The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current I(z) and voltage V(z) must have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for V(z) into the first telegrapher's equation, and see what happens!

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L)I(z)$$

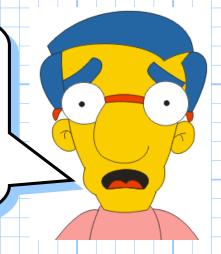
Therefore, rearranging, I(z) must be:

$$I(z) = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z})$$

Q: But wait! I thought we already knew current I(z). Isn't it:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$
 ??

How can both expressions for I(z) be true??



A: Easy! Determine how both expression for current are equal to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z})$$

For the above equation to be true for all z, I_0 and V_0 must be related as:

$$I_0^+ = \frac{\gamma V_0^+}{R + j\omega L}$$
 and $I_0^- = \frac{-\gamma V_0^-}{R + j\omega L}$

Or, written another way:

$$\frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \text{and} \quad \frac{-V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Although V_0^\pm and I_0^\pm are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** V_0^\pm/I_0^\pm is determined by the parameters of the transmission line **only** (R, L, G, C).

This ratio is an important characteristic of a transmission line, called its Characteristic Impedance Z₀.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 I_0^+ e^{-\gamma z} - Z_0 I_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

The line impedance Z(z) is defined as the ratio of the voltage V(z) to the current I(z):

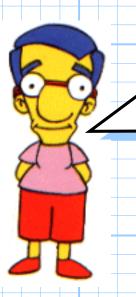
Line Impedance
$$= Z(z) = \frac{V(z)}{I(z)}$$

Therefore, line impedance is a function of position z, as opposed to the constant value Z_0 .

IMPORTANT NOTE!!

The line impedance Z(z) is therefore **not** the same thing as characteristic impedance, and therefore will not be equal to Z_0 (generally speaking), i.e.:

$$Z(z) = \frac{Z_0(V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z})}{V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}} \neq Z_0$$



Remember, characteristic impedance Z_0 is a transmission line **characteristic**, and depends **only** on R, L, G, and C.

Although line impedance Z(z) also depends on the transmission line characteristics, it **additionally** depends on the "things" the line is **attached** to (e.g., sources and loads)!