11.1 Two-Port Power Gains

Reading Assignment: ***pp. 536-542***

Specifying the **gain** of an amplifier is a bit more **ambiguous** than you may think. The problem is that there are so **many** ways to define **power**!

HO: THE POWERS THAT BE

HO: POWER GAIN

Q: *The absorbed powers typically are **less** than the available powers. Isn't there some way to better take advantage of the power available?*

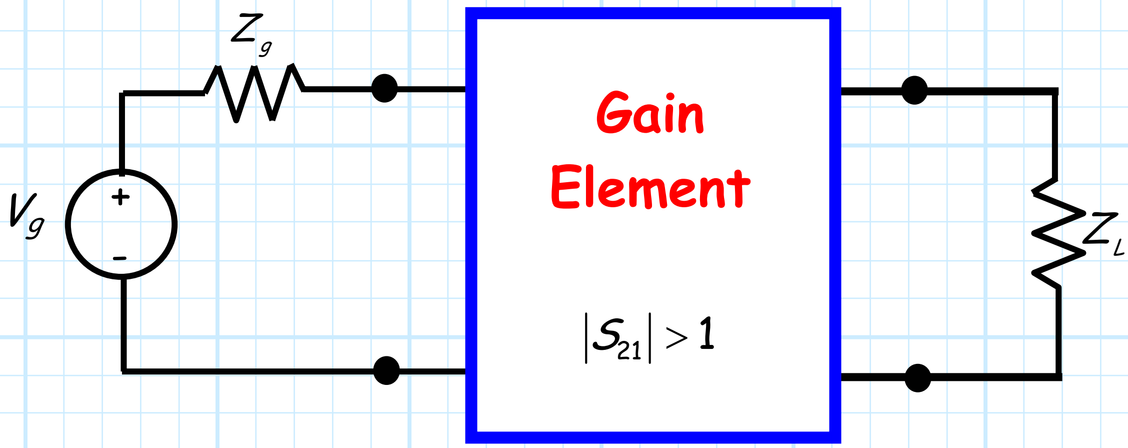
A: We know there is! The answer is **matching networks.**

HO: TURNING A GAIN ELEMENT INTO AN AMPLIFIER

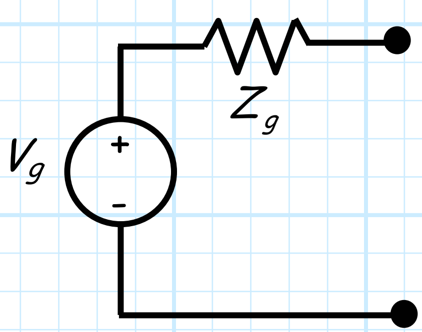
The Powers that Be

To begin our discussion of **amplifiers**, we first must define and derive a number of quantities that describe the **rate of energy flow** (i.e., power).

Consider a source and a load that are connected together by some gain element:

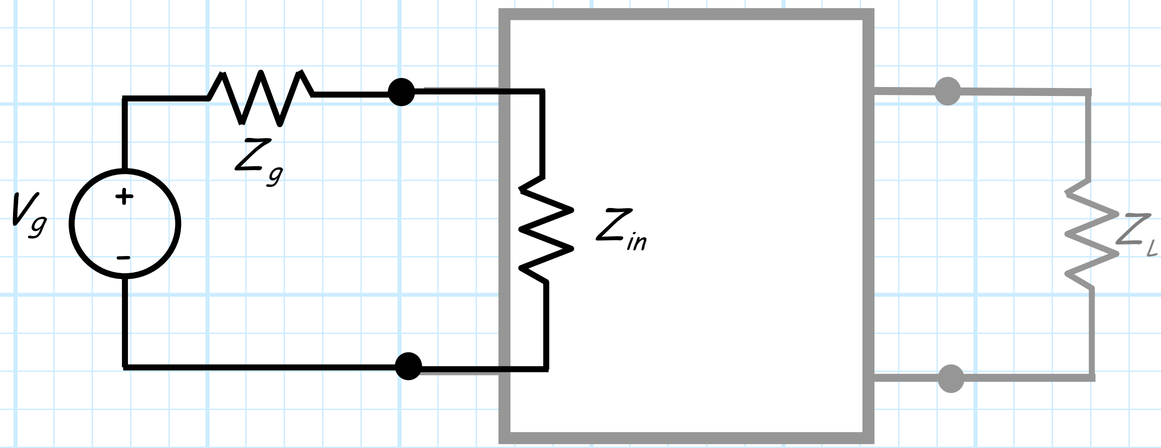


The first power we consider is the **available power** from **the source**:

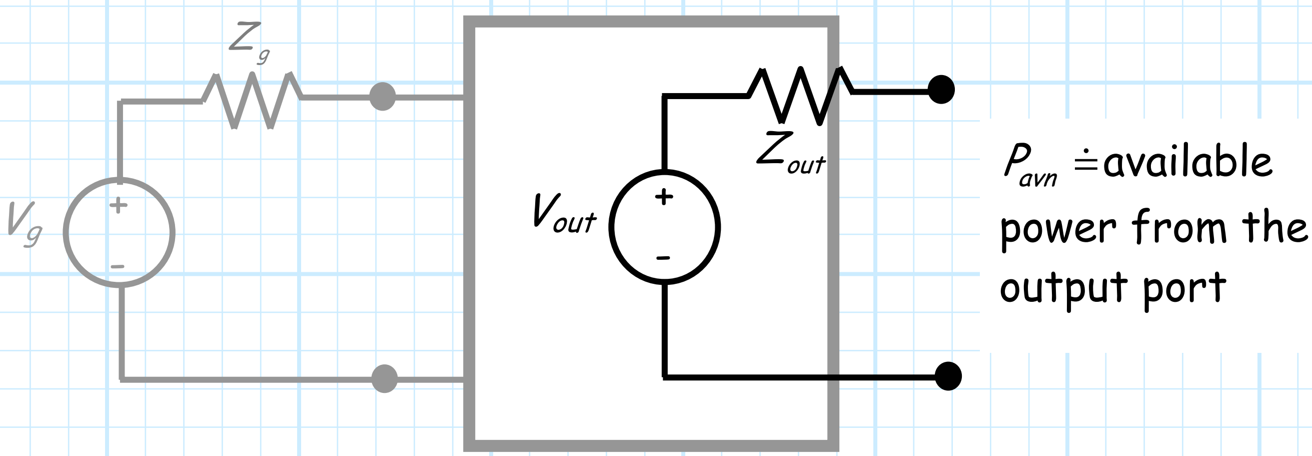


$P_{avs} \doteq$ available power from the source

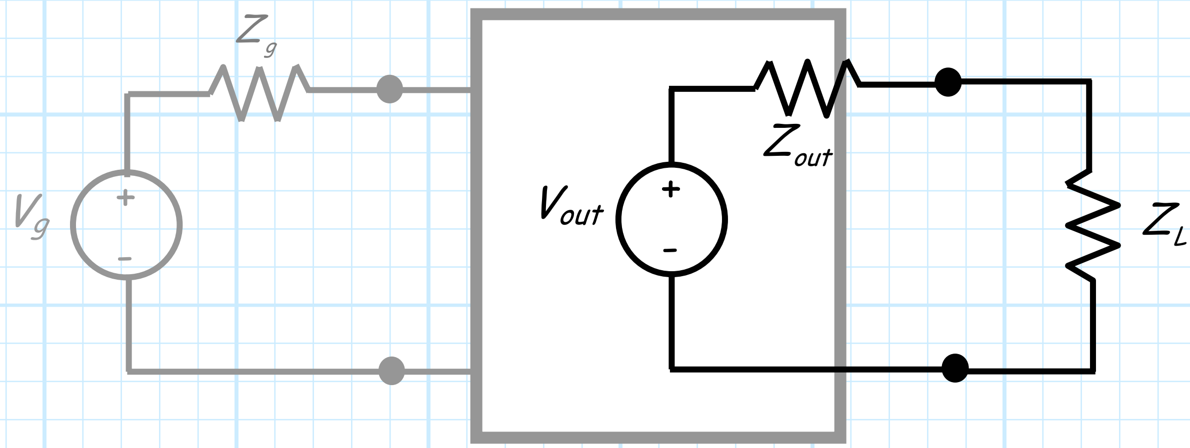
We likewise consider the power P_{in} **delivered** by the source; in other words the power **absorbed** by the input impedance of the gain element with a load attached:



On the output, we consider the power **available** from the **output** of the gain element:



And finally, we consider the power P_L **delivered** by the output port—the power absorbed by load Z_L:



These four power quantities depend (at least in part) on the **source** parameters V_g and Z_g, **load** Z_L, and the **scattering parameters** of $S_{11}, S_{21}, S_{22}, S_{12}$ the gain element.

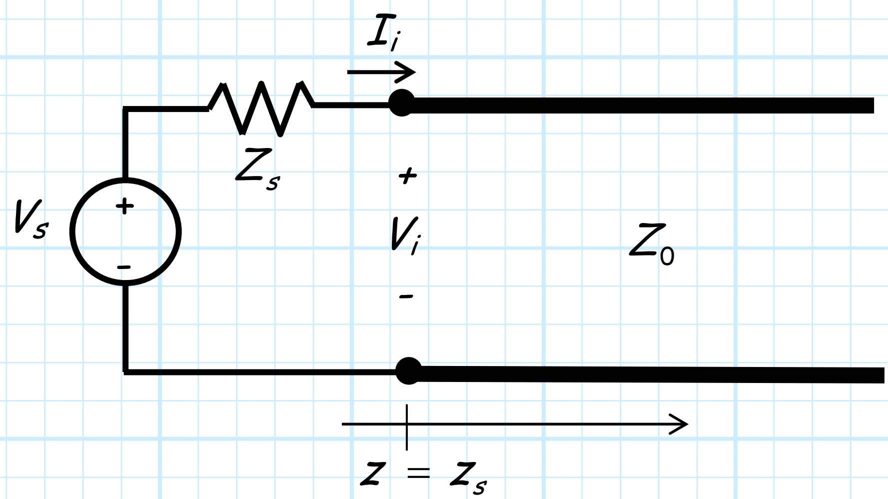
Q: *Yikes! How can we possible* ***determine*** *the power values in terms of these circuit parameters?*

A: Remember, the source, load and gain element (i.e. its scattering matrix) each are described by as set of **equations**. We simply need to **solve** these simultaneous equations!

Your text (pages 537-539) provides an algebraic solution. But you know me; I prefer to graphically solve the algebra using **signal flow graphs**!

Q: *But there's a* ***source*** *in our circuit: How do we handle that in a signal flow graph?*

A: Consider a simple source connected to a transmission line:



From **KVL** we know that:

$$V_s = V_i + Z_s I_i$$

Whereas, from the **telegraphers equations** we know that:

$$V_i = V\left(z = z_s\right) = V_0^+ e^{-j\beta z_s} + V_0^- e^{+j\beta z_s}$$

$$I_i = I\left(z = z_s\right) = \frac{V_0^+}{Z_0} e^{-j\beta z_s} - \frac{V_0^-}{Z_0} e^{+j\beta z_s}$$

Substituting the **definitions**:

$a_s \doteq V_0^- e^{+j\beta z_g}$ (complex amplitude of voltage waveincident on source)

$b_s \doteq V_0^+ e^{-j\beta z_g}$ (complex amplitude of voltage wave exiting source)

we get:

$$V_i = V(z = z_s) = b_s + a_s$$

$$I_i = I(z = z_s) = \frac{b_s}{Z_0} - \frac{a_s}{Z_0}$$

And then our **KVL** equation can be written as:

$$V_s = (b_s + a_s) + \frac{Z_s}{Z_0}(b_s - a_s)$$

And rearranging:

$$b_g = \left(\frac{Z_0}{Z_g + Z_0}\right) V_g + \Gamma_g \, a_g$$

Reluctantly defining a "reflection coeficient":

$$\Gamma_s \doteq \frac{Z_s - Z_0}{Z_s + Z_0} \qquad \text{(Doh!)}$$

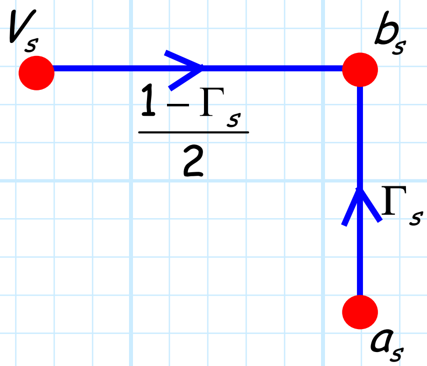
we find by rearranging:

$$\frac{Z_0}{Z_0 + Z_s} = \frac{1 - \Gamma_s}{2}$$

and so:

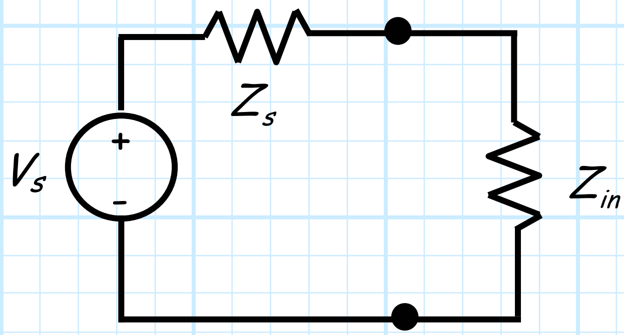
$$b_s = \left(\frac{1 - \Gamma_s}{2}\right) V_s + \Gamma_s \, a_s$$

We can express the above result graphically using a **signal-flow graph**:



$$b_s = \left(\frac{1-\Gamma_s}{2}\right)V_s + \Gamma_s \, a_s$$

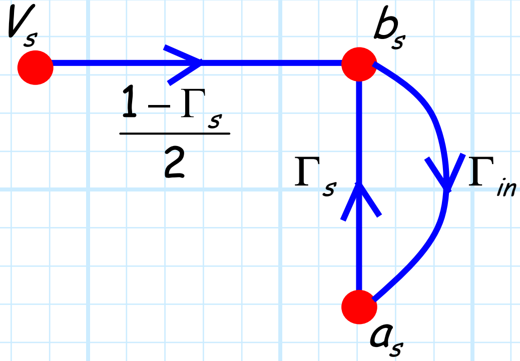
Now, consider the case where we place a **load** (e.g., the input impedance of a two port network) at this source port:



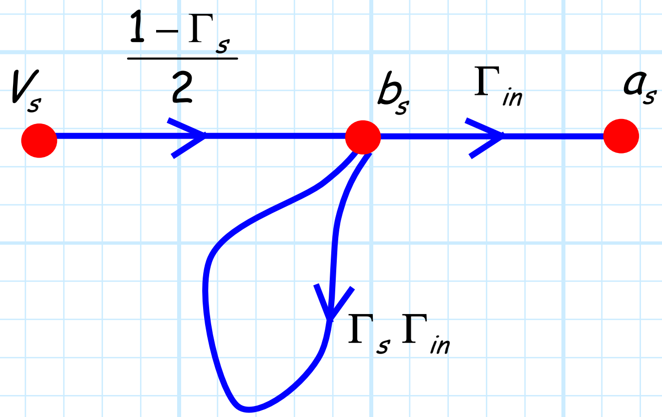
We know from transmission line theory that:

$$\Gamma_{in} = \frac{V_0^- \, e^{+j\beta z_s}}{V_0^+ \, e^{-j\beta z_s}} = \frac{a_s}{b_s} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

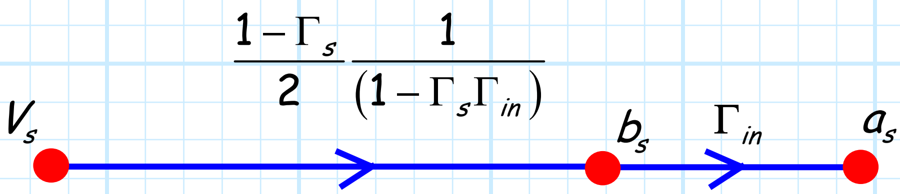
Thus, the relationship $a_s = \Gamma_{in} \, b_s$ can be added to the signal flow graph:



Using the splitting rule:



and then the self-loop rule:



we can directly conclude that:

$$b_s = V_s \frac{1-\Gamma_s}{2} \frac{1}{1-\Gamma_s \Gamma_{in}}$$

$$a_s = V_s \frac{1-\Gamma_s}{2} \frac{\Gamma_{in}}{1-\Gamma_s \Gamma_{in}}$$

Note that the power **incident** on the load can **now** be determined:

$$P_{inc} = \frac{|b_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\Gamma_s \Gamma_{in}|^2}$$

as well as the power **reflected** from the load:

$$P_{ref} = \frac{|a_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0}\frac{|1-\Gamma_s|^2}{|1-\Gamma_s\Gamma_{in}|^2}|\Gamma_{in}|^2$$

so that the power absorbed by the load (i.e. the power **delivered** by the source) is:

$$\begin{aligned} P_{in} &= P_{inc} - P_{ref} \\ &= \frac{|b_s|^2 - |a_s|^2}{2Z_0} \\ &= \frac{|V_s|^2}{8Z_0}\frac{|1-\Gamma_s|^2}{|1-\Gamma_s\Gamma_{in}|^2}\left(1-|\Gamma_{in}|^2\right) \\ &= \frac{|V_s|^2}{2Z_0}\left|\frac{Z_0}{Z_0+Z_s}\right|^2\frac{1-|\Gamma_{in}|^2}{|1-\Gamma_s\Gamma_{in}|^2} \end{aligned}$$

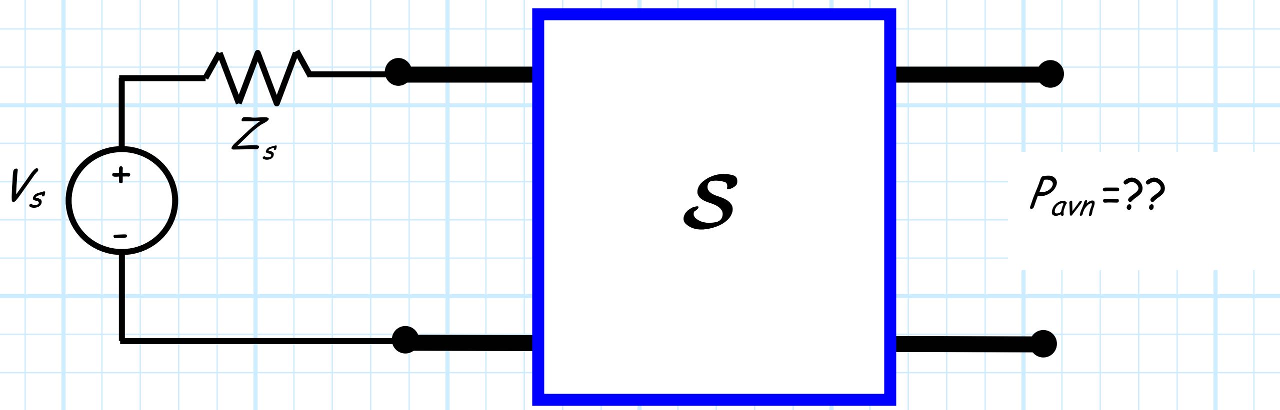
It is evident from the result above that the amount of power delivered is **dependent** on the value of **load impedance**. To maximize this power, we must find the value Γ_{in} that maximizes the term:

$$\frac{1-|\Gamma_{in}|^2}{|1-\Gamma_s\Gamma_{in}|^2}$$

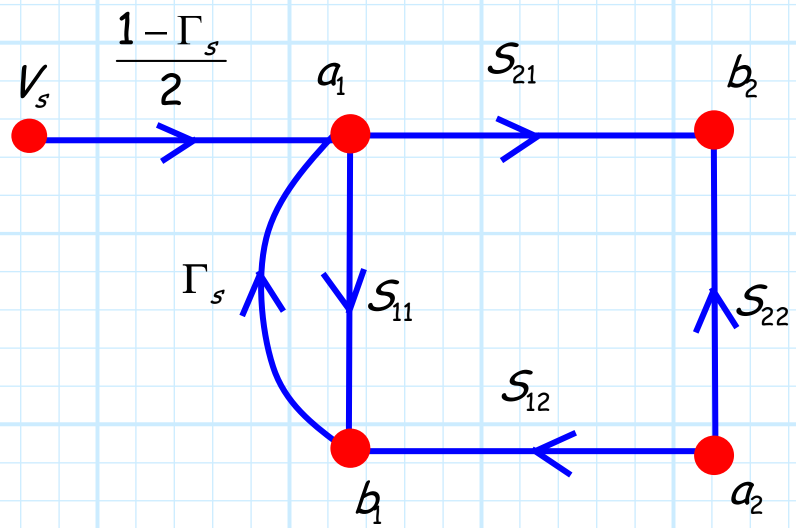
It can be shown that this term is maximized when $\Gamma_{in} = \Gamma_s^*$. No surprise here; the load must be **conjugate matched** to the source in order to maximize power transfer. This maximum value—resulting only when the load is conjugate matched to the source—is referred to as the **available power** of the source:

$$
\begin{aligned}
P_{avs} &= P_{in}\big|_{\Gamma_{in}=\Gamma_s^*} \\
&= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{1-|\Gamma_s|^2} \\
&= \frac{|V_s|^2}{2Z_0} \left|\frac{Z_0}{Z_0+Z_s}\right|^2 \frac{1}{1-|\Gamma_s|^2} \\
&= \frac{1}{2}\,|V_s|^2 \,\frac{1}{4\,Re\{Z_s^*\}}
\end{aligned}
$$

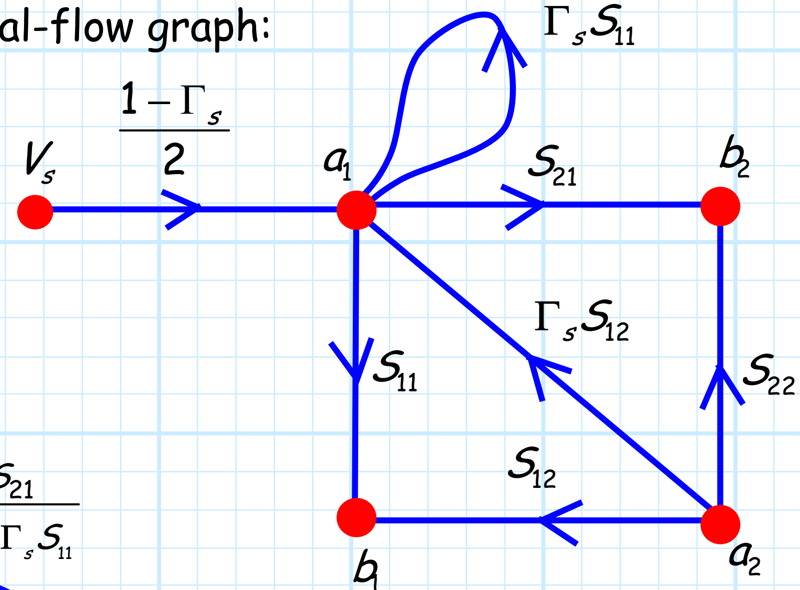
Now, consider the case where we connect some arbitrary **two-port device** to the source. We would like to determine the **available power** P_{avn} from the output port of this two-port device.

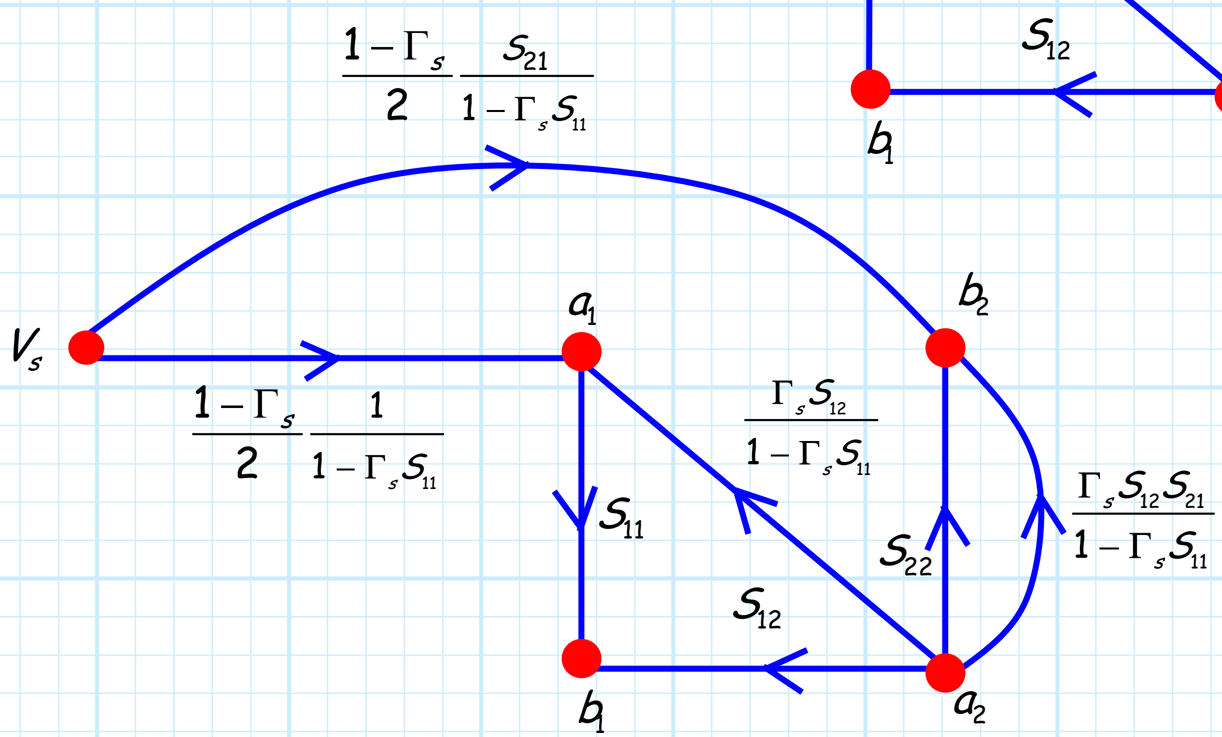


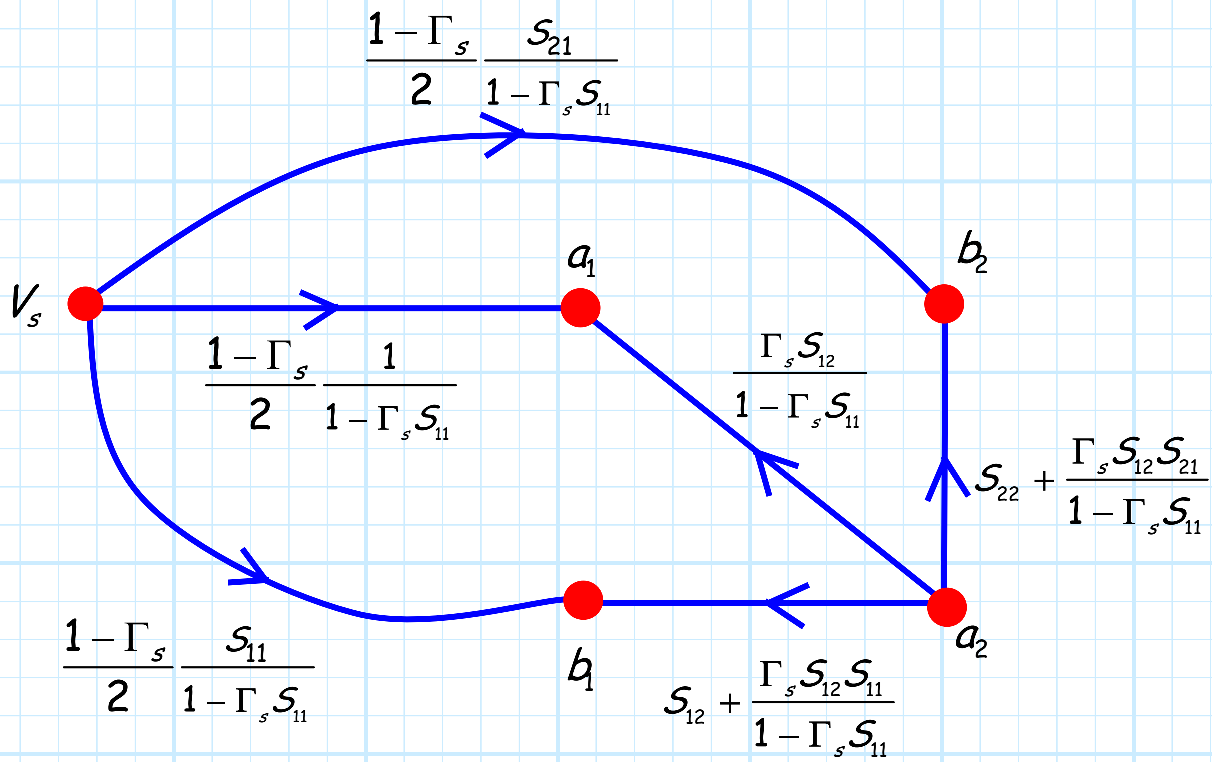
The signal-flow graph for **this** network:



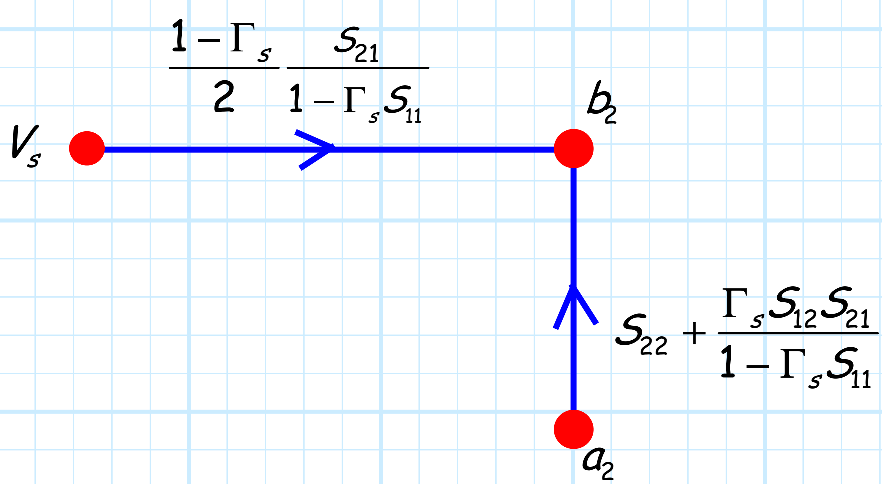
We can **reduce** this signal-flow graph:



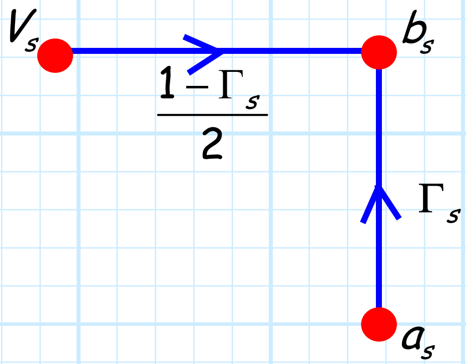




Now, for the purposes of determining the output power at port 2, we can **ignore** nodes a_1 and b_1 (in the **final** signal flow-graph above they are **terminal** nodes, no branches are **leaving** these nodes). Thus, the relevant portion of the ruduced signal flow graph is:



Notice this signal flow graph has the **same form** as the source signal-flow graph:

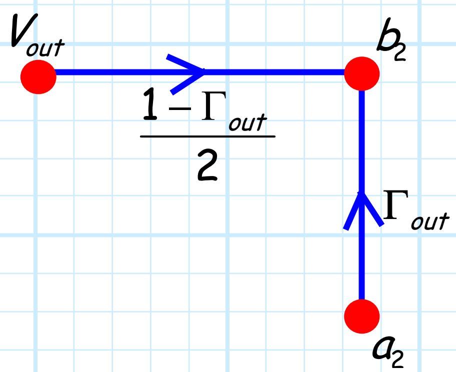


To make this comparison more specific, we **define** variables:

$$V_{out} \doteq V_s \frac{1-\Gamma_s}{1-\Gamma_{out}} \frac{S_{21}}{1-\Gamma_s S_{11}}$$

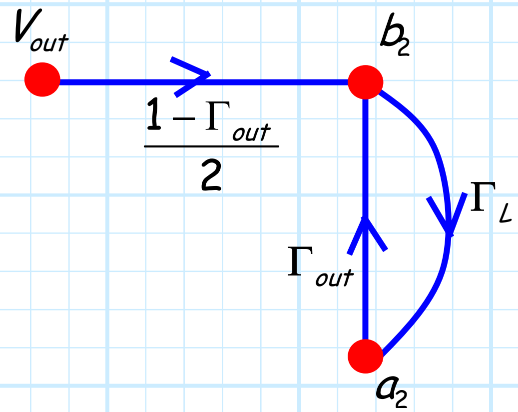
$$\Gamma_{out} \doteq S_{22} + \frac{\Gamma_s S_{12} S_{21}}{1-\Gamma_s S_{11}}$$

And thus, using these definitions, our signal flow graph can be **equivalently** written as:

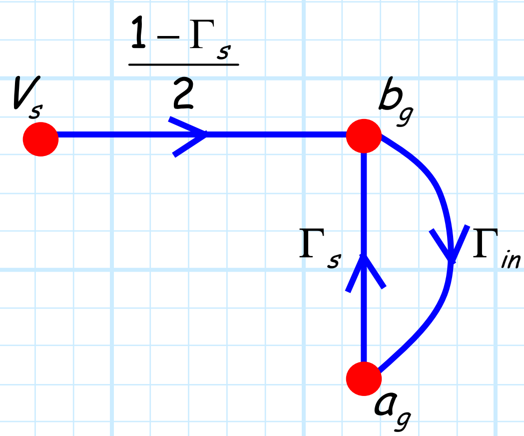


It is apparent that V_{out} and Γ_{out} define an **equivalent** source created when the original source is connected to a two-port device.

Thus, when some **load** is connected to the output of the two-port device, the signal –flow graph is:



Which has **precisely** the same form as:



As a result, the **delivered** power is **precisely** the same as the original case, with the exception that we use the **equivalent** values defined above:

$$P_L = \frac{|b_2|^2 - |a_2|^2}{2Z_0}$$

$$= \frac{|V_{out}|^2}{8Z_0} \frac{|1-\Gamma_{out}|^2}{|1-\Gamma_{out}\Gamma_L|^2}\left(1-|\Gamma_L|^2\right)$$

$$= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2} \frac{|1-\Gamma_s|^2}{|1-\Gamma_{out}\Gamma_L|^2}\left(1-|\Gamma_L|^2\right)$$

Likewise, the **available power** from port 2 is simply the maximum possible power absorbed by a load Γ_L. This **again** is found by maximizing the term:

$$\frac{1-\left|\Gamma_L\right|^2}{\left|1-\Gamma_{out}\,\Gamma_L\right|^2}$$

which **again** occurs when $\Gamma_L = \Gamma_{out}^*$. Thus, maximum power transfer occurs when the load is **conjugate matched** to the **equivalent** source impedance Z_{out} (Γ_{out}). As a result the **available power** from port 2 is:

$$\begin{aligned} P_{avn} &= P_L\big|_{\Gamma_L=\Gamma_{out}^*} \\ &= \frac{\left|V_s\right|^2}{8Z_0}\frac{\left|S_{21}\right|^2}{\left|1-\Gamma_s S_{11}\right|^2}\frac{\left|1-\Gamma_s\right|^2}{\left|1-\Gamma_{out}\,\Gamma_{out}^*\right|^2}\left(1-\left|\Gamma_{out}\right|^2\right) \\ &= \frac{\left|V_s\right|^2}{8Z_0}\frac{\left|S_{21}\right|^2}{\left|1-\Gamma_s S_{11}\right|^2}\frac{\left|1-\Gamma_s\right|^2}{\left(1-\left|\Gamma_{out}\right|^2\right)^2}\left(1-\left|\Gamma_{out}\right|^2\right) \\ &= \frac{\left|V_s\right|^2}{8Z_0}\frac{\left|S_{21}\right|^2}{\left|1-\Gamma_s S_{11}\right|^2}\frac{\left|1-\Gamma_s\right|^2}{1-\left|\Gamma_{out}\right|^2} \end{aligned}$$

Two-Port Power Gains

There are **three** standard ways of defining amplifier gain:

1. Power Gain

Power gain is defined as:

$$G \doteq \frac{P_L}{P_{in}}$$

Thus, it describes the increase in **delivered** (i.e., absorbed) power from input to output. From our power definitions, we find that:

$$\begin{aligned}
G &= \frac{P_L}{P_{in}} \\
&= \frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2} \frac{|1-\Gamma_s|^2}{|1-\Gamma_{out}\Gamma_L|^2} \left(1-|\Gamma_L|^2\right) \frac{|1-\Gamma_s\Gamma_{in}|^2}{|1-\Gamma_s|^2} \frac{1}{1-|\Gamma_{in}|^2} \\
&= \frac{|S_{21}|^2}{1-|\Gamma_{in}|^2} \frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\Gamma_L|^2} \frac{1}{|1-\Gamma_s S_{11}|^2} |1-\Gamma_s\Gamma_{in}|^2 \\
&= \frac{|S_{21}|^2}{1-|\Gamma_{in}|^2} \frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\Gamma_L|^2} \frac{1}{|1-\Gamma_s S_{11}|^2} \frac{|1-\Gamma_s S_{11}|^2 |1-\Gamma_{out}\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2} \\
&= \frac{|S_{21}|^2}{1-|\Gamma_{in}|^2} \frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}
\end{aligned}$$

Where we have used the fact (trust me!) that:

$$\left|1-\Gamma_s \Gamma_{in}\right|^2 = \frac{\left|1-\Gamma_s S_{11}\right|^2 \left|1-\Gamma_{out} \Gamma_L\right|^2}{\left|1-S_{22}\Gamma_L\right|^2}$$

2. Available Gain

Available gain is defined as:

$$G_A \doteq \frac{P_{avn}}{P_{avs}}$$

Thus, it describes the increase in **available** power from input to output. From our power definitions, we find that:

$$\begin{aligned} G_A &= \frac{P_{avn}}{P_{avs}} \\ &= \frac{\left|S_{21}\right|^2}{\left|1-\Gamma_s S_{11}\right|^2} \frac{\left|1-\Gamma_s\right|^2}{1-\left|\Gamma_{out}\right|^2} \frac{1-\left|\Gamma_s\right|^2}{\left|1-\Gamma_s\right|^2} \\ &= \frac{\left|S_{21}\right|^2}{\left|1-\Gamma_s S_{11}\right|^2} \frac{1-\left|\Gamma_s\right|^2}{1-\left|\Gamma_{out}\right|^2} \end{aligned}$$

3. Transducer Gain

Transducer gain is defined as:

$$G_T \doteq \frac{P_L}{P_{avs}}$$

Thus, it relates the power available from the source to the power delivered to the load. It in effect describes how **effectual** the amplifier was in extracting the available power from the source, increasing this power, and then delivering the power to the load.

$$
\begin{aligned}
G_T &= \frac{P_L}{P_{avs}} \\
&= \frac{|S_{21}|^2}{|1-\Gamma_s S_{11}|^2} \frac{|1-\Gamma_s|^2}{|1-\Gamma_{out}\Gamma_L|^2} \left(1-|\Gamma_L|^2\right) \frac{1-|\Gamma_s|^2}{|1-\Gamma_s|^2} \\
&= \frac{|S_{21}|^2 \, 1-|\Gamma_s|^2 \left(1-|\Gamma_L|^2\right)}{1} \frac{1}{|1-\Gamma_{out}\Gamma_L|^2 |1-\Gamma_s S_{11}|^2} \\
&= \frac{|S_{21}|^2 \, 1-|\Gamma_s|^2 \left(1-|\Gamma_L|^2\right)}{1} \frac{1}{|1-\Gamma_s\Gamma_{in}|^2 |1-\Gamma_L S_{22}|^2} \\
&= \frac{|S_{21}|^2 \left(1-|\Gamma_s|^2\right) 1-|\Gamma_L|^2}{|1-\Gamma_s\Gamma_{in}|^2 |1-\Gamma_L S_{22}|^2}
\end{aligned}
$$

There are likewise a few **special cases** that we need to be aware of. If both the source and the load impedance are Z_0, then we find $\Gamma_s = \Gamma_L = 0$, and then not surprisingly:

$$G_T = |S_{21}|^2$$

Additionally, we often find that $S_{12} = 0$ (or least approximately so), and as a result $\Gamma_{in} = S_{11}$, so:

$$G_T = \frac{|S_{21}|^2 \left(1-|\Gamma_s|^2\right) 1-|\Gamma_L|^2}{|1-\Gamma_s S_{11}|^2 \, |1-\Gamma_L S_{22}|^2} \doteq G_{TU}$$

We call this gain the **unilateral transducer power gain** G_{TU}

Q: *I'm so confused! **Which** gain definitions should I use when specifying an amp? **Which** gain definition do amplifier vendors use to specify their performance?*

A: We find that for a **well-designed** amplifier, the three gain values generally do **not** provide significantly differing values. Your book (on page 539-540) provides a typically example, where G=5.58, G_A=5.85, and G_T=5.49.

Most often then, microwave amplifier vendors do **not** explicitly specify the three values (for an assumed Z_0 source and load impedance). Instead, they provide a somewhat ambiguous value that they simply call **gain***.

* If you are inclined to be mischievous, ask an amplifier vendor if their gain spec. is actually **available** gain or **transducer** gain.

Turning a Gain Element into an Amplifier

Say the design criteria for our amplifier is to maximize the power delivered to the load (i.e., maximize P_L). This power is maximized when:

1. The available power from the **source** is entirely delivered to the **input** of the gain element $P_{in} = P_{avs}$.

2. The available power from the **output** of the gain element is entirely delivered to the **load** $P_L = P_{avn}$.

Recall this happy occurrence results when $\Gamma_{in} = \Gamma_s^*$ and $\Gamma_L = \Gamma_{out}^*$.

Q: *But what if this is **not** the case? What if our gain element is not matched to our source, or to our load? Must we simply accept inferior power transfer?*

A: Nope! Remember, we can always build **lossless matching networks** to efficiently transfer power between mismatched sources and loads.

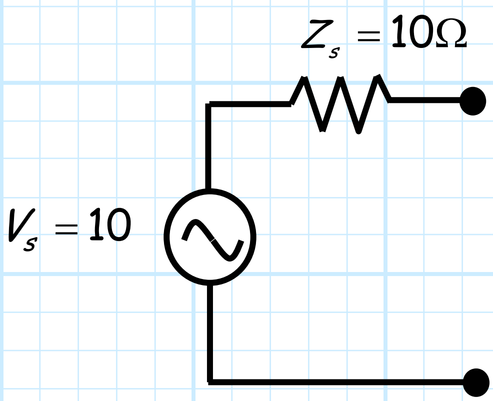
Q: *I see! We need to **modify** the source impedance Z_s and modify the output impedance Z_{out} such that $Z_s = Z_{in}^*$ and $Z_{out} = Z_L^*$. Right?*

A: Not exactly.

Remember, it is true that a lossless matching network can change the source impedance to match a specific load. But the lossless matching network likewise alters the source voltage V_s such that the available power is preserved!

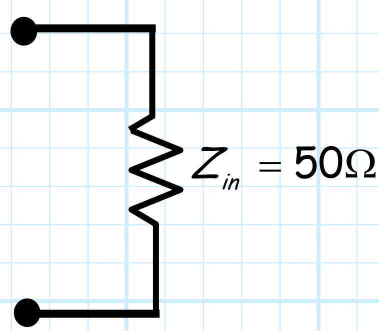
Messing around **directly** with the source impedance will undoubtedly **reduce** the available power of the source (this is bad!).

For **example,** consider this simple problem. Say we have this source, with a robust **available power** of 1.25 W:

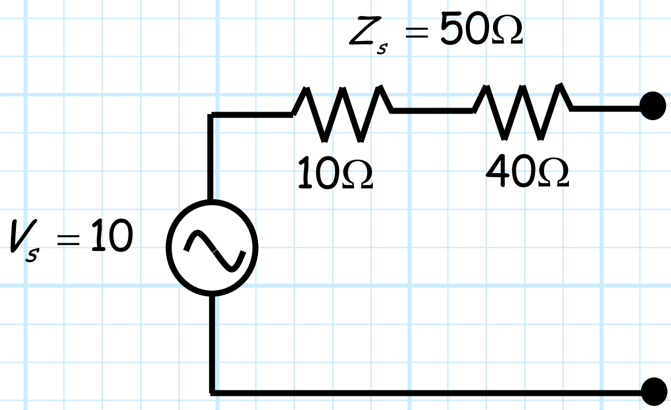


$$P_{avs} = \frac{|V_s|^2}{8\,Re\{Z_s\}} = \frac{10^2}{8(10)} = 1.25 \quad W$$

and wish to deliver this power to an impedance of $Z_{in} = 50\Omega$:



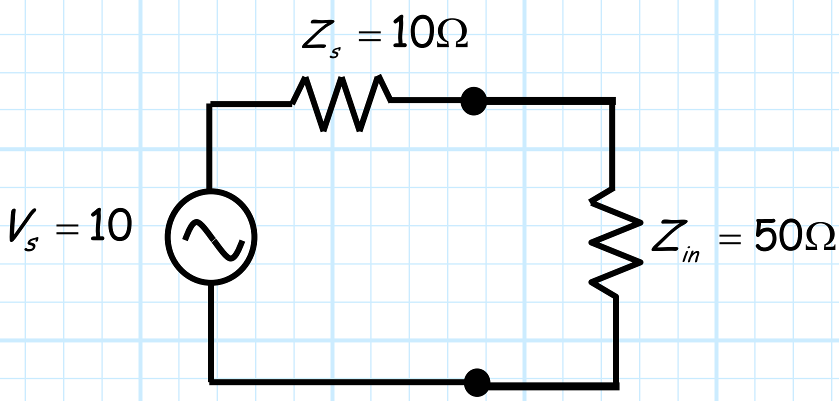
Although increasing the source impedance by 40 Ω would result in a **conjugate match**, it would likewise **reduce** the available power to a **measly 0.25 Watts**.



$$P_{avs} = \frac{|V_s|^2}{8\,Re\{Z_s\}} = \frac{10^2}{8(50)} = 0.25 \quad W$$

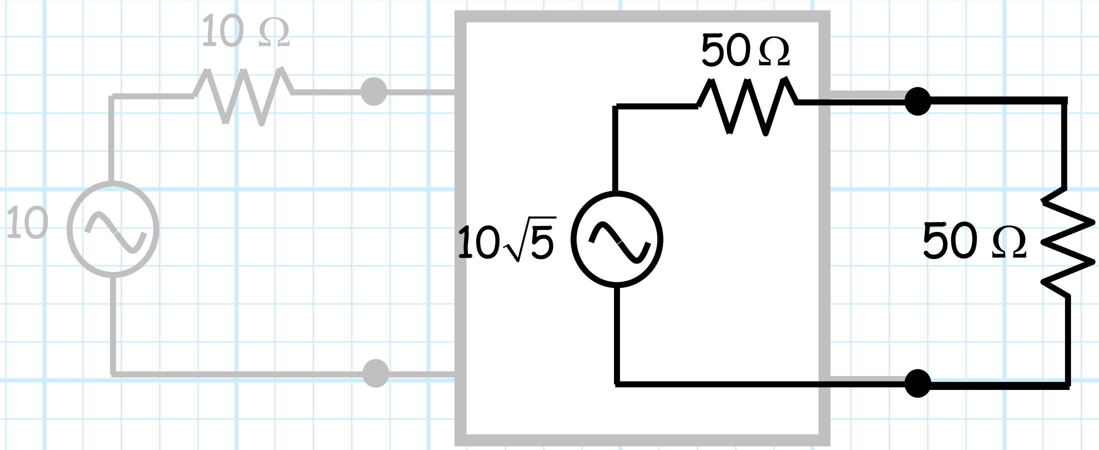
Thus, although finagling the source impedance does result in extracting **all** the available power from the resulting source, it likewise **decreases** this available power by 80% !

Moreover, we find that the delivered power to would be greater if we simply left the darn thing **alone**!



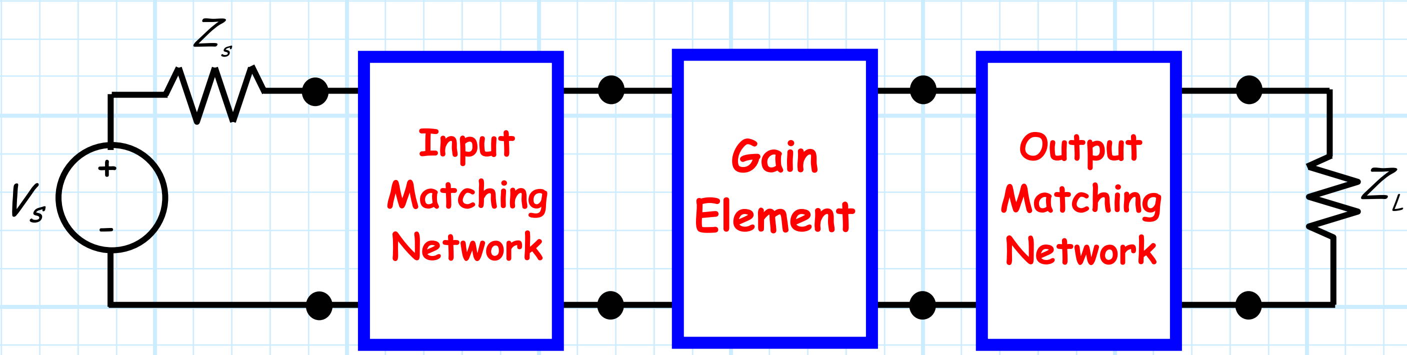
$$P_L = \frac{1}{2}|V_g|^2 \frac{Re\{Z_L\}}{|Z_g + Z_L|^2} = \frac{10^2}{2}\frac{50}{(20+50)^2} = \frac{50^2}{70^2} = 0.51 \quad W$$

In contrast, a properly designed matching network will transform the source impedance to a matched value of 50 Ω, but it **likewise** transforms the source **voltage** such that the absorbed power remains the **same**—1.25 Watts is delivered to the 50 Ω load!

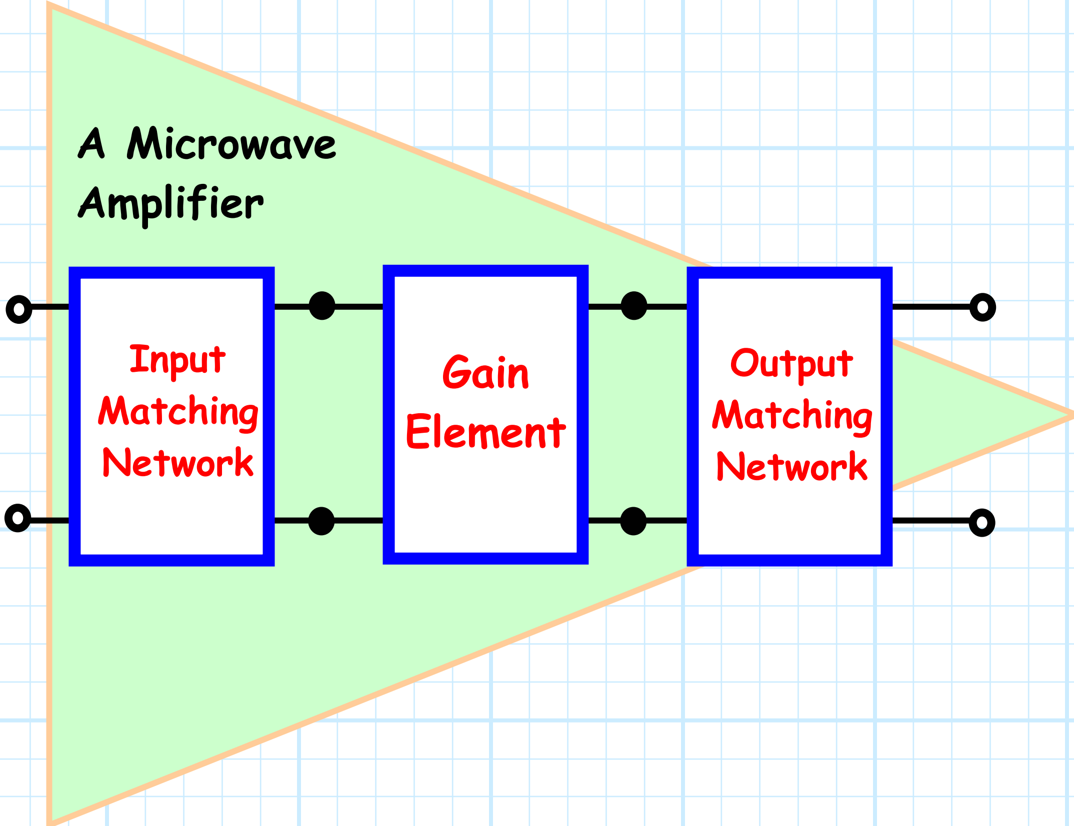


We **have** our cake. We **eat** it too.

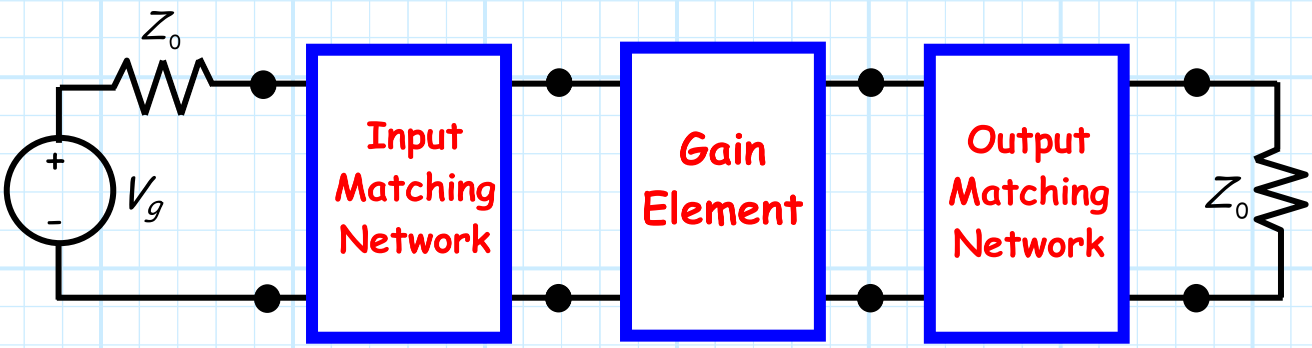
So, to maximize the power delivered to a load, we need to insert **lossless matching networks** between the source and gain element, and between the gain element and the load:



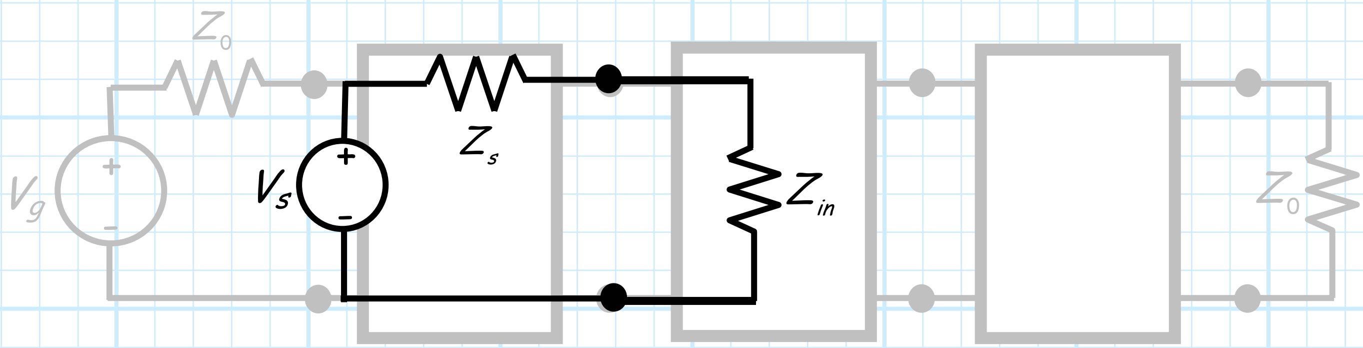
The **three stages** together—input matching network, gain element, and output matching network—form a **microwave amplifier**!



Of course, the impedance of both the source and the load connected to this amp will most certainly be that of transmission line **characteristic impedance** Z_0. Thus, our amplifier circuit is typically:



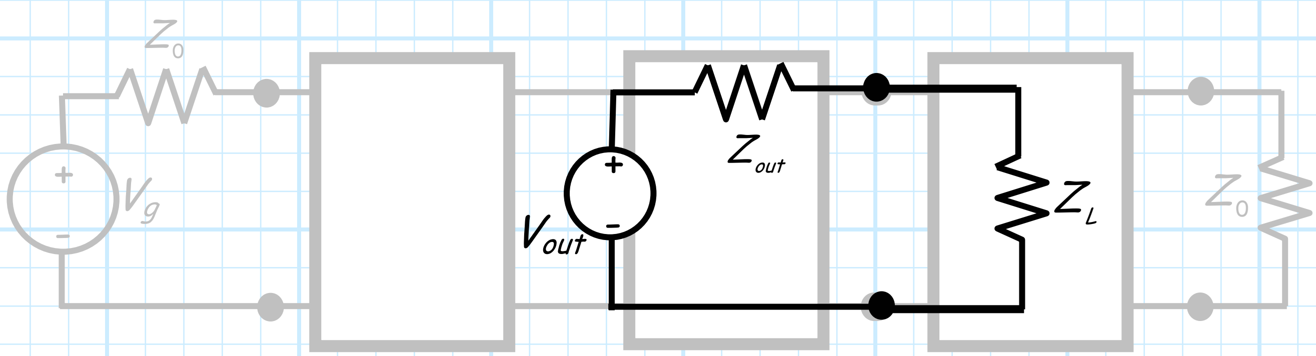
The **input network** is thus required to match Z_0 to the gain element input impedance Z_{in}. For the purposes of amplifier design, we view the input matching network as one that transforms the source impedance Z_0 into a new source impedance Z_s, one that is conjugate matched to the gain element input impedance Z_{in}:



If our input matching network is properly designed, we then find:

$$Z_s = Z_{in}^* \qquad \text{and so} \qquad \Gamma_s = \Gamma_{in}^*$$

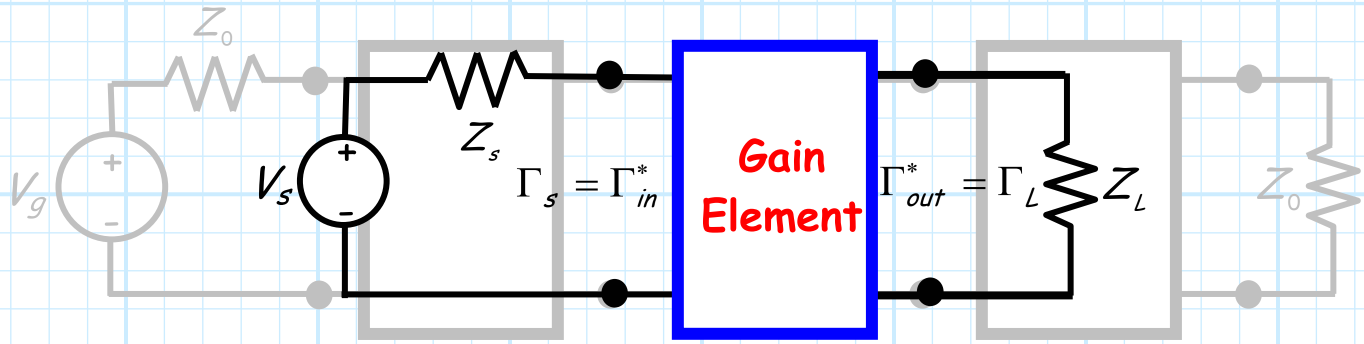
Likewise, the **output matching network** is used to match Z_0 to the gain element output impedance Z_{out}. For the purposes of amplifier design, we view the output matching network as one that transforms the load impedance Z_0 into a new load impedance Z_L, one that is conjugate matched to the gain element output impedance Z_{out}:



Thus, if our input matching network is properly designed, then we find:

$$Z_L = Z_{out}^* \qquad \text{and so} \qquad \Gamma_L = \Gamma_{out}^*$$

And so, our amplifier design problem can be described as:

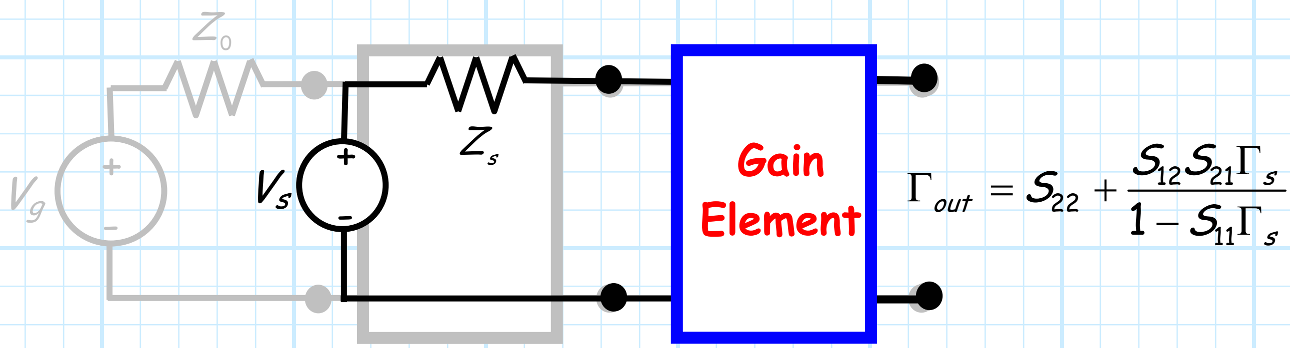


where **the values of** Γ_s **and** Γ_L **depend on the input and output matching networks.**

Q: *Alright, we get it. We* ***know*** *how to make matching networks. Can't we move on to something else?*

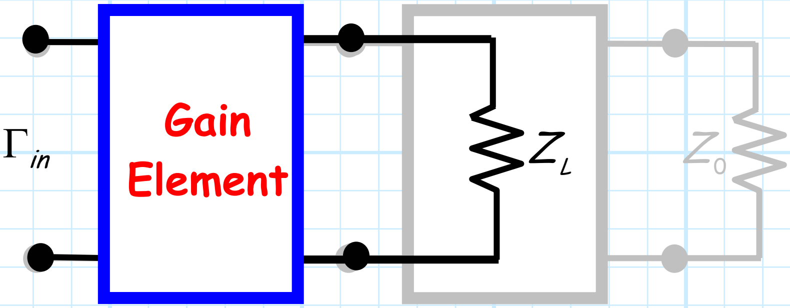
A: Not so fast! There's one little **problem** that makes this procedure more difficult than it otherwise might appear.

Note that the value of Γ_{out} depends on the value of Z_s (i.e., depends on Γ_s).



Likewise, the value of Γ_{in} depends on the value of Z_L (i.e., depends on Γ_L).

$$S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \Gamma_{in}$$



It's a classic **chicken and egg**!

1. We can't design the input matching network until we determine Γ_{in}.

2. We can't determine Γ_{in} until we design the output matching network.

3. We can't determine the output matching network until we determine Γ_{out}.

4. We can't determine Γ_{out} until we design the input matching network.

5. But we can't design the input matching network until we determine Γ_{in}!

Our matching network design problems are thus **coupled**. The solution to this coupled problem is provided in your textbook on page 550, and provides simultaneous solutions for $\Gamma_s = \Gamma_{in}^*$ and $\Gamma_L = \Gamma_{out}^*$.

Now for some **good news**!

Recall that for many gain elements, the value of S_{12} is exceedingly small. Often it is so small that we can approximate as zero.

Q: *So?*

A: Look at what this does to the value of Γ_{in} and Γ_{out}!

$$\Gamma_{out}\Big|_{S_{12}=0} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}\Bigg|_{S_{12}=0} = S_{22}$$

$$\Gamma_{in}\Big|_{S_{12}=0} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}\Bigg|_{S_{12}=0} = S_{11}$$

Thus, for this **unilateral** gain element, the matching network design problem decouples, and our matching network design simplifies to these two independent equations:

$$\Gamma_s = \Gamma_{in}^* \qquad \text{and} \qquad \Gamma_L = \Gamma_{out}^*$$