A Review of Complex Arithmetic

A **complex** value C has both a **real** and **imaginary** component:

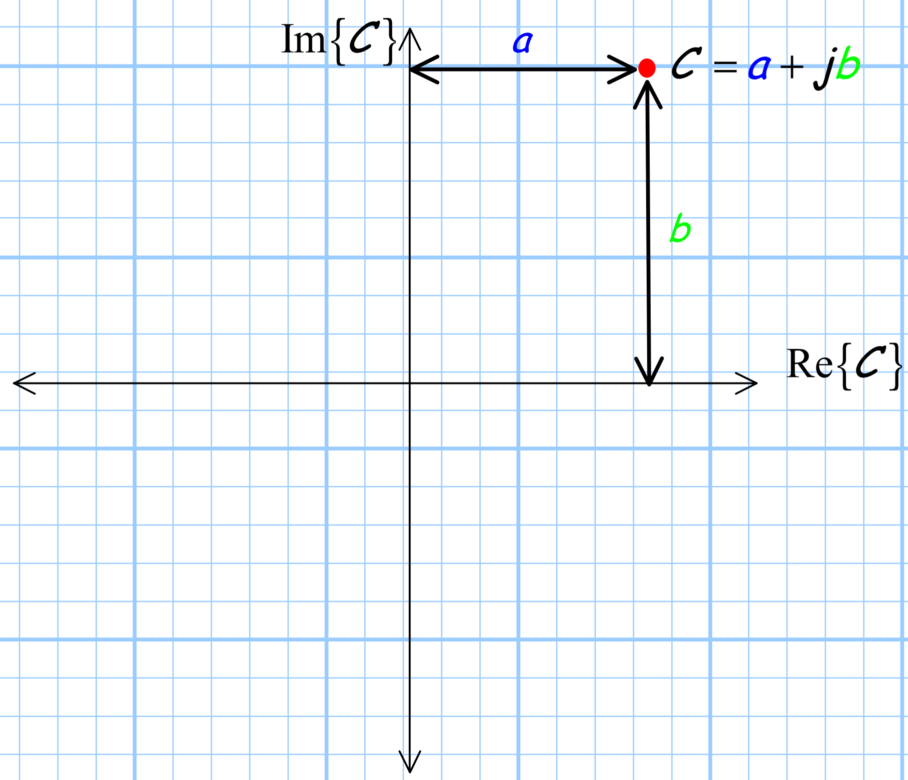
$$a = \text{Re}\{C\} \qquad \text{and} \qquad b = \text{Im}\{C\}$$

so that we can express this complex value as:

$$C = a + jb$$

where $j^2 = -1$.

Just as a **real** value can be expressed as a point on the **real line**, a **complex** value can be expressed as a point on the **complex plane**:

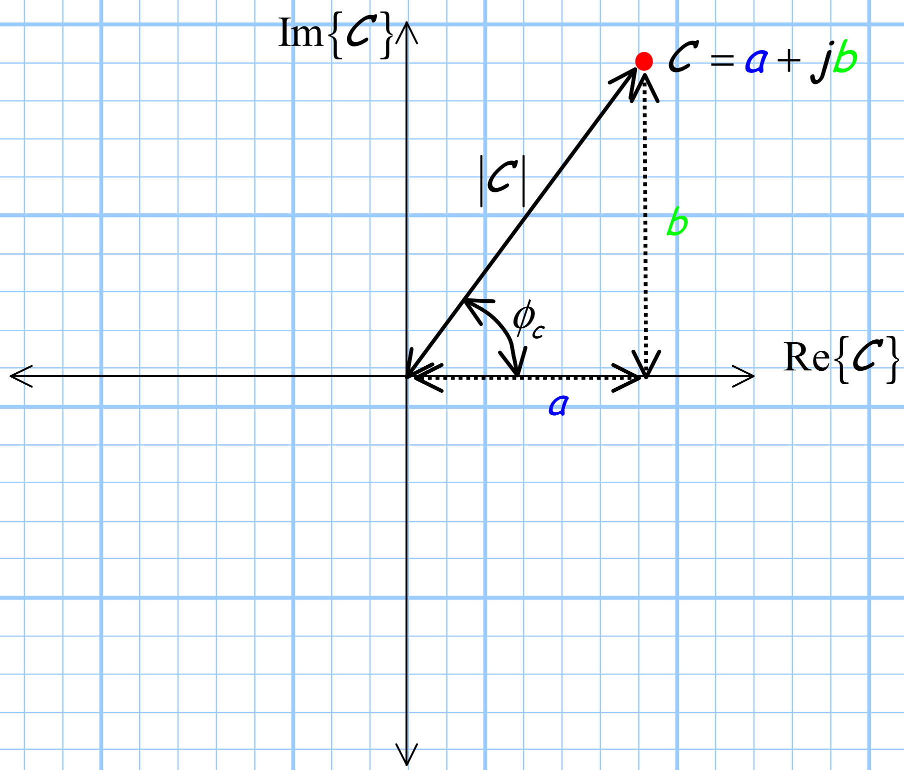


The values (a,b) are a **Cartesian** representation of a point on the complex plane. Recall that we can **alternatively** denote a point on a 2-dimensional plane using **polar** coordinates:

$|C| \doteq$ distance from the origin to the point

$\angle C \doteq \phi_c =$ rotation angle from the horizontal ($\mathrm{Re}\{C\}$) axis

i.e.,



Using our knowledge of **trigonometry**, we can determine the relationship between the Cartesian (a,b) and polar $(|C|, \phi_c)$ representations.

From the **Pythagorean** theorem, we find that:

$$|C| = \sqrt{a^2 + b^2}$$

Likewise, from the definition of ***sine*** (opposite over hypotenuse), we find:

$$\sin\phi_c = \frac{b}{|C|} = \frac{b}{\sqrt{a^2+b^2}}$$

or, using the definition of ***cosine*** (adjacent over hypotenuse):

$$\cos\phi_c = \frac{a}{|C|} = \frac{a}{\sqrt{a^2+b^2}}$$

Combining these results, we can determine the ***tangent*** (opposite over adjacent) of ϕ_c:

$$\tan\phi_c = \frac{\sin\phi_c}{\cos\phi_c} = \frac{b}{a}$$

Thus, we can write the polar coordinates **in terms of** the Cartesian coordinates:

$$|C| = \sqrt{a^2+b^2}$$

$$\phi_c = \tan^{-1}\left(b/a\right) = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) = \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

Likewise, we can use trigonometry to write the **Cartesian** coordinates in terms of the **polar** coordinates.

For example, we can use the definition of ***sine*** to determine *b:*

$$b = |C| \sin\phi_c$$

and the definition of ***cosine*** to determine *a:*

$$a = |C| \cos\phi_c$$

Summarizing:

$$a = |C| \cos\phi_c$$

$$b = |C| \sin\phi_c$$

Note that we can explicitly write the complex value C in terms of its **magnitude** $|C|$ and **phase angle** ϕ_c:

$$\begin{aligned} C &= a + jb \\ &= |C| \cos\phi_c + j|C| \sin\phi_c \\ &= |C|\left(\cos\phi_c + j \sin\phi_c\right) \end{aligned}$$

Hey! we can use ***Euler's equation*** *to simplify this further!*

Recall that Euler's equation states:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

so complex value C is:

$$\begin{aligned} C &= a + jb \\ &= |C|(\cos\phi_c + j\sin\phi_c) \\ &= |C|\, e^{j\phi_c} \end{aligned}$$

Now we have **two** ways of expressing a complex value C!

$$C = a + jb \qquad \text{and/or} \qquad C = |C|\, e^{j\phi_c}$$

Note that both representations are **equally valid** mathematically—**either one** can be successfully used in complex analysis and computation.

Typically, we find that the **Cartesian** representation is **easiest** to use **if** we are doing **arithmetic** calculations (e.g., addition and subtraction).

For example, if:

$$C_1 = a_1 + j\,b_1 \qquad \text{and} \qquad C_2 = a_2 + j\,b_2$$

then:

$$C_1 + C_2 = (a_1 + a_2) + j\,(b_1 + b_2)$$

$$C_1 - C_2 = (a_1 - a_2) + j\,(b_1 - b_2)$$

Conversely, for **geometric** calculations (multiplication and division), it is **easier** to use the **polar** representation:

For example, if:

$$C_1 = |C_1|\,e^{j\phi_1} \qquad \text{and} \qquad C_2 = |C_2|\,e^{j\phi_2}$$

then:

$$\begin{aligned} C_1\,C_2 &= |C_1|\,e^{j\phi_1}\;|C_2|\,e^{j\phi_2} \\ &= |C_1|\,|C_2|\;e^{j\phi_1}\,e^{j\phi_2} \\ &= |C_1|\,|C_2|\;e^{j(\phi_1+\phi_2)} \end{aligned}$$

and:

$$\begin{aligned} \frac{C_1}{C_2} &= \frac{|C_1|\,e^{j\phi_1}}{|C_2|\,e^{j\phi_2}} \\ &= \frac{|C_1|\,e^{j\phi_1}\,e^{-j\phi_2}}{|C_2|} \\ &= \frac{|C_1|}{|C_2|}\,e^{j(\phi_1-\phi_2)} \end{aligned}$$

Note in the above calculations we have used the **general** facts:

$$x^y x^z = x^{(y+z)} \qquad \text{and} \qquad \frac{x^y}{x^z} = x^{(y-z)}$$

Additionally, we note that **powers** and **roots** are most easily accomplished using the **polar** form of C:

$$\begin{aligned} C^n &= \left(|C| e^{j\phi_c}\right)^n \\ &= |C|^n \left(e^{j\phi_c}\right)^n \\ &= |C|^n e^{jn\phi_c} \end{aligned}$$

and

$$\begin{aligned} \sqrt[n]{C} &= \left(|C| e^{j\phi_c}\right)^{1/n} \\ &= |C|^{1/n} \left(e^{j\phi_c}\right)^{1/n} \\ &= |C|^{1/n} e^{j\left(\phi_c/n\right)} \end{aligned}$$

Therefore:

$$C^2 = \left(|C| \ e^{j\phi_c}\right)^2 = |C|^2 e^{j(2\phi_c)}$$

and:

$$\sqrt{C} = \left(|C| e^{j\phi_c}\right)^{1/2} = \sqrt{|C|}\, e^{j\left(\phi_c/2\right)}$$

Finally, we define the **complex conjugate** (C^*) of a complex value C:

$$\begin{aligned} C^* &\doteq \text{Complex Conjugate of } C \\ &= a - jb \\ &= |C|\, e^{-j\phi_c} \end{aligned}$$

A **very important** application of the complex conjugate is for determining the **magnitude** of a complex value:

$$|C|^2 = C\, C^*$$

Typically, the **proof** of this relationship is given as:

$$\begin{aligned} C\, C^* &= (a + jb)(a - jb) \\ &= a(a - jb) + jb(a - jb) \\ &= a^2 + jab - jba - j^2 b^2 \\ &= a^2 + b^2 \\ &= |C|^2 \end{aligned}$$

However, it is more **easily** shown as:

$$
\begin{aligned}
C\,C^* &= \left(|C|e^{j\phi_c}\right)\left(|C|e^{-j\phi_c}\right)\\
&= |C|^2\, e^{j(\phi_c-\phi_c)}\\
&= |C|^2\, e^{j0}\\
&= |C|^2
\end{aligned}
$$

Another important relationship involving complex conjugate is:

$$
\begin{aligned}
C + C^* &= (a + jb) + (a - jb)\\
&= (a+a) + j(b-b)\\
&= 2a
\end{aligned}
$$

Thus, the **sum** of a complex value and its complex conjugate is a purely **real** value.

Additionally, the **difference** of complex value and its complex conjugate results in a purely **imaginary** value:

$$
\begin{aligned}
C - C^* &= (a + jb) - (a - jb)\\
&= (a-a) + j(b+b)\\
&= j2b
\end{aligned}
$$

Note from these results we can derive the relationships:

$$
a = \mathrm{Re}\{C\} = \frac{C + C^*}{2}
$$

$$
b = \mathrm{Im}\{C\} = \frac{C - C^*}{j2}
$$