The Transmission Line
Wave Equation

So, what functions $I(z)$ and $V(z)$ do satisfy both telegrapher’s equations??

To make this easier, we will combine the telegrapher equations to form one differential equation for $V(z)$ and another for $I(z)$.

First, take the derivative with respect to $z$ of the first telegrapher equation:

$$
\frac{\partial}{\partial z} \left( \frac{\partial V(z)}{\partial z} \right) = -(R + j\omega L) I(z)
$$

$$
= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L) \frac{\partial I(z)}{\partial z}
$$

Note that the second telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$
\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z)
$$

Combining these two equations, we get an equation involving $V(z)$ only:

$$
\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C) V(z)
$$
$$
= \gamma^2 V(z)
$$
where it is apparent that $\gamma^2 = (R + j\omega L)(G + j\omega C)$.

In a similar manner (i.e., begin by taking the derivative of the second telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z)$$

We have decoupled the telegrapher's equations, such that we now have two equations involving one function only:

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 V(z)$$
$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z)$$

BUT! Again we ask, what functions satisfy these differential equations??

Note only special functions satisfy these equations: if we take the double derivative of the function, the result is the original function (to within a constant)!

Such functions do exist! For example, $e^{-\gamma z}$ and $e^{+\gamma z}$.
Therefore, the **general** solution to these differential equations (and thus the telegrapher equations) are a **linear superposition** of these two solutions:

\[
V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \\
I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}
\]

where \( V_0^+, V_0^-, I_0^+, I_0^- \), and \( \gamma \) are **complex** constants.

**Q:** *How do we determine \( V_0^+, V_0^-, I_0^+, I_0^- \)??

**A:** *We apply boundary conditions!*

The solutions describe **two waves** propagating in the transmission line, one propagating in a direction \((+z)\) and one propagating in the other direction \((-z)\).

Therefore, we call the differential equations introduced in this handout the transmission line **wave equations**.