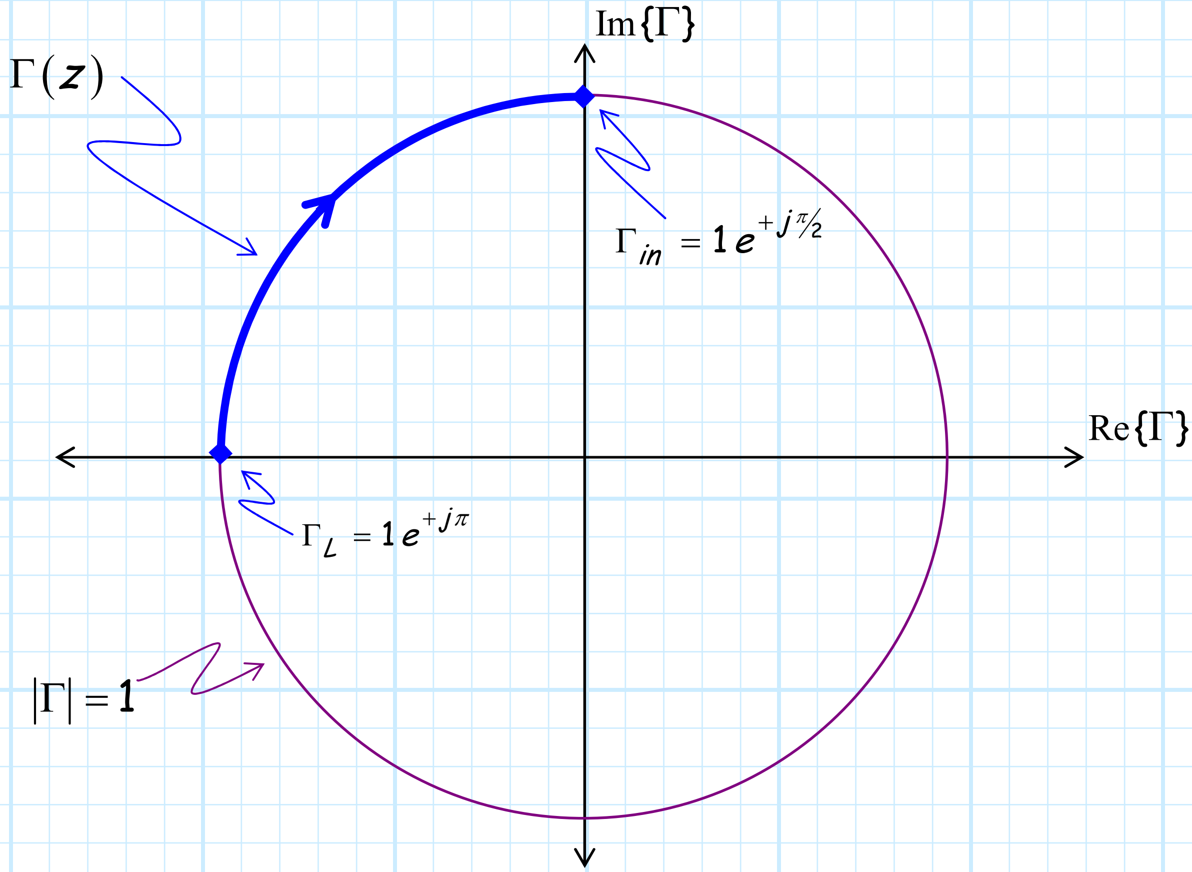



Since adding a length of transmission line to a load $\Gamma_L$ **modifies** the **phase** $\theta_\Gamma$ but **not** the **magnitude** $|\Gamma_L|$, we trace a **circular arc** as we parametrically plot $\Gamma(z)$! This arc has a **radius** $|\Gamma_L|$ and an **arc angle** $2\beta\ell$ radians.

With this knowledge, we can **easily** solve many interesting transmission line problems **graphically**—using the complex $\Gamma$ plane! For **example**, say we wish to determine $\Gamma_{in}$ for a transmission line length $\ell = \lambda/8$ and terminated with a **short** circuit.

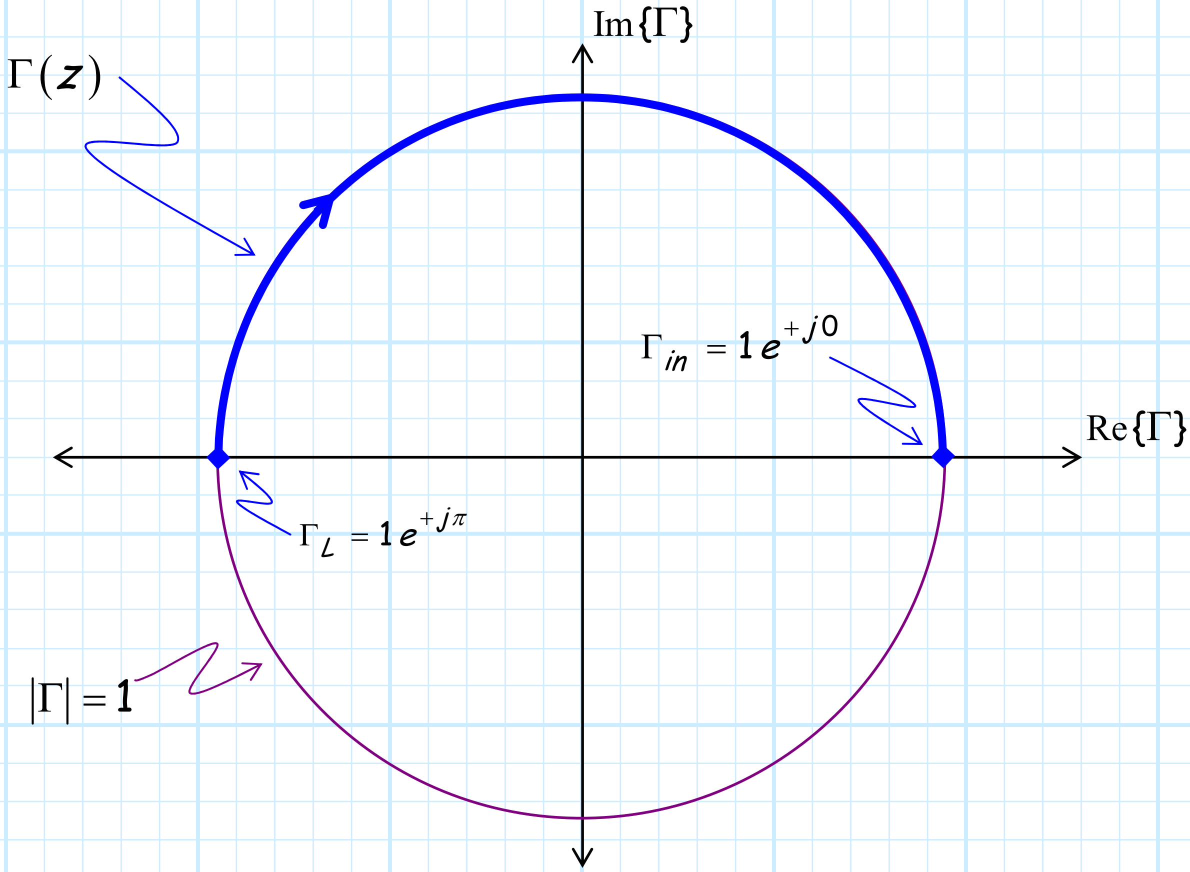



The reflection coefficient of a **short** circuit is $\Gamma_L = -1 = 1\, e^{j\pi}$, and therefore we **begin** at that point on the complex $\Gamma$ plane. We then move along a **circular arc** $-2\beta\ell = -2(\pi/4) = -\pi/2$ radians (i.e., rotate **clockwise** $90^\circ$).



When we **stop**, we find we are at the point for $\Gamma_{in}$; in this case $\Gamma_{in} = 1e^{j\pi/2}$ (i.e., magnitude is **one**, phase is **90°**).

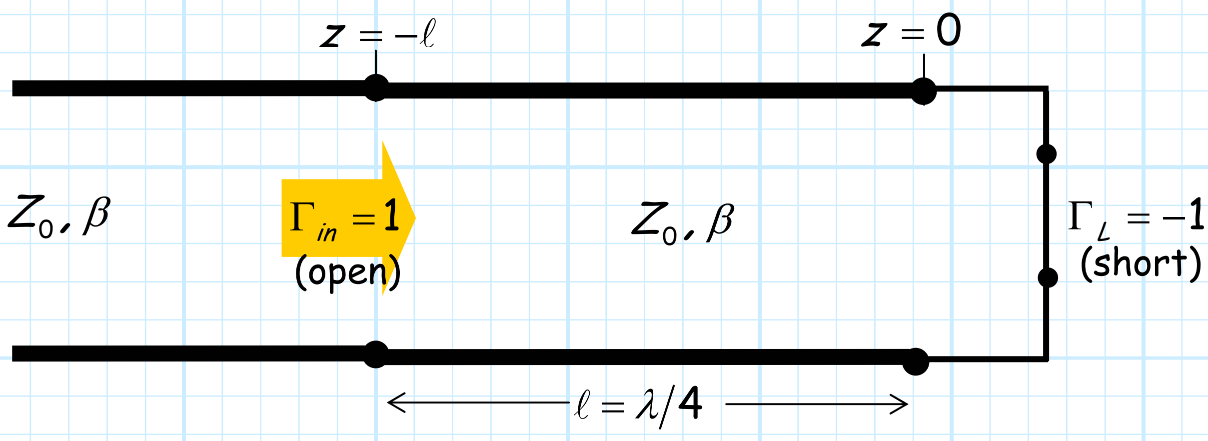
Now, let's **repeat** this same problem, only with a **new** transmission line **length** of $\ell = \lambda/4$. Now we rotate **clockwise** $2\beta\ell = \pi$ radians (180°).




For this case, the **input** reflection coefficient is $\Gamma_{in} = 1e^{j0} = 1$ : the reflection coefficient of an **open circuit**!

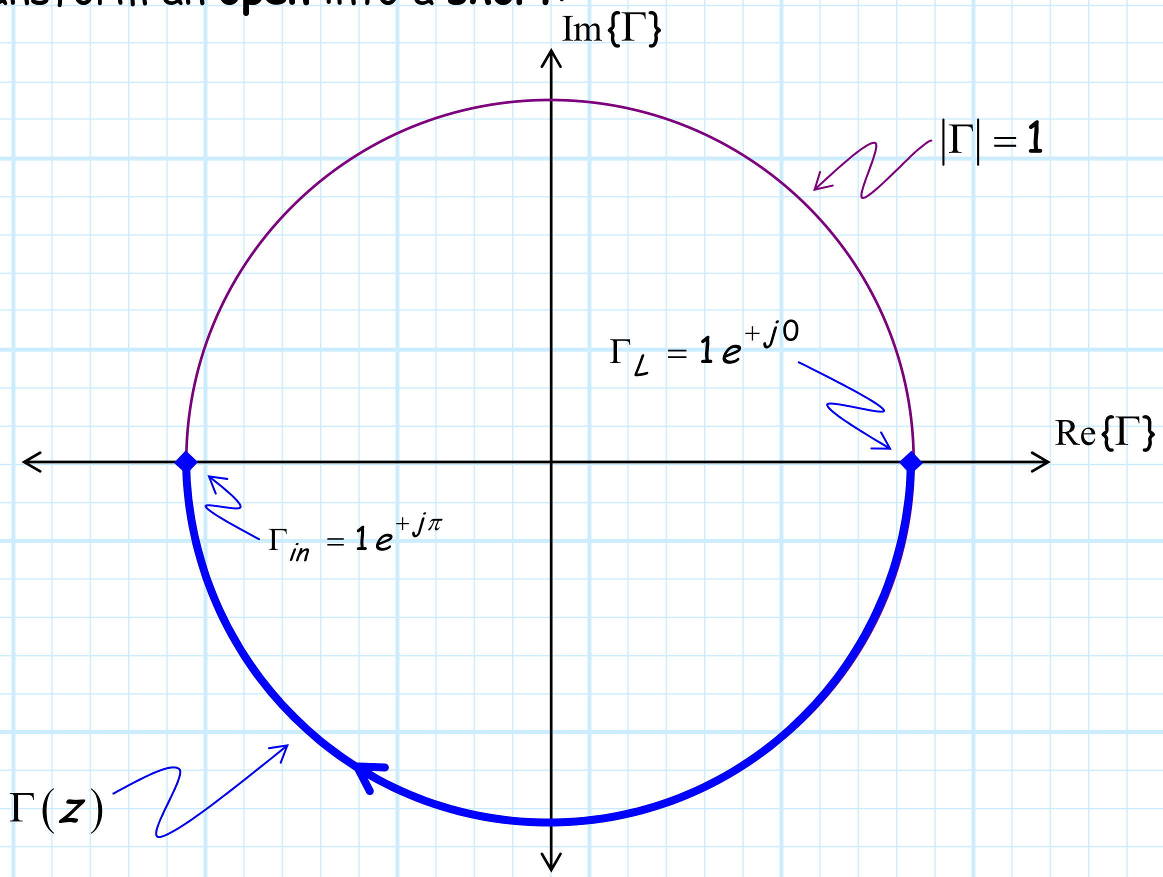
Our **short**-circuit load has been transformed into an **open** circuit with a **quarter-wavelength** transmission line!

But, you **knew** this would happen—**right**?

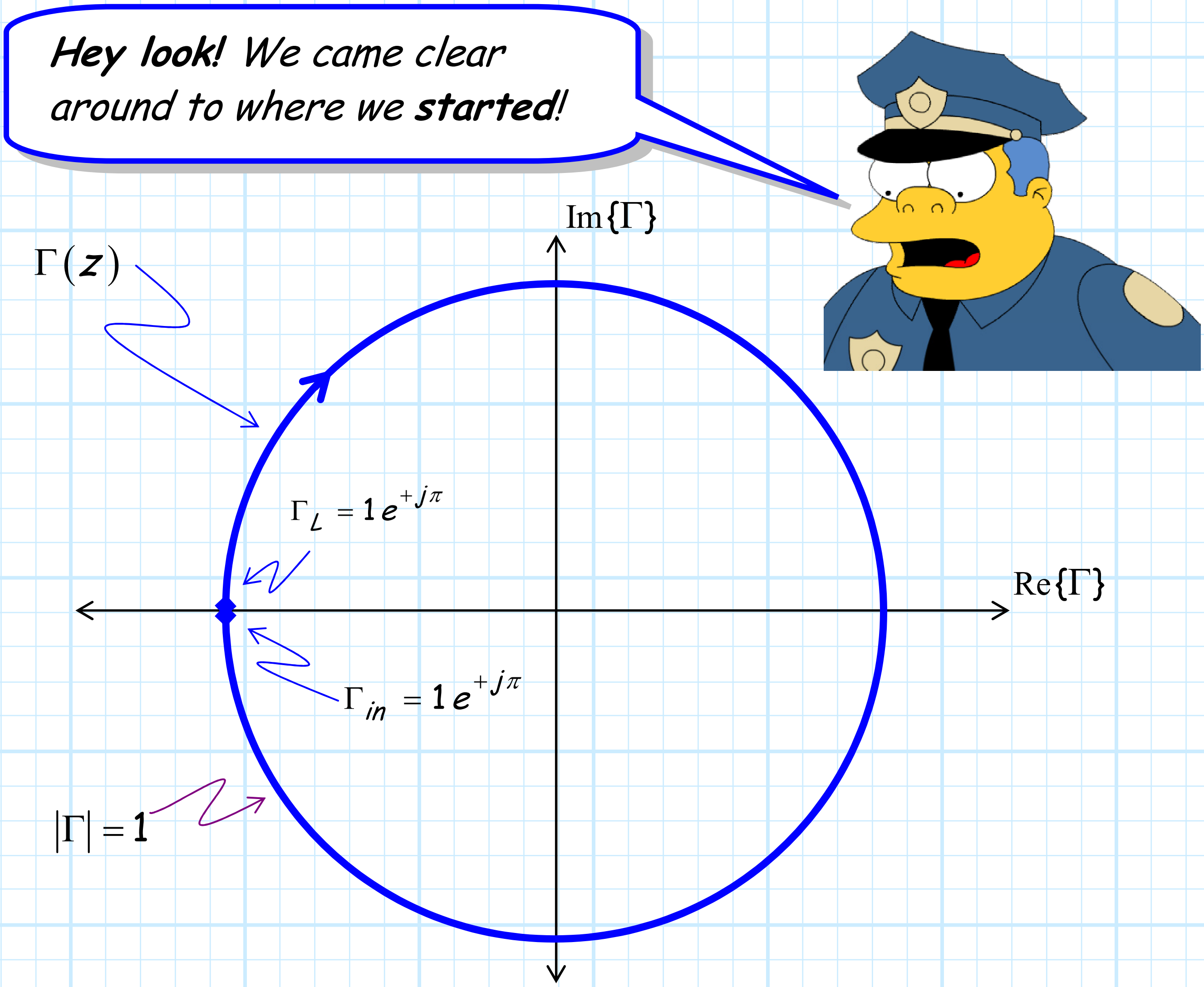



Recall that a **quarter**-wave transmission line was one of the **special cases** we considered earlier. Recall we found that the input impedance was proportional to the **inverse** of the load impedance.

Thus, a **quarter-wave** transmission line transforms a **short** into an **open**. Conversely, a quarter-wave transmission can also transform an **open** into a **short**:



Finally, let's **again** consider the problem where $\Gamma_L = -1$ (i.e., short), only this time with a transmission line length $\ell = \lambda/2$ ( a **half** wavelength!). We rotate **clockwise** $2\beta\ell = 2\pi$ radians ($360^\circ$).




Thus, we find that $\Gamma_{in} = \Gamma_L$ **if** $\ell = \lambda/2$--but you knew **this** too!

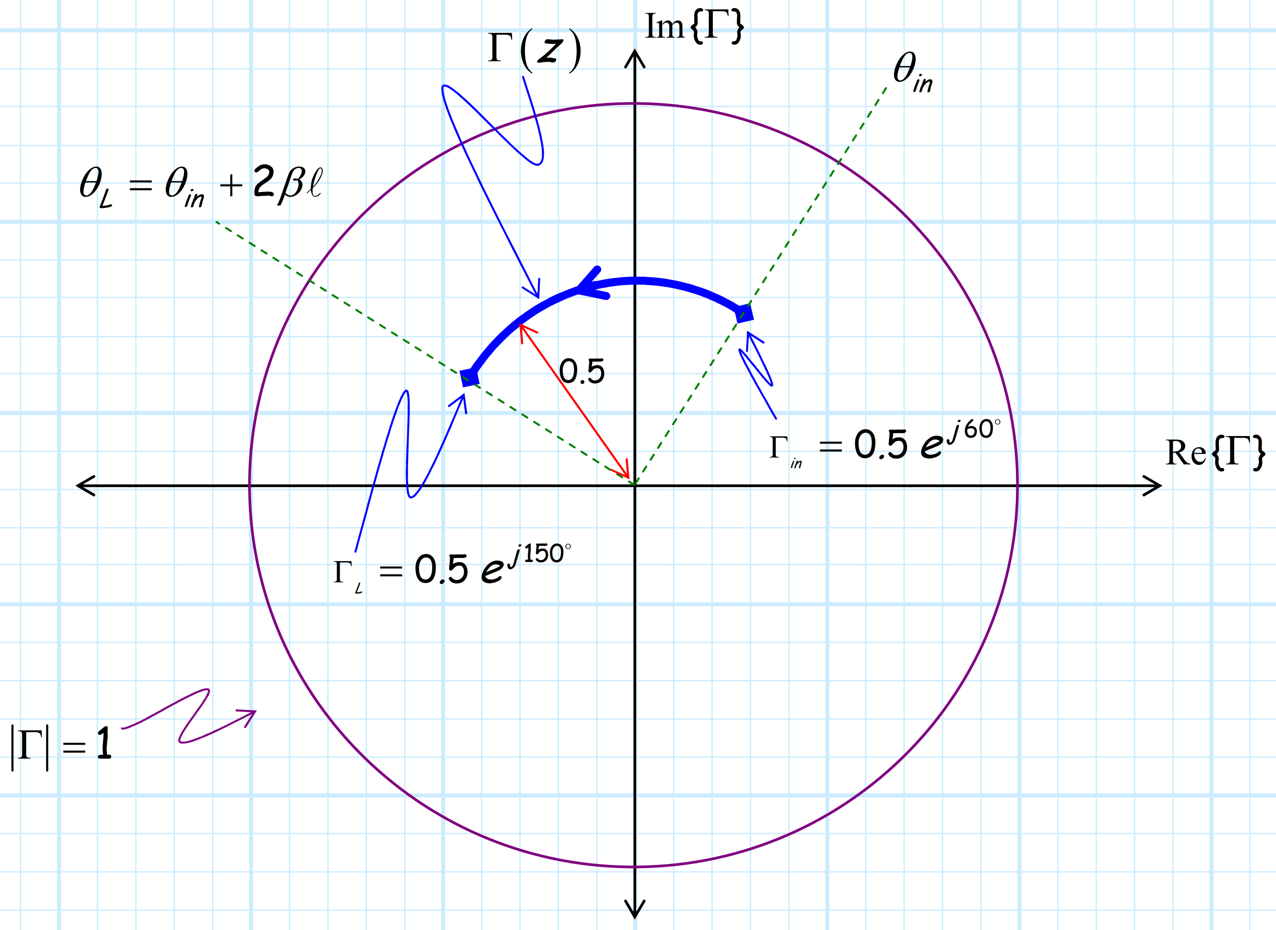
Recall that the **half**-wavelength transmission line is likewise a **special case**, where we found that $Z_{in} = Z_L$. This result, of course, likewise means that $\Gamma_{in} = \Gamma_L$.

Now, let's consider the **opposite** problem. Say we know that the **input** impedance at the **beginning** of a transmission line with length $\ell = \lambda/8$ is:

$$\Gamma_{in} = 0.5\, e^{j60^\circ}$$

**Q:** *What is the reflection coefficient of the **load**?*

**A:** In this case, we begin at $\Gamma_{in}$ and rotate **COUNTER-CLOCKWISE** along a circular arc (radius 0.5) $2\beta\ell = \pi/2$ radians (i.e., $60^\circ$). Essentially, we are **removing** the phase shift associated with the transmission line!




The reflection coefficient of the **load** is therefore:

$$\Gamma_L = 0.5\, e^{j150^\circ}$$