Zin Calculations using the Smith Chart

The normalized input impedance $z_{in}'$ of a transmission line length $\ell$, when terminated in normalized load $z_L'$, can be determined as:

$$z_{in}' = \frac{Z_{in}}{Z_0} = \frac{1}{Z_0} \left( \frac{Z_L + j Z_0 \tan \beta \ell}{Z_0 + j Z_L \tan \beta \ell} \right)$$

$$= \frac{Z_L/Z_0 + j \tan \beta \ell}{1 + j Z_L/Z_0 \tan \beta \ell}$$

$$= \frac{z_L' + j \tan \beta \ell}{1 + j z_L' \tan \beta \ell}$$

Q: Evaluating this unattractive expression looks not the least bit pleasant. Isn’t there a less disagreeable method to determine $z_{in}'$?
A: Yes there is! Instead, we could determine this normalized input impedance by following these three steps:

1. Convert $z'_L$ to $\Gamma_L$, using the equation:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$$

$$= \frac{z'_L - 1}{z'_L + 1}$$

2. Convert $\Gamma_L$ to $\Gamma_{in}$, using the equation:

$$\Gamma_{in} = \Gamma_L e^{-j2\beta t}$$

3. Convert $\Gamma_{in}$ to $z'_{in}$, using the equation:

$$z'_{in} = Z_{in}/Z_0 = \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$$

Q: But performing these three calculations would be even more difficult than the single step you described earlier. What sort of dimwit would ever use (or recommend) this approach?
A: The benefit in this last approach is that each of the three steps can be executed using a Smith Chart—no complex calculations are required!

1. Convert $z'_L$ to $\Gamma_L$

Find the point $z'_L$ from the impedance mappings on your Smith Chart. Place you pencil at that point—you have now located the correct $\Gamma_L$ on your complex $\Gamma$ plane!

For example, say $z'_L = 0.6 - j1.4$. We find on the Smith Chart the circle for $r = 0.6$ and the circle for $x = -1.4$. The intersection of these two circles is the point on the complex $\Gamma$ plane corresponding to normalized impedance $z'_L = 0.6 - j1.4$.

This point is a distance of 0.685 units from the origin, and is located at angle of -65 degrees. Thus the value of $\Gamma_L$ is:

$$\Gamma_L = 0.685 e^{-j65^\circ}$$

2. Convert $\Gamma_L$ to $\Gamma_{in}$

Since we have correctly located the point $\Gamma_L$ on the complex $\Gamma$ plane, we merely need to rotate that point clockwise around a circle ($|\Gamma| = 0.685$) by an angle $2 \beta \ell$.

When we stop, we are located at the point on the complex $\Gamma$ plane where $\Gamma = \Gamma_{in}$!
For example, if the length of the transmission line terminated in $z'_{L} = 0.6 - j1.4$ is $\ell = 0.307\lambda$, we should rotate around the Smith Chart a total of $2\beta\ell = 1.228\pi$ radians, or 221°. We are now at the point on the complex $\Gamma$ plane:

$$\Gamma = 0.685 e^{+j74^\circ}$$

This is the value of $\Gamma_{in}$!

3. Convert $\Gamma_{in}$ to $z'_{in}$

When you get finished rotating, and your pencil is located at the point $\Gamma = \Gamma_{in}$, simply lift your pencil and determine the values $r$ and $x$ to which the point corresponds!

For example, we can determine directly from the Smith Chart that the point $\Gamma_{in} = 0.685 e^{+j74^\circ}$ is located at the intersection of circles $r = 0.5$ and $x = 1.2$. In other words:

$$z'_{in} = 0.5 + j1.2$$
Step 1

\[ |\Gamma| = 0.685 \]

\[ \Gamma_L = 0.685 e^{-j65^\circ} \]

\[ \theta_r = -65^\circ \]
Step 2

\[ \ell_2 = 0.147 \lambda \]

\[ \ell_1 = 0.16 \lambda \]

\[ \ell = \ell_1 + \ell_2 = 0.160 \lambda + 0.147 \lambda = 0.307 \lambda \]

\[ 2 \beta \ell = 221^\circ \]
Step 3

\[ z'_{\text{in}} = 0.5 + j1.2 \]