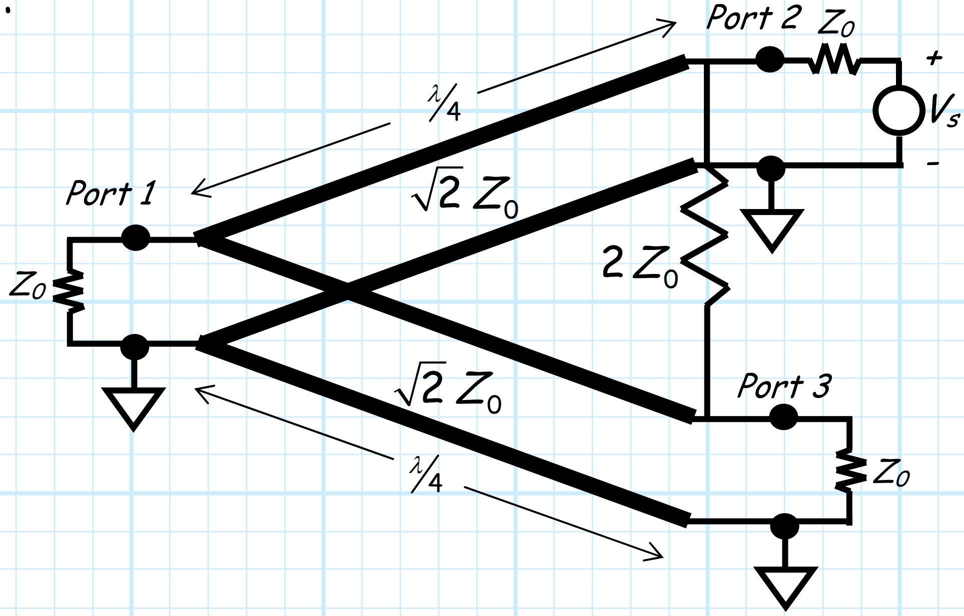
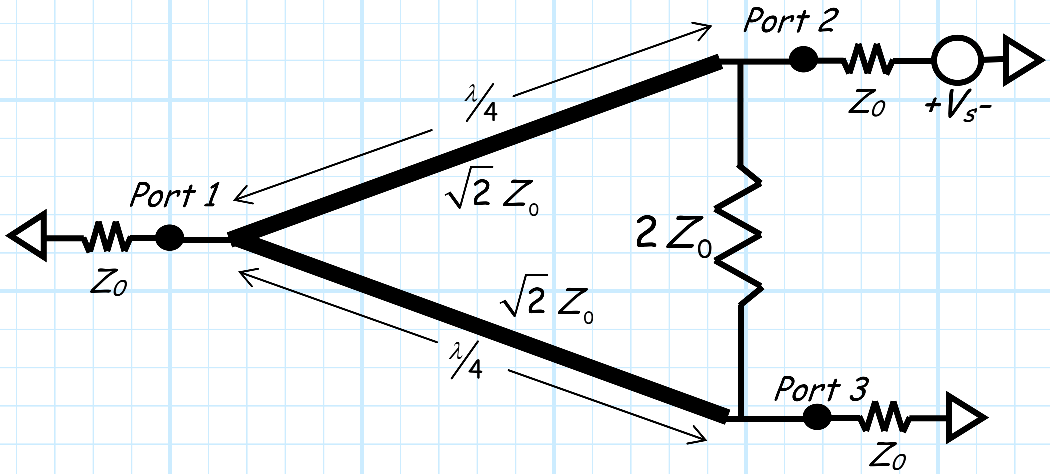
Even/Odd Mode Analysis of the Wilkinson Divider

Consider a matched **Wilkinson power divider**, with a **source at port 2**:



Too **simplify** this schematic, we **remove** the ground plane, which includes the **bottom conductor** of the transmission lines:



Q: *How do we **analyze** this circuit ?*

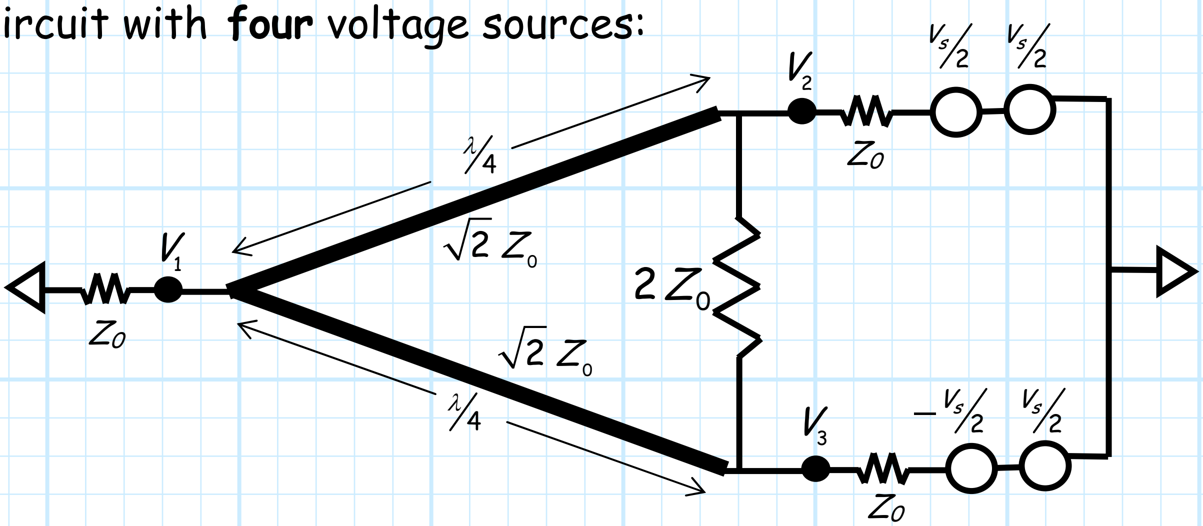
A: Use **Even-Odd mode** analysis!

Remember, even-odd mode analysis uses **two** important principles:

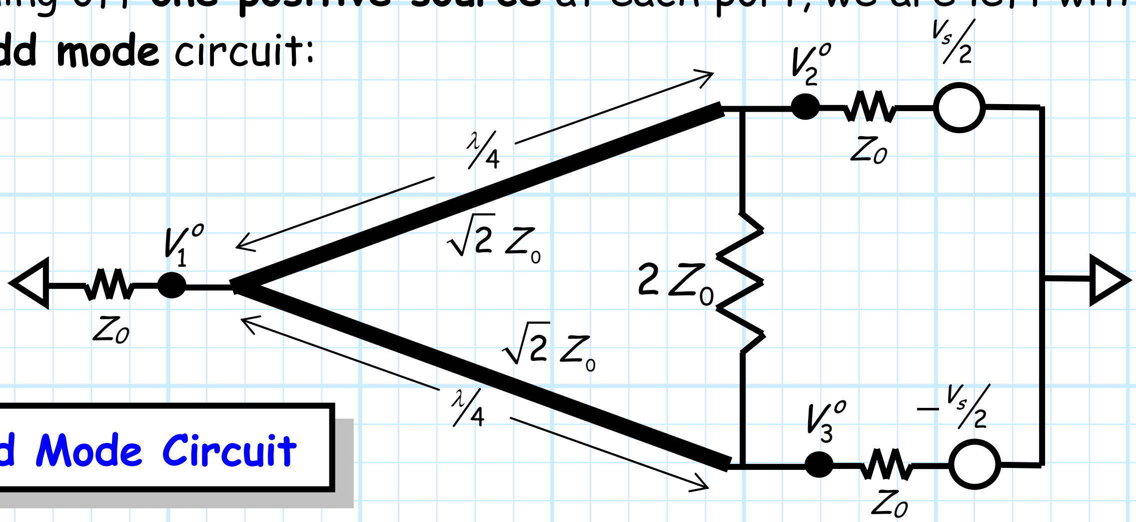
a) superposition

b) circuit **symmetry**

To see how we apply these principles, let's first rewrite the circuit with **four** voltage sources:

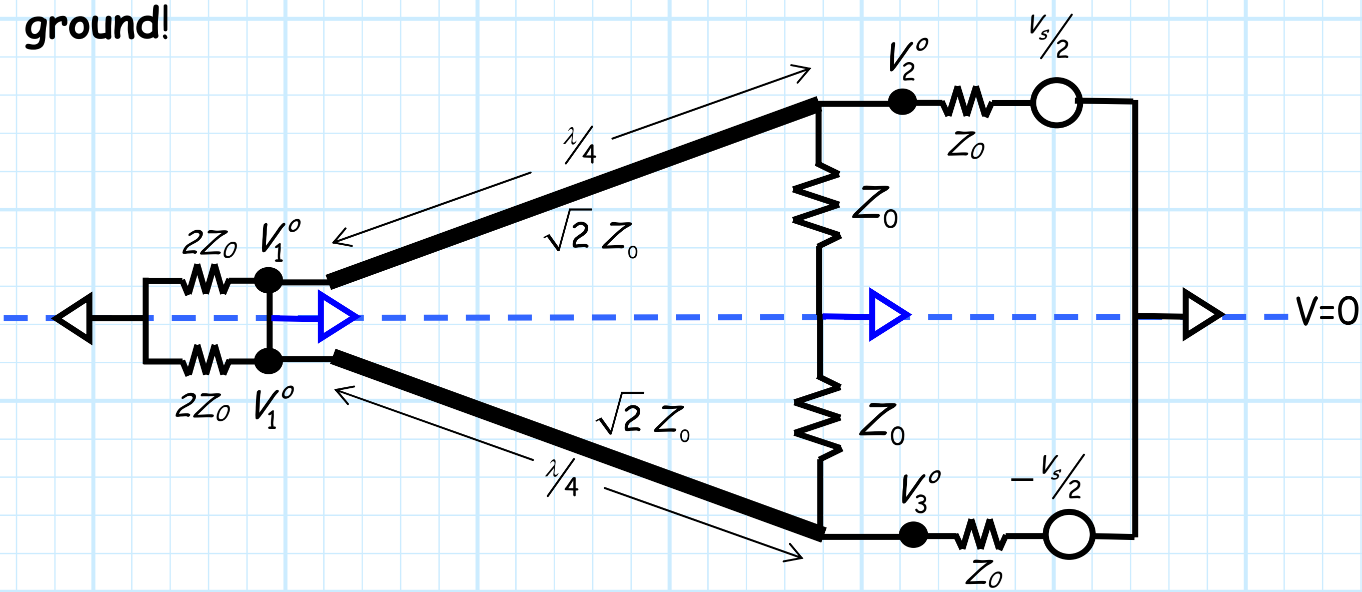


Turning off **one positive source** at each port, we are left with an **odd mode** circuit:

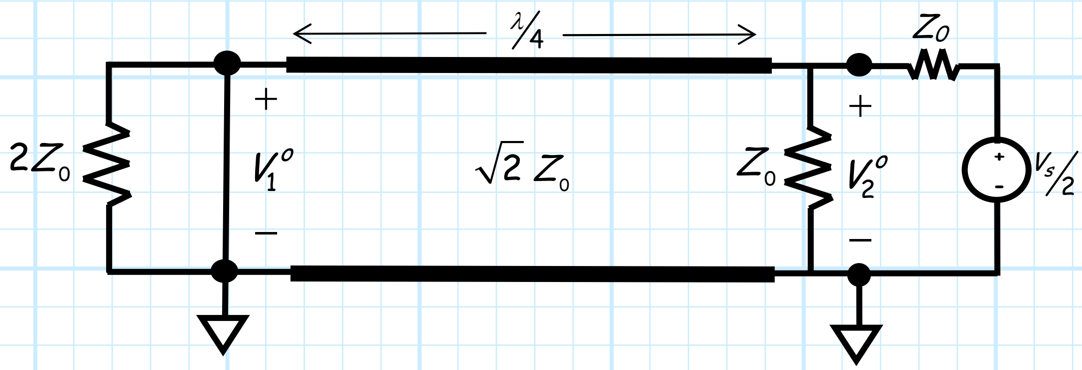


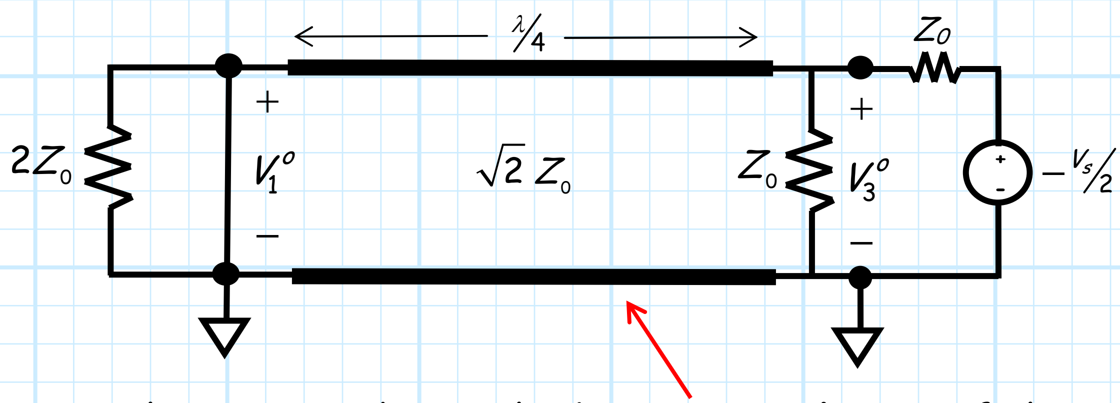
Odd Mode Circuit

Note the circuit has **odd symmetry**, and thus the plane of symmetry becomes a **virtual short**, and in this case, a virtual **ground**!



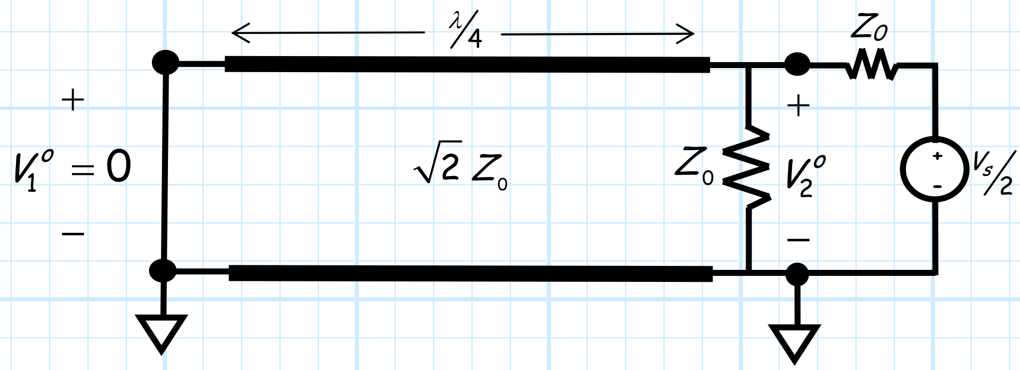
Dividing the circuit into two **half-circuits**, we get:





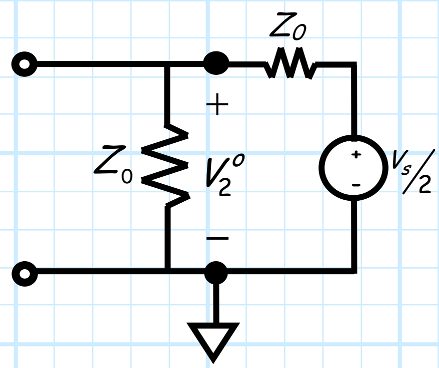
Note we have again drawn the **bottom conductor** of the transmission line (a **ground plane**) to enhance **clarity** (I **hope**!).

Analyzing the top circuit, we find that the transmission line is terminated in a **short** circuit in **parallel** with a **resistor** of value $2Z_0$. Thus, the transmission line is terminated in a **short circuit**!



This of course makes determining V_1^o **trivial** (**hint:** $V_1^o = 0$).

Now, since the transmission line is a **quarter wavelength**, this **short** circuit at the **end** of the transmission line transforms to an **open** circuit at the **beginning**!



As a result, determining voltage V_2^o is nearly as **trivial** as determining voltage V_1^o. **Hint:**

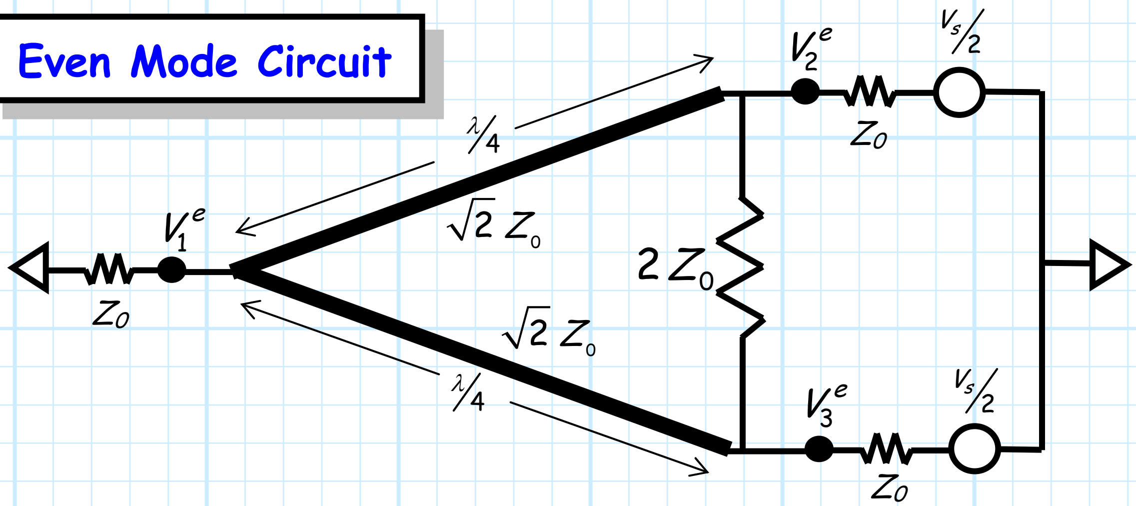
$$V_2^o = \frac{V_s}{2}\frac{Z_0}{Z_0 + Z_0} = \frac{V_s}{4}$$

And from the **odd symmetry** of the circuit, we likewise know:

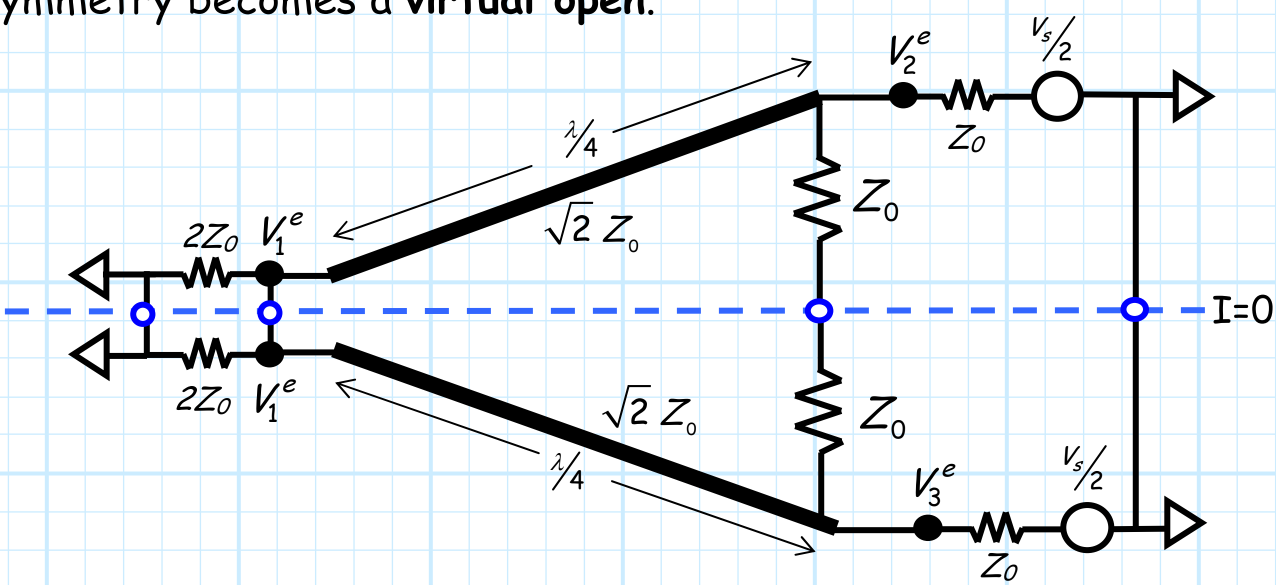
$$V_3^o = -V_2^o = -\frac{V_s}{4}$$

Now, let's turn **off** the **odd mode sources**, and turn back **on** the **even mode sources**.

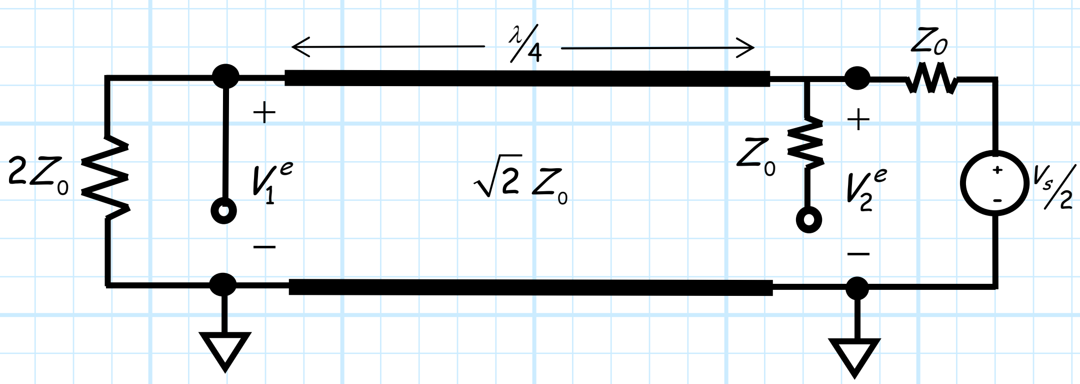
Even Mode Circuit

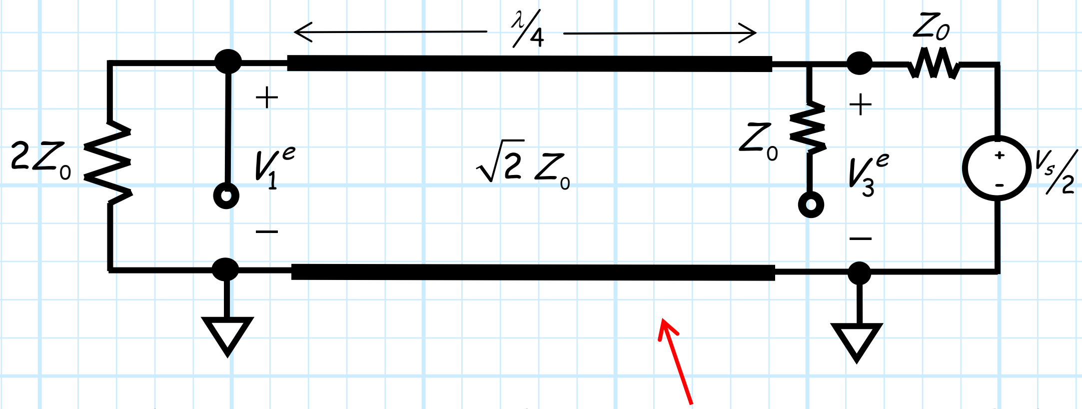


Note the circuit has **even symmetry**, and thus the plane of symmetry becomes a **virtual open**.



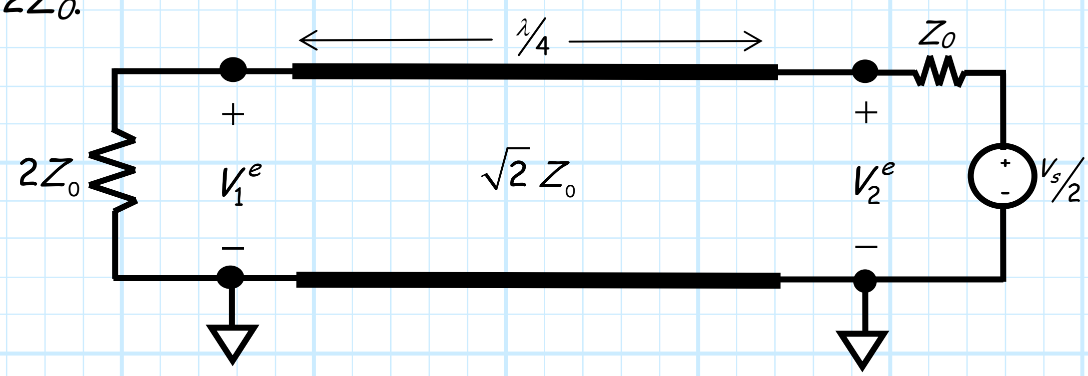
Dividing the circuit into two **half-circuits**, we get:





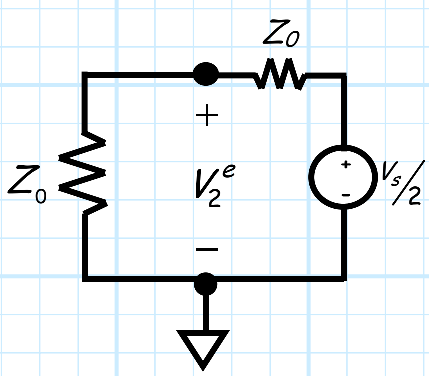
Note we have **again** drawn the **bottom conductor** of the transmission line (a **ground plane**).

Analyzing the top circuit, we find that the transmission line is terminated in a **open** circuit in **parallel** with a **resistor** of value $2Z_0$. Thus, the transmission line is terminated in a **resistor** valued $2Z_0$.



Now, since the transmission line is a **quarter wavelength**, the $2Z_0$ resistor at the **end** of the transmission line transforms to **this** value at the **beginning**:

$$Z_{in} = \frac{\left(\sqrt{2}Z_0\right)^2}{2Z_0} = Z_0$$



Voltage V_2^e can again be determined by **voltage division**:

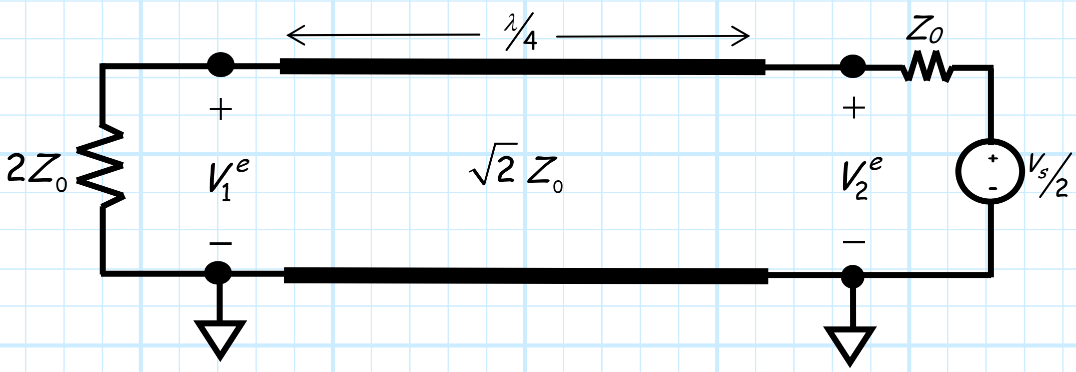
$$V_2^e = \frac{V_s}{2}\frac{Z_0}{Z_0 + Z_0} = \frac{V_s}{4}$$

And then due to the **even symmetry** of the circuit, we know:

$$V_3^e = V_2^e = \frac{V_s}{4}$$

Q: *What about voltage V_1^e? What is its value?*

A: Well, there's **no** direct or easy way to find this value. We must apply our transmission line theory (i.e., the solution to the **telegrapher's equations** + **boundary conditions**) to find this value. This means **applying** the knowledge and skills acquired during our **scholarly** examination of **Chapter 2**!



If we **carefully** and **patiently** analyze the above transmission line circuit, we find that (see if **you** can verify this!):

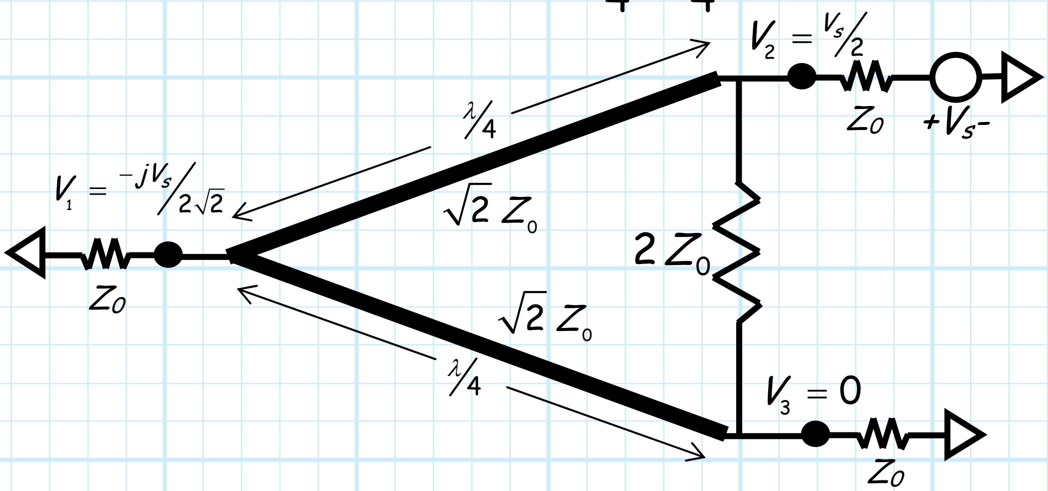
$$V_1^e = \frac{-j\,V_s}{2\sqrt{2}}$$

And thus, completing our **superposition** analysis, the voltages and currents within the circuit is simply found from the **sum** of the solutions of each mode:

$$V_1 = V_1^o + V_1^o = 0 - \frac{jV_s}{2\sqrt{2}} = -\frac{jV_s}{2\sqrt{2}}$$

$$V_2 = V_2^o + V_2^o = \frac{V_s}{4} + \frac{V_s}{4} = \frac{V_s}{2}$$

$$V_3 = V_3^o + V_3^o = -\frac{V_s}{4} + \frac{V_s}{4} = 0$$



Note that the voltages we calculated are **total voltages**—the **sum** of the **incident** and **exiting** waves at each port:

$$V_1 \doteq V_1\left(z_1 = z_{1P}\right) = V_1^+\left(z_1 = z_{1P}\right) + V_1^-\left(z_1 = z_{1P}\right)$$

$$V_2 \doteq V_2\left(z_2 = z_{2P}\right) = V_2^+\left(z_2 = z_{2P}\right) + V_2^-\left(z_2 = z_{2P}\right)$$

$$V_3 \doteq V_3\left(z_3 = z_{3P}\right) = V_3^+\left(z_3 = z_{3P}\right) + V_3^-\left(z_3 = z_{3P}\right)$$

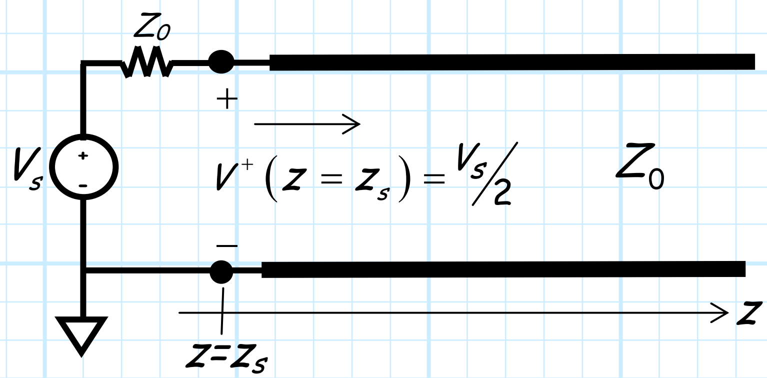
Since ports 1 and 3 are terminated in **matched loads**, we know that the **incident** wave on those ports are **zero**. As a result, the **total** voltage is equal to the value of the exiting waves at those ports:

$$V_1^+(z_1 = z_{1P}) = 0 \qquad V_1^-(z_1 = z_{1P}) = \frac{-jV_s}{2\sqrt{2}}$$

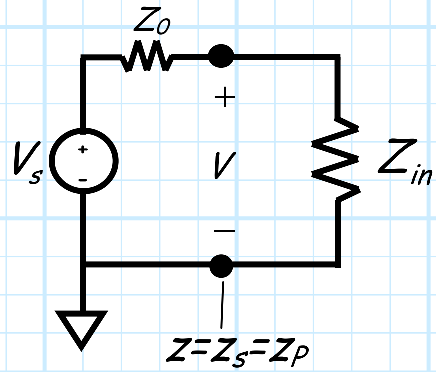
$$V_3^+(z_3 = z_{3P}) = 0 \qquad V_3^-(z_3 = z_{3P}) = 0$$

The problem now is to determine the values of the **incident** and **exiting** waves at port 2 (i.e., $V_2^+(z_2 = z_{2P})$ and $V_2^-(z_2 = z_{2P})$).

Recall however, the specific case where the **source impedance** is **matched** to transmission line characteristic impedance (i.e., $Z_s = Z_0$). We found for this specific case, the incident wave "launched" by the source **always** has the value $V_s/2$ at the source:



Now, if the length of the transmission line connecting a source to a port (or load) is **electrically very small** (i.e., $\beta\ell \ll 1$), then the source is effectively **connected directly** to the source (i.e, $\beta z_s = \beta z_P$):



And thus the **total** voltage is:

$$\begin{aligned} V &= V^+(z = z_P) + V^-(z = z_P) \\ &= V^+(z = z_S) + V^-(z = z_P) \\ &= V_s/2 + V^-(z = z_P) \end{aligned}$$

For the case where a **matched source** (i.e. $Z_s = Z_0$) is connected directly to a port, we can thus conclude:

$$V^{+}(z = z_P) = V_s/2$$

$$V^{-}(z = z_P) = V - V_s/2$$

Thus, for port 2 we find:

$$V_2^{+}(z_2 = z_{2P}) = V_s/2$$

$$V_2^{-}(z_2 = z_{2P}) = V_2 - V_s/2 = V_s/2 - V_s/2 = 0$$

Now, we can **finally** determine the scattering parameters S_{12}, S_{22}, S_{32}:

$$S_{12} = \frac{V_1^{-}(z_1 = z_{1P})}{V_2^{+}(z_2 = z_{2P})} = \left(\frac{-jV_s}{2\sqrt{2}}\right)\frac{2}{V_s} = \frac{-j}{\sqrt{2}}$$

$$S_{22} = \frac{V_2^{-}(z_2 = z_{2P})}{V_2^{+}(z_2 = z_{2P})} = (0)\frac{2}{V_s} = 0$$

$$S_{32} = \frac{V_3^{-}(z_3 = z_{3P})}{V_2^{+}(z_2 = z_{2P})} = (0)\frac{2}{V_s} = 0$$

Q: *Wow! That seemed like a **lot** of hard work, and we're only ⅓ of the way done. Do we **have** to move the source to port 1 and then port 3 and perform similar analyses?*

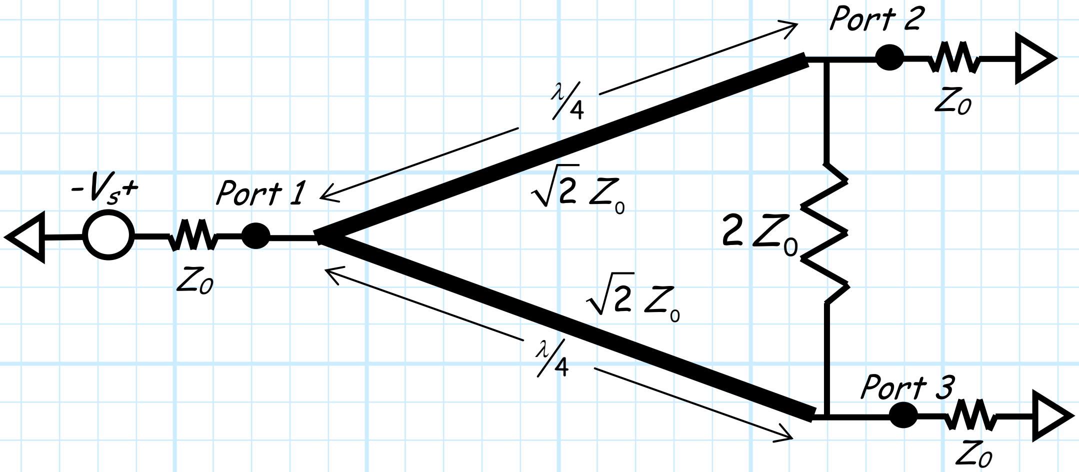
A: **Nope!** Using the bilateral **symmetry** of the circuit ($1 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 2$), we can conclude:

$$S_{13} = S_{12} = \frac{-j}{\sqrt{2}} \qquad S_{33} = S_{22} = 0 \qquad S_{23} = S_{32} = 0$$

and from **reciprocity**:

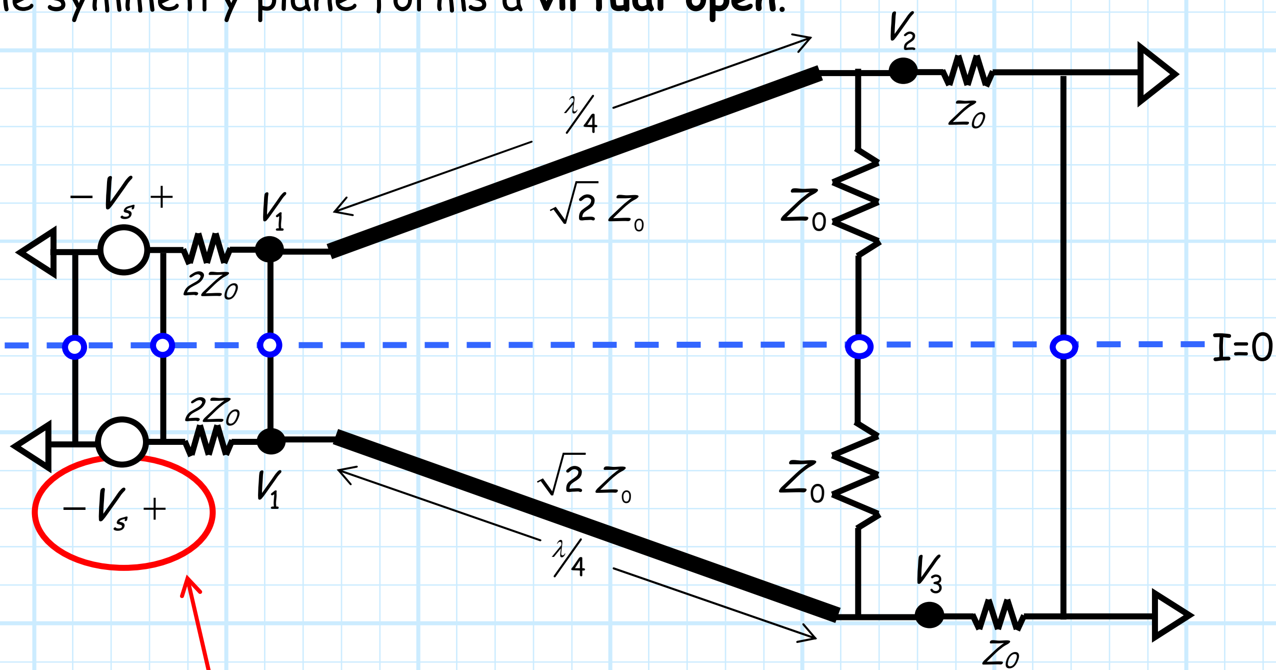
$$S_{21} = S_{12} = \frac{-j}{\sqrt{2}} \qquad S_{31} = S_{13} = \frac{-j}{\sqrt{2}}$$

We thus have determined **8** of the **9** scattering parameters needed to characterize this 3-port device. The **remaining** holdout is the scattering parameter S_{11}. To find this value, we must move the **source to port 1** and analyze.

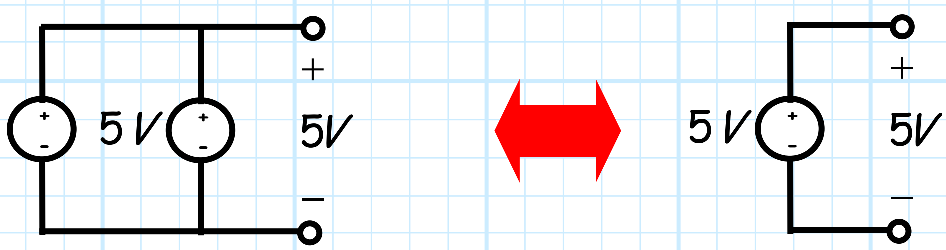


Note this source does **not** alter the bilateral symmetry of the circuit. We can thus use this symmetry to **help analyze** the circuit, **without** having to specifically define odd and even mode sources.

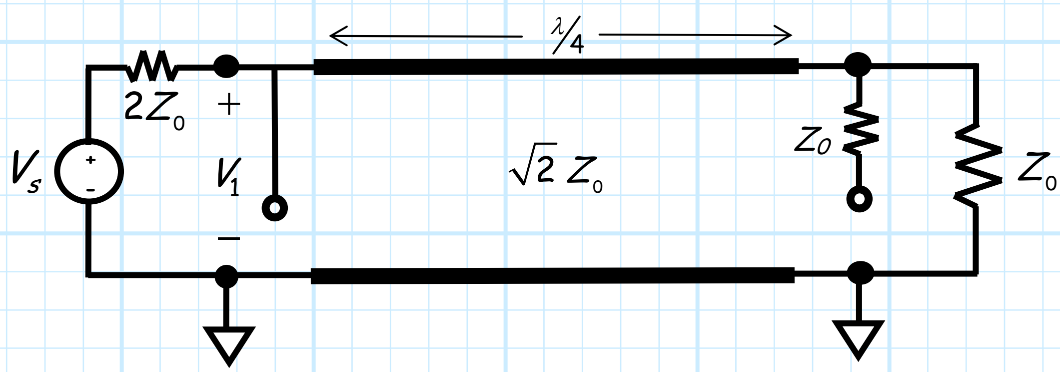
Since the circuit has (even) bilateral symmetry, we know that the symmetry plane forms a **virtual open**.



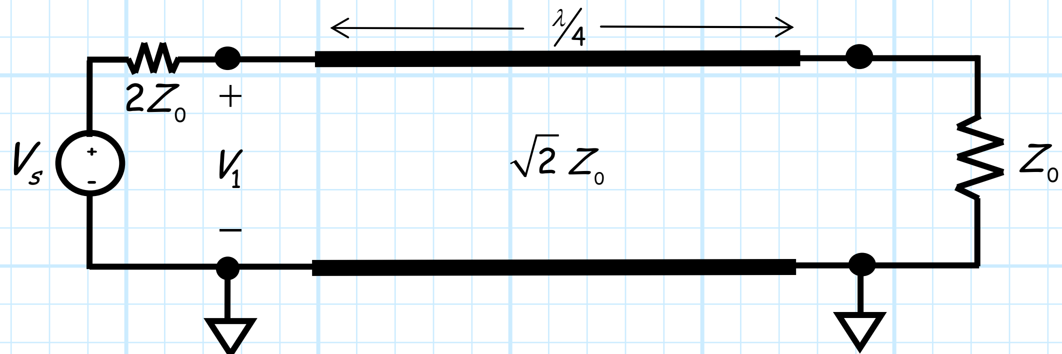
Note the **value** of the voltage sources. They have a value of V_s (as **opposed** to, say, $2\,V_s$ or $V_s/2$) because two equal voltage sources in **parallel** is equivalent to one voltage source of the **same value**. E.G.:



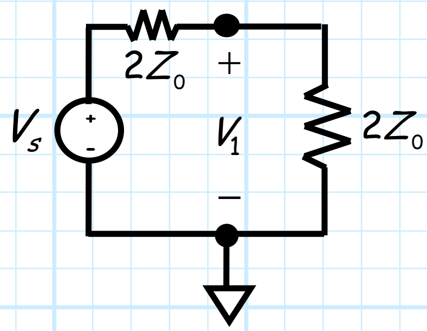
Now **splitting** the circuit into **two** half-circuits, we find the **top** half-circuit to be:



Which **simplifies** to:



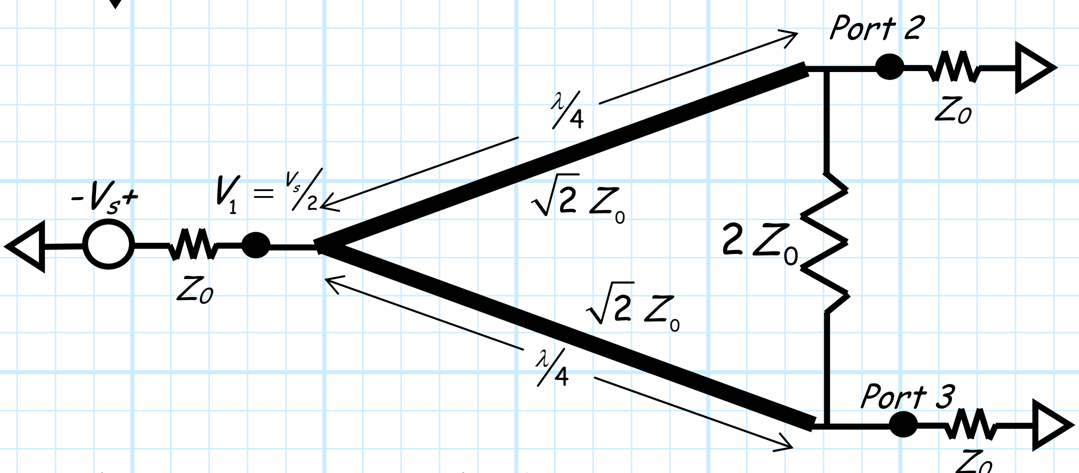
And **transforming** the load resistor at the end of the λ/4 wave line back to its **beginning**:



Finally, we use **voltage division** to determine that:

$$V_1 = V_s \left(\frac{2Z_0}{2Z_0 + 2Z_0} \right) = \frac{V_s}{2}$$

Thus,



And since the **source is matched**:

$$V_1^+ \left(z_1 = z_{1P} \right) = V_s/2$$

$$V_1^- \left(z_1 = z_{1P} \right) = V_1 - V_s/2 = V_s/2 - V_s/2 = 0$$

So our **final** scattering element is revealed!

$$S_{11} = \frac{V_1^-(z_1 = z_{1P})}{V_1^+(z_1 = z_{1P})} = (0)\frac{2}{V_s} = 0$$

So the scattering matrix of a **Wilkinson power divider** has been **confirmed**:

$$\mathcal{S} = \begin{bmatrix} 0 & {}^{-j}\!/\!{}_{\sqrt{2}} & {}^{-j}\!/\!{}_{\sqrt{2}} \\ {}^{-j}\!/\!{}_{\sqrt{2}} & 0 & 0 \\ {}^{-j}\!/\!{}_{\sqrt{2}} & 0 & 0 \end{bmatrix}$$