A Review of Complex Arithmetic

A complex value $C$ has both a real and imaginary component:

$$a = \text{Re}\{C\} \quad \text{and} \quad b = \text{Im}\{C\}$$

so that we can express this complex value as:

$$C = a + jb$$

where $j^2 = -1$.

Just as a real value can be expressed as a point on the real line, a complex value can be expressed as a point on the complex plane:
The values \((a,b)\) are a **Cartesian** representation of a point on the complex plane. Recall that we can **alternatively** denote a point on a 2-dimensional plane using **polar** coordinates:

- \(|C| = \text{distance from the origin to the point}
- \angle C = \phi_c = \text{rotation angle from the horizontal (Re\{C\}) axis}

i.e.,

Using our knowledge of **trigonometry**, we can determine the relationship between the Cartesian \((a,b)\) and polar \((|C|, \phi_c)\) representations.

From the **Pythagorean** theorem, we find that:

\[ |C| = \sqrt{a^2 + b^2} \]
Likewise, from the definition of *sine* (opposite over hypotenuse), we find:

\[
\sin \phi_c = \frac{b}{|C|} = \frac{b}{\sqrt{a^2 + b^2}}
\]

or, using the definition of *cosine* (adjacent over hypotenuse):

\[
\cos \phi_c = \frac{a}{|C|} = \frac{a}{\sqrt{a^2 + b^2}}
\]

Combining these results, we can determine the *tangent* (opposite over adjacent) of \( \phi_c \):

\[
\tan \phi_c = \frac{\sin \phi_c}{\cos \phi_c} = \frac{b}{a}
\]

Thus, we can write the polar coordinates in terms of the Cartesian coordinates:

\[
|C| = \sqrt{a^2 + b^2}
\]

\[
\phi_c = \tan^{-1}\left(\frac{b}{a}\right) = \cos^{-1}\left(\frac{a}{\sqrt{a^2 + b^2}}\right) = \sin^{-1}\left(\frac{b}{\sqrt{a^2 + b^2}}\right)
\]
Likewise, we can use trigonometry to write the Cartesian coordinates in terms of the polar coordinates.

For example, we can use the definition of sine to determine $b$:

$$b = |C| \sin \phi_c$$

and the definition of cosine to determine $a$:

$$a = |C| \cos \phi_c$$

Summarizing:

$$a = |C| \cos \phi_c$$

$$b = |C| \sin \phi_c$$

Note that we can explicitly write the complex value $C$ in terms of its magnitude $|C|$ and phase angle $\phi_c$:

$$C = a + jb = |C| \cos \phi_c + j |C| \sin \phi_c = |C| (\cos \phi_c + j \sin \phi_c)$$

*Hey! we can use Euler's equation to simplify this further!*
Recall that Euler’s equation states:

\[ e^{j\phi} = \cos \phi + j \sin \phi \]

so complex value \( C \) is:

\[
C = a + jb \\
= |C|(\cos \phi_c + j \sin \phi_c) \\
= |C|e^{j\phi_c}
\]

Now we have two ways of expressing a complex value \( C \):

\[
C = a + jb \quad \text{and/or} \quad C = |C|e^{j\phi_c}
\]

Note that both representations are equally valid mathematically—either one can be successfully used in complex analysis and computation.

Typically, we find that the Cartesian representation is easiest to use if we are doing arithmetic calculations (e.g., addition and subtraction).
For example, if:

\[ C_1 = a_1 + j b_1 \quad \text{and} \quad C_2 = a_2 + j b_2 \]

then:

\[ C_1 + C_2 = (a_1 + a_2) + j (b_1 + b_2) \]

\[ C_1 - C_2 = (a_1 - a_2) + j (b_1 - b_2) \]

Conversely, for geometric calculations (multiplication and division), it is easier to use the polar representation:

For example, if:

\[ C_1 = |C_1| e^{j \phi_1} \quad \text{and} \quad C_2 = |C_2| e^{j \phi_2} \]

then:

\[ C_1 C_2 = |C_1| e^{j \phi_1} \cdot |C_2| e^{j \phi_2} = |C_1| |C_2| e^{j(\phi_1 + \phi_2)} \]

and:

\[ \frac{C_1}{C_2} = \frac{|C_1| e^{j \phi_1}}{|C_2| e^{j \phi_2}} = \frac{|C_1| e^{j \phi_1} e^{-j \phi_2}}{|C_2|} = \frac{|C_1|}{|C_2|} e^{j(\phi_1 - \phi_2)} \]
Note in the above calculations we have used the **general facts**:

\[ x^y x^z = x^{(y+z)} \quad \text{and} \quad \frac{x^y}{x^z} = x^{(y-z)} \]

Additionally, we note that **powers** and **roots** are most easily accomplished using the **polar form** of \( C \):

\[ C^n = (|C| e^{j\phi_c})^n \]
\[ = |C|^n (e^{j\phi_c})^n \]
\[ = |C|^n e^{jn\phi_c} \]

and

\[ \sqrt[n]{C} = (|C| e^{j\phi_c})^{1/n} \]
\[ = |C|^{1/n} (e^{j\phi_c})^{1/n} \]
\[ = |C|^{1/n} e^{j(\phi_c/n)} \]

Therefore:

\[ C^2 = (|C| e^{j\phi_c})^2 = |C|^2 e^{j(2\phi_c)} \]

and:

\[ \sqrt{C} = (|C| e^{j\phi_c})^{1/2} = \sqrt{|C|} e^{j(\phi_c/2)} \]
Finally, we define the complex conjugate ($C^*$) of a complex value $C$:

$$C^* = \text{Complex Conjugate of } C$$

$$= a - jb$$

$$= |C|e^{-j\phi_c}$$

A very important application of the complex conjugate is for determining the magnitude of a complex value:

$$|C|^2 = C^* C$$

Typically, the proof of this relationship is given as:

$$C C^* = (a + jb)(a - jb)$$

$$= a(a - jb) + jb(a - jb)$$

$$= a^2 + jba - jba - j^2b^2$$

$$= a^2 + b^2$$

$$= |C|^2$$

However, it is more easily shown as:
Another important relationship involving complex conjugate is:

\[
C C^* = (|C| e^{i\phi_c})(|C| e^{-j\phi_c}) = |C|^2 e^{j(\phi_c + \phi_c)} = |C|^2 e^{j0}
\]

Thus, the sum of a complex value and its complex conjugate is a purely real value.

Additionally, the difference of complex value and its complex conjugate results in a purely imaginary value:

\[
C - C^* = (a + jb) - (a - jb) = (a - a) + j(b + b) = j2b
\]

Note from these results we can derive the relationships:

\[
a = \text{Re}\{C\} = \frac{C + C^*}{2}
\]
\[
b = \text{Im}\{C\} = \frac{C - C^*}{j2}
\]