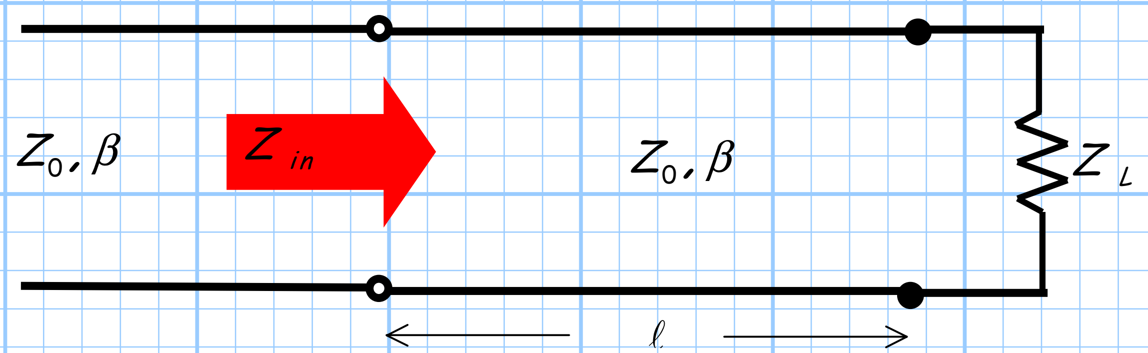
The Reflection Coefficient Transformation

The **load** at the end of some length of a transmission line (with characteristic impedance Z_0) can be specified in terms of its impedance Z_L **or** its reflection coefficient Γ.

Note **both** values are complex, and **either one** completely specifies the load—if you know one, you know the other!

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad Z_L = Z_0\left(\frac{1+\Gamma}{1-\Gamma}\right)$$

Recall that we determined how a length of transmission line **transformed** the load **impedance** into an input **impedance** of a (generally) different value:

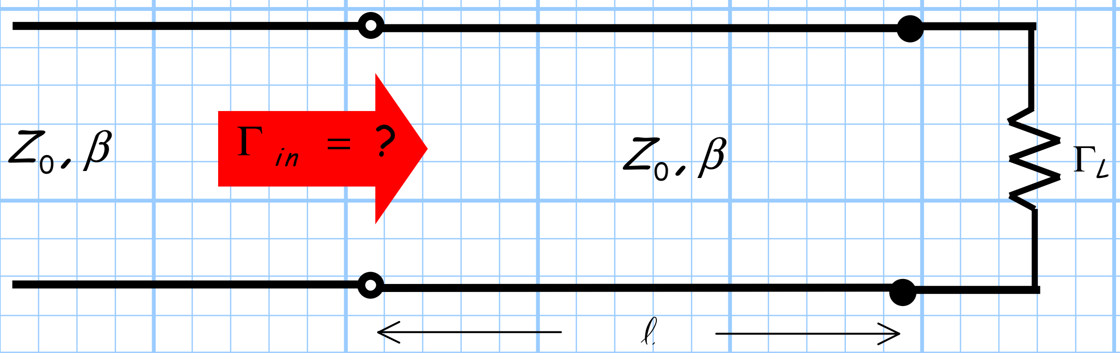


where:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan\beta\ell}{Z_0 + j\, Z_L \tan\beta\ell} \right)$$

Q: *Say we know the load in terms of its* ***reflection coefficient*** *(call this Γ_L). How can we express the* ***input*** *impedance in terms its* ***reflection coefficient*** *(call this Γ_{in})?*



A: Well, we **could** execute these **three** steps:

1. Convert Γ_L to Z_L:

$$Z_L = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right)$$

2. Transform Z_L down the line to Z_{in}:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

3. Convert Z_{in} to Γ_{in}:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Q: *Yikes! This is a **ton** of complex arithmetic—isn't there an **easier** way?*

A: Actually, there **is**!

Recall in an **earlier handout** that the input impedance of a transmission line length ℓ, terminated with a load Γ_L, is:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0\left(\frac{e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}}\right)$$

Note this **directly** relates Γ_L to Z_{in} (steps 1 and 2 combined!).

If we directly **insert** this equation into:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

we get an equation directly relating Γ_L to Γ_{in}:

$$\begin{aligned}\Gamma_{in} &= \frac{Z_0\left(e^{+j\beta\ell}+\Gamma_L e^{-j\beta\ell}\right)-\left(e^{+j\beta\ell}-\Gamma_L e^{-j\beta\ell}\right)}{Z_0\left(e^{+j\beta\ell}+\Gamma_L e^{-j\beta\ell}\right)+\left(e^{+j\beta\ell}-\Gamma_L e^{-j\beta\ell}\right)}\\ &= \frac{2\Gamma_L e^{-j\beta\ell}}{2e^{+j\beta\ell}}\\ &= \Gamma_L e^{-j\beta\ell}e^{-j\beta\ell}\\ &= \Gamma_L e^{-j2\beta\ell}\end{aligned}$$

Recall that Γ_L is a **complex value**. As such, we can express it in terms of its real and imaginary components, or by its **magnitude** $|\Gamma_L|$ and **phase** θ_Γ, i.e. :

$$\Gamma_L = |\Gamma_L| e^{j\theta_\Gamma}$$

Note then that the input reflection coefficient is related to the load as:

$$\boxed{\begin{aligned}\Gamma_{in} &= \Gamma_L e^{-j2\beta\ell}\\ &= |\Gamma_L| e^{j\theta_\Gamma} e^{-j2\beta\ell}\\ &= |\Gamma_L| e^{j(\theta_\Gamma - 2\beta\ell)}\end{aligned}}$$

Thus, the reflection coefficient at the **beginning** of a transmission line (Γ_{in}) is simply the same as that at the **end** of the line (Γ_L), only **phase-shifted** by a value of $-\beta\ell$ radians.

In other words, the magnitude $|\Gamma_{in}|$ is the **same** as $|\Gamma_L|$!

$$\begin{aligned}|\Gamma_{in}| &= |\Gamma_L|\left|e^{j(\theta_\Gamma - 2\beta\ell)}\right| \\ &= |\Gamma_L|\,(1) \\ &= |\Gamma_L|\end{aligned}$$

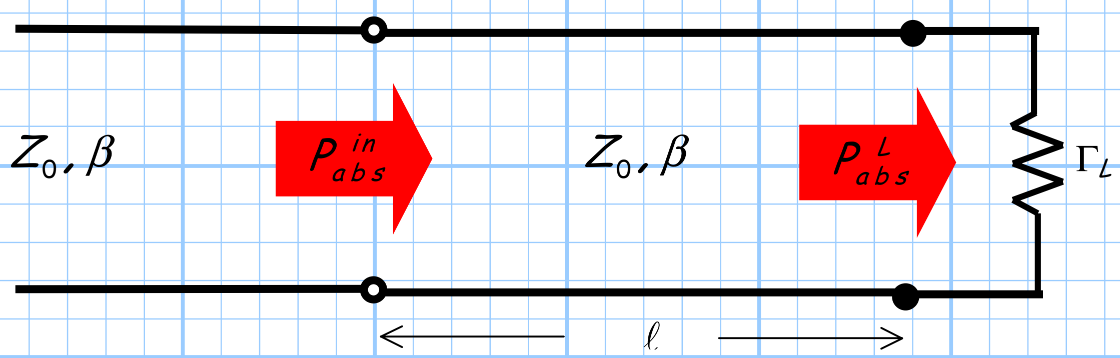
If we **think** about this, it makes **perfect sense**!

Recall that the power **absorbed** by the load Γ_{in} would be:

$$P_{abs}^{in} = \frac{|V_0^+|^2}{2\,Z_0}\left(1 - |\Gamma_{in}|^2\right)$$

while that absorbed by the **load** Γ_L is:

$$P_{abs}^{L} = \frac{|V_0^+|^2}{2\,Z_0}\left(1 - |\Gamma_L|^2\right)$$



Recall, however, that a lossless transmission line can absorb **no** power! By adding a length of transmission line to load Γ_L, we have added only **reactance**. Therefore, the power absorbed by load Γ_{in} is **equal** to the power absorbed by Γ_L:

$$P_{abs}^{in} = P_{abs}^{L}$$

$$\frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_{in}\right|^2\right) = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right)$$

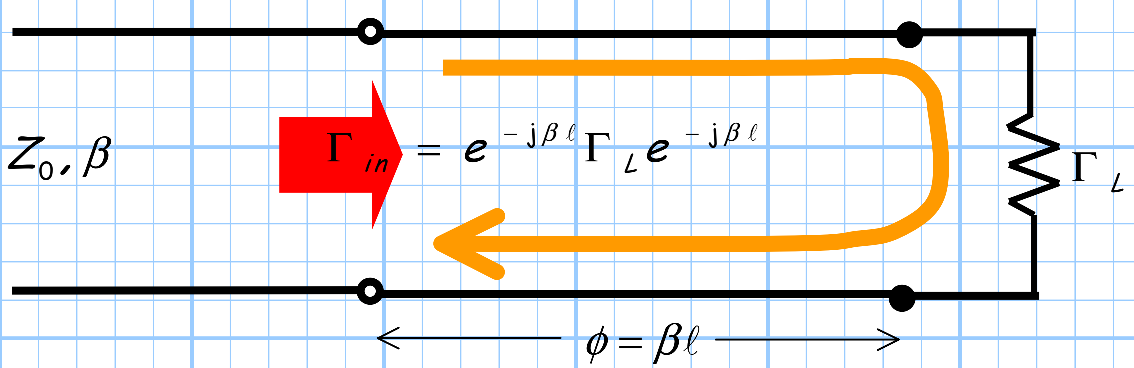
$$1-\left|\Gamma_{in}\right|^2 = 1-\left|\Gamma_L\right|^2$$

Thus, we can conclude from **conservation of energy** that:

$$\left|\Gamma_{in}\right| = \left|\Gamma_L\right|$$

Which of course is **exactly** the result we reached earlier!

Finally, the **phase shift** associated with transforming the load Γ_L down a transmission line can be attributed to the phase shift associated with the wave propagating a length ℓ down the line, reflecting from load Γ_L, and then propagating a length ℓ back up the line:



To emphasize this wave interpretation, we note that we can write Γ_{in} as:

$$
\begin{aligned}
\Gamma_{in} &= \Gamma_L \, e^{-j2\beta\ell} \\
&= \frac{V_0^-}{V_0^+} \, e^{-j2\beta\ell} \\
&= \frac{V_0^- \, e^{-j\beta\ell}}{V_0^+ \, e^{+j\beta\ell}} \\
&= \frac{V^-(z=-\ell)}{V^+(z=-\ell)}
\end{aligned}
$$

Where we have defined $V^-(z)$ as the function describing the wave propagating **away from** the load, and $V^+(z)$ the wave propagating **toward** it:

$$V^-(z) = V_0^- \, e^{+j\beta z}$$

$$V^+(z) = V_0^+ \, e^{-j\beta z}$$

We can therefore define a **general reflection coefficient function**, describing the **line impedance** $Z(z)$ in terms of the reflection coefficient Γ at an **arbitrary line location** z:

$$\boxed{\Gamma(z) = \frac{V^+(z)}{V^-(z)} = \Gamma_L \, e^{j\beta z}}$$

where $\Gamma_L = \Gamma(z=0)$.