The Complex Propagation Constant $\gamma$

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

where $Z_0$ and $\gamma$ are complex constants that describe the properties of a transmission line. Since $\gamma$ is complex, we can consider both its real and imaginary components.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = \alpha + j\beta$$

where $\alpha = \text{Re} \{\gamma\}$ and $\beta = \text{Im} \{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta) z} = e^{-\alpha z} e^{-j\beta z}$$

Since $|e^{j\beta z}| = 1$, then $e^{-\alpha z}$ alone determines the magnitude of $e^{-\gamma z}$. 
I.E., \( |e^{-\gamma z}| = e^{-\alpha z} \).

Therefore, \( \alpha \) expresses the attenuation of the signal due to the loss in the transmission line.

Since \( e^{-\alpha z} \) is a real function, it expresses the magnitude of \( e^{-\gamma z} \) only. The relative phase \( \phi(z) \) of \( e^{-\gamma z} \) is therefore determined by \( e^{-j\beta z} = e^{-j\phi(z)} \) only (recall \( |e^{-j\beta z}| = 1 \)).

From Euler's equation:

\[
e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j \sin(\beta z)
\]

Therefore, \( \beta z \) represents the relative phase \( \phi(z) \) of the oscillating signal, as a function of transmission line position \( z \). Since phase \( \phi(z) \) is expressed in radians, and \( z \) is distance (in meters), the value \( \beta \) must have units of:

\[
\beta = \frac{\phi}{z} \text{ radians/meter}
\]
The **wavelength** $\lambda$ of the signal is the distance $\Delta z_{2\pi}$ over which the relative phase changes by $2\pi$ radians. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \Delta z_{2\pi} = \beta \lambda$$

or, rearranging:

$$\beta = \frac{2\pi}{\lambda}$$

Since the signal is oscillating in **time** at rate $\omega$ rad/sec, the **propagation velocity** of the wave is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega \lambda}{2\pi} = f \lambda \quad \left(\frac{m}{\text{sec}} = \frac{\text{rad}}{\text{sec} \cdot \text{rad}}\right)$$

where $f$ is **frequency** in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., $v(z, t)$ and $i(z, t)$). We assumed the time function was sinusoidal, oscillating with frequency $\omega$:

$$v(z, t) = \Re\{V(z)e^{j\omega t}\}$$

$$i(z, t) = \Re\{I(z)e^{j\omega t}\}$$
Now that we know $V(z)$ and $I(z)$, we can write the original functions as:

$$v(z, t) = \text{Re} \left\{ V_0^+ e^{\alpha z} e^{-j(\beta z - \omega t)} + V_0^- e^{\alpha z} e^{j(\beta z - \omega t)} \right\}$$

$$i(z, t) = \text{Re} \left\{ \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j(\beta z - \omega t)} - \frac{V_0^+}{Z_0} e^{\alpha z} e^{j(\beta z - \omega t)} \right\}$$

The first term in each equation describes a wave propagating in the $+z$ direction, while the second describes a wave propagating in the opposite $(-z)$ direction.

Each wave has wavelength:

$$\lambda = \frac{2\pi}{\beta}$$

And velocity:

$$v_p = \frac{\omega}{\beta}$$