The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current I(z) and voltage V(z) **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for V(z) into the first telegrapher's equation, and **see what happens** !

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L) I(z)$$

Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

Q: ***But wait*** *! I thought we already knew current I(z). Isn't it:*

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for I(z) be true??*

A: Easy ! Determine how both expression for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for all z, I_0 and V_0 must be related as:

$$I_0^+ = \frac{\gamma V_0^+}{R + j\omega L} \qquad \text{and} \qquad I_0^- = \frac{-\gamma V_0^-}{R + j\omega L}$$

Or, written another way:

$$\frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \text{and} \qquad \frac{-V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the transmission line **only** (R, L, G, C).

This ratio is an important characteristic of a transmission line, called its **Characteristic Impedance Z_0.**

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

The **line impedance** $Z(z)$ is defined as the ratio of the voltage $V(z)$ to the current $I(z)$:

$$\text{Line Impedance } \doteq Z(z) = \frac{V(z)}{I(z)}$$

Therefore, line impedance is a **function of position** z, as opposed to the **constant** value Z_0 .

IMPORTANT NOTE !!

The line impedance Z(z) is therefore **not** the same thing as characteristic impedance, and therefore will not be equal to Z_0 (generally speaking), i.e.:

$$Z(z) = \frac{Z_0\left(V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}\right)}{V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}} \neq Z_0$$

Remember, characteristic impedance Z_0 is a transmission line **characteristic**, and depends **only** on R, L, G, and C.

Although line impedance Z(z) also depends on the transmission line characteristics, it **additionally** depends on the "things" the line is **attached** to (e.g., sources and loads)!