**EECS 723 - Microwave Engineering**

**Teacher:** "Bart, do you even know your multiplication tables?"

**Bart:** "Well, I know of them".

Like Bart and his multiplication tables, many electrical engineers know of the concepts of microwave engineering.

Concepts such as characteristic impedance, scattering parameters, Smith Charts and the like are familiar, but often we find that a complete, thorough and unambiguous understanding of these concepts can be somewhat lacking.

Thus, the goals of this class are for you to:

1. Obtain a complete, thorough, and unambiguous understanding of the fundamental concepts on microwave engineering.

2. Apply these concepts to the design and analysis of useful microwave devices.
2.1 - The Lumped Element Circuit Model for Transmission Lines

Reading Assignment: pp. 1-5, 49-52

The most important fact about microwave devices is that they are connected together using transmission lines.

Q: So just what is a transmission line?

A: A passive, linear, two port device that allows bounded E. M. energy to flow from one device to another.

→ Sort of an “electromagnetic pipe”!

Q: Oh, so it’s simply a conducting wire, right?

A: NO! At high frequencies, things get much more complicated!

HO: The Telegraphers Equations

HO: Time-Harmonic Solutions for Linear Circuits

Q: So, what complex functions $I(z)$ and $V(z)$ do satisfy both telegrapher equations?

A: The solutions to the transmission line wave equations!
**HO: The Transmission Line Wave Equations**

**Q:** Are the solutions for \( I(z) \) and \( V(z) \) completely independent, or are they related in any way?

**A:** The two solutions are related by the transmission line characteristic impedance.

**HO: The Transmission Line Characteristic Impedance**

**Q:** So what is the significance of the complex constant \( \gamma \)? What does it tell us?

**A:** It describes the propagation of each wave along the transmission line.

**HO: The Complex Propagation Constant**

**Q:** Now, you said earlier that characteristic impedance \( Z_0 \) is a complex value. But I recall engineers referring to a transmission line as simply a "50 Ohm line", or a "300 Ohm line". But these are real values; are they not referring to characteristic impedance \( Z_0 \)?

**A:** These real values are in fact some standard \( Z_0 \) values. They are real values because the transmission line is lossless (or nearly so!).

**HO: The Lossless Transmission Line**
Q: *Is characteristic impedance $Z_0$ the same as the concept of impedance I learned about in circuits class?*

A: *NO! The $Z_0$ is a wave impedance. However, we can also define line impedance, which is the same as that used in circuits.*

**HO: Line Impedance**

Q: *These wave functions $V^+(z)$ and $V^-(z)$ seem to be important. How are they related?*

A: *They are in fact very important! They are related by a function called the reflection coefficient.*

**HO: The Reflection Coefficient**

Q: *Does this mean I can describe transmission line activity in terms of (complex) voltage, current, and impedance, or alternatively in terms of an incident wave, reflected wave, and reflection coefficient?*

A: *Absolutely! A microwave engineer has a choice to make when describing transmission line activity.*

**HO: $V$, $I$, $Z$ OR $V^+$, $V^-$, $\Gamma$?**
The Telegrapher Equations

Consider a section of “wire”:

\[ i(z,t) \quad \rightarrow \quad i(z + \Delta z, t) \]

\[ v(z,t) \quad \rightarrow \quad v(z + \Delta z, t) \]

\[ + \quad + \]

\[ v(z,t) \quad \rightarrow \quad v(z + \Delta z, t) \]

\[ - \quad - \]

\[ \Delta z \]

Where:

\[ i(z,t) \neq i(z + \Delta z, t) \]

\[ v(z,t) \neq v(z + \Delta z, t) \]

Q: No way! Kirchoff’s Laws tells me that:

\[ i(z,t) = i(z + \Delta z, t) \]

\[ v(z,t) = v(z + \Delta z, t) \]

How can the voltage/current at the end of the line (at \( z + \Delta z \)) be different than the voltage/current at the beginning of the line (at \( z \))??
A: Way. The structure above actually exhibits some non-zero value of inductance, capacitance, conductance, and admittance! A more accurate transmission line model is therefore:

\[ v(z,t) + G \Delta z + L \Delta z i(z,t) + R \Delta z i(z,t) + \Delta v(z,t) = \frac{\partial i(z,t)}{\partial t} \neq 0 \]

and from KCL:

\[ i(z + \Delta z,t) - i(z,t) = -G \Delta z v(z,t) - C \Delta z \frac{\partial v(z,t)}{\partial t} \neq 0 \]

Where:
- \( R \): resistance/unit length
- \( L \): inductance/unit length
- \( C \): capacitance/unit length
- \( G \): conductance/unit length

\[ \therefore \text{ resistance of wire length } \Delta z \text{ is } R \Delta z \]
Dividing the first equation by $\Delta z$, and then taking the limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z, t)}{\partial z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z, t)}{\partial z} = -G v(z, t) - C \frac{\partial v(z, t)}{\partial t}$$

These coupled differential equations are quite famous! Derived by Oliver Heavyside, they are known as the telegrapher’s equations, and are essentially the Maxwell’s equations of transmission lines.

Although **mathematically** the functions $v(z, t)$ and current $i(z, t)$ can take any form, they can **physically** exist only if they satisfy the both of the differential equations shown above!
Time-Harmonic Solutions for Linear Circuits

There are an unaccountably infinite number of solutions \( v(z,t) \) and \( i(z,t) \) for the telegrapher’s equations! However, we can simplify the problem by assuming that the function of time is time harmonic (i.e., sinusoidal), oscillating at some radial frequency \( \omega \) (e.g., \( \cos \omega t \)).

Q: Why on earth would we assume a sinusoidal function of time? Why not a square wave, or triangle wave, or a “sawtooth” function?

A: We assume sinusoids because they have a very special property!

Sinusoidal time functions—and only a sinusoidal time functions—are the eigen functions of linear, time-invariant systems.

Q: ???

A: If a sinusoidal voltage source with frequency \( \omega \) is used to excite a linear, time-invariant circuit (and a transmission line is both linear and time invariant!), then the voltage at each
and every point with the circuit will likewise vary sinusoidally—at the same frequency $\omega$!

**Q:** *So what? Isn’t that obvious?*

**A:** Not at all! If you were to excite a linear circuit with a square wave, or triangle wave, or sawtooth, you would find that—generally speaking—nowhere else in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth. The linear circuit will effectively distort the input signal into something else!

![Graph](image)

**Q:** *Into what function will the input signal be distorted?*

**A:** It depends—both on the original form of the input signal, and the parameters of the linear circuit. At different points within the circuit we will discover different functions of time—unless, of course, we use a sinusoidal input. Again, for a sinusoidal excitation, we find at every point within circuit an undistorted sinusoidal function!

**Q:** *So, the sinusoidal function at every point in the circuit is exactly the same as the input sinusoid?*
A: Not quite exactly the same. Although at every point within the circuit the voltage will be precisely sinusoidal (with frequency $\omega$), the magnitude and relative phase of the sinusoid will generally be different at each and every point within the circuit.

Thus, the voltage along a transmission line—when excited by a sinusoidal source—must have the form:

$$v(z, t) = v(z) \cos(\omega t + \varphi(z))$$

Thus, at some arbitrary location $z$ along the transmission line, we must find a time-harmonic oscillation of magnitude $v(z)$ and relative phase $\varphi(z)$.

Now, consider Euler's equation, which states:

$$e^{j\psi} = \cos \psi + j \sin \psi$$

Thus, it is apparent that:

$$\text{Re} \{ e^{j\psi} \} = \cos \psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$v(z, t) = v(z) \cos(\omega t + \varphi(z))$$

$$= \text{Re} \{ v(z) e^{j(\omega t + \varphi(z))} \}$$

$$= \text{Re} \{ v(z) e^{j\varphi(z)} e^{j\omega t} \}$$
Thus, we can specify the time-harmonic voltage at each and every location $z$ along a transmission line with the complex function $V(z)$:

$$V(z) = v(z)e^{-j\phi(z)}$$

where the magnitude of the complex function is the magnitude of the sinusoid:

$$v(z) = |V(z)|$$

and the phase of the complex function is the relative phase of the sinusoid:

$$\phi(z) = \arg\{V(z)\}$$

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to explicitly write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any one location (such as at the excitation source) then this must be time function at all transmission line locations $z$!

The only unknown is the complex function $V(z)$. Once we determine $V(z)$, we can always (if we so desire) “recover” the real function $v(z,t)$ as:

$$v(z,t) = \text{Re}\{V(z)e^{j\omega t}\}$$
Thus, if we assume a time-harmonic source, finding the transmission line solution $v(z,t)$ reduces to solving for the complex function $V(z)$. 
The Transmission Line Wave Equation

Let’s assume that $v(z,t)$ and $i(z,t)$ each have the time-harmonic form:

$$v(z,t) = \text{Re}\{V(z)e^{j\omega t}\} \quad \text{and} \quad i(z,t) = \text{Re}\{I(z)e^{j\omega t}\}$$

The time-derivative of these functions are:

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\{j\omega V(z)e^{j\omega t}\}$$

$$\frac{\partial i(z,t)}{\partial t} = \text{Re}\left\{I(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\{j\omega I(z)e^{j\omega t}\}$$

Inserting these results into the telegrapher’s equations, we find:

$$\text{Re}\left\{\frac{\partial V(z)}{\partial z} e^{j\omega t}\right\} = \text{Re}\{-(R + j\omega L)I(z)e^{j\omega t}\}$$

$$\text{Re}\left\{\frac{\partial I(z)}{\partial z} e^{j\omega t}\right\} = \text{Re}\{-(G + j\omega C)V(z)e^{j\omega t}\}$$

Simplifying, we have the complex form of telegrapher’s equations:
\[
\frac{\partial V(z)}{\partial z} = -(R + j\omega L) I(z)
\]
\[
\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z)
\]

Note that these complex differential equations are not a function of time \( t \)!

* The functions \( I(z) \) and \( V(z) \) are complex, where the magnitude and phase of the complex functions describe the magnitude and phase of the sinusoidal time function \( e^{j\omega t} \).

* Thus, \( I(z) \) and \( V(z) \) describe the current and voltage along the transmission line, as a function as position \( z \).

* Remember, not just any function \( I(z) \) and \( V(z) \) can exist on a transmission line, but rather only those functions that satisfy the telegraphers equations.

Our task, therefore, is to solve the telegrapher equations and find all solutions \( I(z) \) and \( V(z) \)!
Q: So, what functions $I(z)$ and $V(z)$ do satisfy both telegrapher’s equations??

A: To make this easier, we will combine the telegrapher equations to form one differential equation for $V(z)$ and another for $I(z)$.

First, take the derivative with respect to $z$ of the first telegrapher equation:

$$\frac{\partial}{\partial z} \left\{ \frac{\partial V(z)}{\partial z} \right\} = -(R + j \omega L) I(z)$$

$$\frac{\partial^2 V(z)}{\partial z^2} = -(R + j \omega L) \frac{\partial I(z)}{\partial z}$$

Note that the second telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G + j \omega C) V(z)$$

Combining these two equations, we get an equation involving $V(z)$ only:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j \omega L)(G + j \omega C) V(z)$$

We can simplify this equation by defining the complex value $\gamma$:

$$\gamma = \sqrt{(R + j \omega L)(G + j \omega C)}$$
So that:

\[ \frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 V(z) \]

In a similar manner (i.e., begin by taking the derivative of the second telegrapher equation), we can derive the differential equation:

\[ \frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z) \]

We have decoupled the telegrapher's equations, such that we now have two equations involving one function only:

\[ \frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 V(z) \]
\[ \frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z) \]

These are known as the transmission line wave equations.

Note that value \( \gamma \) is complex, and is determined by taking the square-root of a complex value. Likewise, \( \gamma^2 \) is a complex value. Do you know how to square a complex number? Can you determine the square root of a complex number?
Note only special functions satisfy these wave equations; if we take the double derivative of the function, the result is the original function (to within a constant $\gamma^2$)!

A: Such functions do exist!

For example, the functions $V(z) = e^{\gamma z}$ and $V(z) = e^{-\gamma z}$ each satisfy this transmission line wave equation (insert these into the differential equation and see for yourself!).

Likewise, since the transmission line wave equation is a linear differential equation, a weighted superposition of the two solutions is also a solution (again, insert this solution to and see for yourself!):

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

In fact, it turns out that any and all possible solutions to the differential equations can be expressed in this simple form!

Therefore, the general solution to these complex wave equations (and thus the telegrapher equations) are:
\[ V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \]
\[ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \]

where \( V_0^+ \), \( V_0^- \), \( I_0^+ \), and \( I_0^- \) are complex constants.

\( \Rightarrow \) It is unfathomably important that you understand what this result means!

It means that the functions \( V(z) \) and \( I(z) \), describing the current and voltage at all points \( z \) along a transmission line, can always be completely specified with just four complex constants \( (V_0^+, V_0^-, I_0^+, I_0^-) \)!!

We can alternatively write these solutions as:

\[ V(z) = V^+(z) + V^-(z) \]
\[ I(z) = I^+(z) + I^-(z) \]

where:
\[ V^+(z) \doteq V_0^+ e^{-\gamma z} \quad V^-(z) \doteq V_0^- e^{+\gamma z} \]
\[ I^+(z) \doteq I_0^+ e^{-\gamma z} \quad I^-(z) \doteq I_0^- e^{+\gamma z} \]
The two terms in each solution describe two waves propagating in the transmission line, one wave \((V^+(z) \text{ or } I^+(z))\) propagating in one direction \((+z)\) and the other wave \((V^-(z) \text{ or } I^-(z))\) propagating in the opposite direction \((-z)\).

\[
V^-(z) = V_0^- e^{+\gamma z} \quad \quad V^+(z) = V_0^+ e^{-\gamma z}
\]

Q: So just what are the complex values \(V_0^+, V_0^-, I_0^+, I_0^-\)?

A: Consider the wave solutions at one specific point on the transmission line—the point \(z = 0\). For example, we find that:

\[
V^+(z = 0) = V_0^+ e^{-(\gamma z = 0)} = V_0^+ e^{(0)} = V_0^+ (1) = V_0^+
\]

In other words, \(V_0^+\) is simply the complex value of the wave function \(V^+(z)\) at the point \(z = 0\) on the transmission line!

Likewise, we find:
\[ V_0^- = V^-(z = 0) \]
\[ I_0^+ = I^+(z = 0) \]
\[ I_0^- = I^-(z = 0) \]

Again, the four complex values \( V_0^+, I_0^+, V_0^-, I_0^- \) are all that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, each of these four complex constants completely specifies one of the four transmission line wave functions \( V^+(z), I^+(z), V^-(z), I^-(z) \).

**Q:** But what determines these wave functions? How do we find the values of constants \( V_0^+, I_0^+, V_0^-, I_0^- \)?

**A:** As you might expect, the voltage and current on a transmission line is determined by the devices attached to it on either end (e.g., active sources and/or passive loads)!

The precise values of \( V_0^+, I_0^+, V_0^-, I_0^- \) are therefore determined by satisfying the **boundary conditions** applied at each end of the transmission line—much more on this later!
The Characteristic Impedance of a Transmission Line

So, from the telegrapher’s differential equations, we know that the complex current \( I(z) \) and voltage \( V(z) \) must have the form:

\[
V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}
\]
\[
I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}
\]

Let’s insert the expression for \( V(z) \) into the first telegrapher’s equation, and see what happens!

\[
\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R + j\omega L)I(z)
\]

Therefore, rearranging, \( I(z) \) must be:

\[
I(z) = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z})
\]
Q: But wait! I thought we already knew current $I(z)$. Isn’t it:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

How can both expressions for $I(z)$ be true??

A: Easy! Both expressions for current are equal to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z})$$

For the above equation to be true for all $z$, $I_0$ and $V_0$ must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \quad \text{and} \quad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the two propagating waves:

$$I^+(z) = \left(\frac{+\gamma}{R + j\omega L}\right) V^+(z) \quad \text{and} \quad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$
We find that:

$$\frac{\gamma}{R + j\omega L} = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{R + j\omega L} = \sqrt{G + j\omega C}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^+(z)}{I^+(z)} = \sqrt{R + j\omega L} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \sqrt{G + j\omega C}$$

**Q:** *What’s so startling about this conclusion?*

**A:** Note that although the magnitude and phase of each propagating wave is a **function** of transmission line position $z$ (e.g., $V^+(z)$ and $I^+(z)$), the **ratio** of the voltage and current of each wave is independent of position—a **constant** with respect to position $z$!

Although $V^+_0$ and $I^+_0$ are determined by **boundary conditions** (i.e., what’s connected to either end of the transmission line), the ratio $V^+_0/I^+_0$ is determined by the parameters of the transmission line **only** ($R, L, G, C$).

⇒ This ratio is an important **characteristic** of a transmission line, called its **Characteristic Impedance** $Z_0$. 
We can therefore describe the current and voltage along a transmission line as:

\[
V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}
\]

\[
I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}
\]

or equivalently:

\[
V(z) = Z_0 I_0^+ e^{-\gamma z} - Z_0 I_0^- e^{+\gamma z}
\]

\[
I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}
\]

Note that instead of characterizing a transmission line with real parameters \( R, G, L, \) and \( C, \) we can (and typically do!) describe a transmission line using complex parameters \( Z_0 \) and \( \gamma. \)
The Complex Propagation Constant $\gamma$

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as wave functions:

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$V^-(z) = V_0^- e^{+\gamma z}$$

where $\gamma$ is a complex constant that describe the properties of a transmission line. Since $\gamma$ is complex, we can consider both its real and imaginary components.

$$\gamma = \sqrt{(R + j \omega L)(G + j \omega C)} = \alpha + j \beta$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$V^+(z) = V_0^+ e^{-\gamma z} = V_0^+ e^{-(\alpha + j \beta) z} = V_0^+ e^{-\alpha z} e^{-j \beta z}$$

**Q:** What are these constants $\alpha$ and $\beta$? What do they physically represent?
A: Remember, a complex value can be expressed in terms of its **magnitude** and **phase**. For example:

\[ V_0^+ = |V_0^+| e^{j\phi_0} \]

Likewise:

\[ V^+(z) = |V^+(z)| e^{j\phi(z)} \]

And since:

\[ V^+(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} = |V_0^+| e^{j\phi_0} e^{-\alpha z} e^{-j\beta z} = |V_0^+| e^{-\alpha z} e^{j(\phi_0 - \beta z)} \]

we find:

\[ |V^+(z)| = |V_0^+| e^{-\alpha z} \quad \phi^+(z) = \phi_0 - \beta z \]

It is evident that \( e^{-\alpha z} \) alone determines the **magnitude** of wave \( V^+(z) = V_0^+ e^{-\alpha z} \) as a function of position \( z \).
Therefore, $\alpha$ expresses the attenuation of the signal due to the loss in the transmission line. The larger the value of $\alpha$, the greater the exponential attenuation.

**Q:** So what is the constant $\beta$? What does it physically mean?

**A:** Recall

$$\phi^+(z) = \phi_0^+ - \beta z$$

represents the relative phase of wave $V^+(z)$; a function of transmission line position $z$. Since phase $\phi$ is expressed in radians, and $z$ is distance (in meters), the value $\beta$ must have units of:

$$\beta = \frac{\phi}{z} \text{ radians/meter}$$

Thus, if the value $\beta$ is small, we will need to move a significant distance $\Delta z$ down the transmission line in order to observe a change in the relative phase of the oscillation.

Conversely, if the value $\beta$ is large, a significant change in relative phase can be observed if traveling a short distance $\Delta z_{2\pi}$ down the transmission line.

**Q:** How far must we move along a transmission line in order to observe a change in relative phase of $2\pi$ radians?

**A:** We can easily determine this distance ($\Delta z_{2\pi}$, say) from the transmission line characteristic $\beta$.

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \Delta z_{2\pi}$$
or, rearranging:

\[ \Delta z_{2\pi} = \frac{2\pi}{\beta} \quad \Rightarrow \quad \beta = \frac{2\pi}{\Delta z_{2\pi}} \]

The distance \( \Delta z_{2\pi} \) over which the relative phase changes by \( 2\pi \) radians, is more specifically known as the wavelength \( \lambda \) of the propagating wave (i.e., \( \lambda \neq \Delta z_{2\pi} \)):

\[
\lambda = \frac{2\pi}{\beta} \quad \Rightarrow \quad \beta = \frac{2\pi}{\lambda}
\]

The value \( \beta \) is thus essentially a spatial frequency, in the same way that \( \omega \) is a temporal frequency:

\[ \omega = \frac{2\pi}{T} \]

Note \( T \) is the time required for the phase of the oscillating signal to change by a value of \( 2\pi \) radians, i.e.:

\[ \omega T = 2\pi \]

And the period of a sinewave, and related to its frequency in Hertz (cycles/second) as:

\[ T = \frac{2\pi}{\omega} = \frac{1}{f} \]
Compare these results to:

\[ \beta = \frac{2\pi}{\lambda} \quad 2\pi = \beta \lambda \quad \lambda = \frac{2\pi}{\beta} \]

Q: So, just how fast does this wave propagate down a transmission line?

We describe wave velocity in terms of its phase velocity—in other words, how fast does a specific value of absolute phase \( \phi \) seem to propagate down the transmission line.

Since velocity is change in distance with respect to time, we need to first express our propagating wave in its real form:

\[
v^+ (z, t) = \text{Re} \{ V^+ (z) e^{-j\omega t} \} = |V_0^+| \cos (\omega t - \beta z + \phi_0^+) \]

Thus, the absolute phase is a function of both time and frequency:

\[
\phi^+ (z, t) = \omega t - \beta z + \phi_0^+ 
\]

Now let's set this phase to some arbitrary value of \( \phi_c \) radians.

\[
\omega t - \beta z + \phi_0^+ = \phi_c
\]

For every time \( t \), there is some location \( z \) on a transmission line that has this phase value \( \phi_c \). That location is evidently:
\[ z = \frac{\omega t + \phi_0^+ - \phi_c}{\beta} \]

Note as time increases, so to does the location \( z \) on the line where \( \phi^+(z, t) = \phi_c \).

The velocity \( v_p \) at which this phase point moves down the line can be determined as:

\[
v_p = \frac{dz}{dt} = \frac{d\left(\frac{\omega t + \phi_0^+ - \phi_c}{\beta}\right)}{dt} = \frac{\omega}{\beta}
\]

This wave velocity is the velocity of the propagating wave!

Note that the value:

\[
\frac{v_p}{\lambda} = \frac{\omega \beta}{\beta 2\pi} = \frac{\omega}{2\pi} = f
\]

and thus we can conclude that:

\[ v_p = f \lambda \]

as well as:

\[ \beta = \frac{\omega}{v_p} \]

Q: But these results were derived for the \( V^+(z) \) wave; what about the other wave \( V^-(z) \)?
A: The results are essentially the same, as each wave depends on the same value $\beta$.

The only subtle difference comes when we evaluate the phase velocity. For the wave $V^-(z)$, we find:

$$\phi^-(z, t) = \omega t + \beta z + \phi_0^-$$

Note the plus sign associated with $\beta z$!

We thus find that some arbitrary phase value will be located at location:

$$z = -\frac{\phi_0^- + \phi_c - \omega t}{\beta}$$

Note now that an increasing time will result in a decreasing value of position $z$. In other words this wave is propagating in the direction of decreasing position $z$—in the opposite direction of the $V^+(z)$ wave!

This is further verified by the derivative:

$$v_p = \frac{dz}{dt} = -\frac{d}{dt} \left( \frac{-\phi_0^- + \phi_c - \omega t}{\beta} \right) = -\frac{\omega}{\beta}$$

Where the minus sign merely means that the wave propagates in the $-z$ direction. Otherwise, the wavelength and velocity of the two waves are precisely the same!
The Lossless Transmission Line

Say a transmission line is **lossless** (i.e., \( R = G = 0 \)); the transmission line equations are then **significantly** simplified!

*Characteristic Impedance*

\[
Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}
\]

Note the characteristic impedance of a **lossless** transmission line is purely **real** (i.e., \( \text{Im}(Z_0) = 0 \)).

*Propagation Constant*

\[
\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(j\omega L)(j\omega C)} = \sqrt{-\omega^2 LC} = j\omega \sqrt{LC}
\]

The wave propagation constant is purely **imaginary**!
In other words, for a **lossless** transmission line:

\[ \alpha = 0 \quad \text{and} \quad \beta = \omega \sqrt{LC} \]

Note that since \( \alpha = 0 \), **neither** propagating wave is **attenuated** as they travel down the line—a wave at the **end** of the line is as large as it was at the **beginning**!

And this **makes sense**!

Wave attenuation occurs when **energy is extracted** from the propagating wave and turned into **heat**. This can **only** occur if resistance and/or conductance are present in the line. If \( R = G = 0 \), then **no attenuation** occurs—that why we call the line **lossless**.

**Voltage and Current**

The **complex functions** describing the magnitude and phase of the voltage/current at every location \( z \) along a transmission line are for a **lossless** line are:

\[
V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}
\]

\[
I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}
\]
**Wavelength and Phase Velocity**

We can now explicitly write the wavelength and propagation velocity of the two transmission line waves in terms of transmission line parameters \( L \) and \( C \):

\[
\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}} \quad \quad \quad \quad \quad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}
\]

Q: *Oh please, continue wasting my valuable time. We both know that a perfectly lossless transmission line is a physical impossibility.*

A: True! However, a low-loss line is possible—in fact, it is typical! If \( R \ll \omega L \) and \( G \ll \omega C \), we find that the lossless transmission line equations are excellent approximations!

Unless otherwise indicated, we will use the lossless equations to approximate the behavior of a low-loss transmission line.
The lone exception is when determining the attenuation of a long transmission line. For that case we will use the approximation:

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right)$$

where $Z_0 = \sqrt{L/C}$.

A summary of lossless transmission line equations

$$Z_0 = \sqrt{\frac{L}{C}} \quad \gamma = j\omega\sqrt{LC}$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \quad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

$$V^+(z) = V_0^+ e^{-j\beta z} \quad V^-(z) = V_0^- e^{+j\beta z}$$

$$\beta = \omega\sqrt{LC} \quad \lambda = \frac{1}{f\sqrt{LC}} \quad v_p = \frac{1}{\sqrt{LC}}$$
Line Impedance

Now let’s define line impedance $Z(z)$, a complex function which is simply the ratio of the complex line voltage and complex line current:

$$Z(z) = \frac{V(z)}{I(z)}$$

**Q:** Hey! I know what this is! The ratio of the voltage to current is simply the *characteristic impedance* $Z_0$, right???

**A:** NO! The line impedance $Z(z)$ is (generally speaking) NOT the transmission line characteristic impedance $Z_0$ !!!

→ **It is unfathomably important** that you understand this!!!!

To see why, recall that:

$$V(z) = V^+(z) + V^-(z)$$
And that:
\[ I(z) = \frac{V^+(z) - V^-(z)}{Z_0} \]

Therefore:
\[ Z(z) = \frac{V(z)}{I(z)} = Z_0 \left( \frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right) \neq Z_0 \]

Or, more specifically, we can write:
\[ Z(z) = Z_0 \left( \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} \right) \]

Q: I'm confused! Isn't:
\[ V^+(z)/I^+(z) = Z_0 \] ???

A: Yes! That is true! The ratio of the voltage to current for each of the two propagating waves is \( \pm Z_0 \). However, the ratio of the sum of the two voltages to the sum of the two currents is not equal to \( Z_0 \) (generally speaking)!

This is actually confirmed by the equation above. Say that \( V^-(z) = 0 \), so that only one wave \( (V^+(z)) \) is propagating on the line.
In this case, the ratio of the total voltage to the total current is simply the ratio of the voltage and current of the one remaining wave—the characteristic impedance $Z_0$!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \left(\frac{V^+(z)}{V^+(z)}\right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad \text{(when } V^-(z) = 0)$$

Q: So, it appears to me that characteristic impedance $Z_0$ is a transmission line parameter, depending only on the transmission line values $L$ and $C$.

Whereas line impedance is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$—values that depend not only on the transmission line, but also on the two things attached to either end of the transmission line!

Right!?

A: Exactly! Moreover, note that characteristic impedance $Z_0$ is simply a number, whereas line impedance $Z(z)$ is a function of position $(z)$ on the transmission line.
The Reflection Coefficient

So, we know that the transmission line voltage $V(z)$ and the transmission line current $I(z)$ can be related by the line impedance $Z(z)$:

$$V(z) = Z(z) I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Q: Piece of cake! I fully understand the concepts of voltage, current and impedance from my circuits classes. Let's move on to something more important (or, at the very least, more interesting).

Expressing the “activity” on a transmission line in terms of voltage, current and impedance is of course perfectly valid.

However, let us look closer at the expression for each of these quantities:
\[ V(z) = V^+(z) + V^-(z) \]

\[ I(z) = \frac{V^+(z) - V^-(z)}{Z_0} \]

\[ Z(z) = Z_0 \left( \frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right) \]

It is evident that we can alternatively express all “activity” on the transmission line in terms of the two transmission line waves \( V^+(z) \) and \( V^-(z) \).

\[ \text{Q: I know } V(z) \text{ and } I(z) \text{ are related by line impedance } Z(z): \]

\[ Z(z) = \frac{V(z)}{I(z)} \]

But how are \( V^+(z) \) and \( V^-(z) \) related?
A: Similar to line impedance, we can define a new parameter—the reflection coefficient $\Gamma(z)$—as the ratio of the two quantities:

$$\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)} \quad \Rightarrow \quad V^-(z) = \Gamma(z) V^+(z)$$

More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = \frac{V_0^-}{V_0^+} e^{+j2\beta z}$$

Note then, the value of the reflection coefficient at $z=0$ is:

$$\Gamma(z = 0) = \frac{V^-(z = 0)}{V^+(z = 0)} e^{+j2\beta(0)} = \frac{V_0^-}{V_0^+}$$

We define this value as $\Gamma_0$, where:

$$\Gamma_0 \doteq \Gamma(z = 0) = \frac{V_0^-}{V_0^+}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z}$$
So now we have two different but equivalent ways to describe transmission line activity!

We can use (total) voltage and current, related by line impedance:

\[ Z(z) = \frac{V(z)}{I(z)} \quad \therefore \quad V(z) = Z(z) I(z) \]

Or, we can use the two propagating voltage waves, related by the reflection coefficient:

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} \quad \therefore \quad V^-(z) = \Gamma(z) V^+(z) \]

These are equivalent relationships—we can use either when describing a transmission line.

Based on your circuits experience, you might well be tempted to always use the first relationship. However, we will find it useful (as well as simple) indeed to describe activity on a transmission line in terms of the second relationship—in terms of the two propagating transmission line waves!
**V, I, Z or V+, V-, \Gamma?**

**Q:** How do I choose *which* relationship to use when describing/analyzing transmission line activity? What if I make the *wrong* choice? How will I know if my analysis is correct?

**A:** Remember, the two relationships are equivalent. There is no explicitly wrong or right choice—*both* will provide you with precisely the same correct answer!

For example, we know that the total voltage and current can be determined from knowledge wave representation:

\[
V(z) = V^+(z) + V^+(z)
\]

\[
= V^+(z)(1 + \Gamma(z))
\]

\[
I(z) = \frac{V^+(z) - V^+(z)}{Z_0}
\]

\[
= \frac{V^+(z)(1 - \Gamma(z))}{Z_0}
\]
Or explicitly using the wave solutions $V^+(z) = V_0^+ e^{-j\beta z}$ and $V^-(z) = V_0^- e^{+j\beta z}$:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$
$$= V_0^+ (e^{-j\beta z} + \Gamma_0 e^{+j\beta z})$$

$$I(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}{Z_0}$$
$$= \frac{V_0^+ (e^{-j\beta z} - \Gamma_0 e^{+j\beta z})}{Z_0}$$

More importantly, we find that line impedance $Z(z) = V(z)/I(z)$ can be expressed as:

$$Z(z) = Z_0 \frac{V^+(z) + V^+(z)}{V^+(z) - V^+(z)}$$
$$= Z_0 \left( \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)$$

**Look** what happened—the line impedance can be completely and unambiguously expressed in terms of reflection coefficient $\Gamma(z)$!

More explicitly:

$$Z(z) = Z_0 \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} = Z_0 \frac{1 + \Gamma_0 e^{+j2\beta z}}{1 - \Gamma_0 e^{+j2\beta z}}$$
With a little algebra, we find likewise that the wave functions can be determined from $V(z), I(z)$ and $Z(z)$:

\[
V^+(z) = \frac{V(z) + I(z)Z_0}{2} = \frac{V(z)}{Z(z)} \left( \frac{Z(z) + Z_0}{2} \right)
\]

\[
V^-(z) = \frac{V(z) - I(z)Z_0}{2} = \frac{V(z)}{Z(z)} \left( \frac{Z(z) - Z_0}{2} \right)
\]

From this result we easily find that the reflection coefficient $\Gamma(z)$ can likewise be written directly in terms of line impedance:

\[
\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}
\]

Thus, the values $\Gamma(z)$ and $Z(z)$ are equivalent parameters— if we know one, then we can directly determine the other!
Q: So, if they are equivalent, why wouldn’t I always use the current, voltage, line impedance representation? After all, I am more familiar and more confident those quantities. The wave representation sort of scares me!

A: Perhaps I can convince you of the value of the wave representation.

Remember, the time-harmonic solution to the telegraphers equation simply boils down to two complex constants—$V_0^+$ and $V_0^-$. Once these complex values have been determined, we can describe completely the activity all points along our transmission line.

For the wave representation we find:

$$V^+(z) = V_0^+ e^{-j \beta z}$$

$$V^-(z) = V_0^+ e^{+j \beta z}$$

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{+j 2\beta z}$$
Note that the magnitudes of the complex functions are in fact constants (with respect to position $z$):

\[ |V^+(z)| = |V_0^+| \]
\[ |V^-(z)| = |V_0^-| \]
\[ |\Gamma(z)| = \left| \frac{V_0^-}{V_0^+} \right| \]

While the relative phase of these complex functions are expressed as a simple linear relationship with respect to $z$:

\[ \arg \{ V^+(z) \} = -\beta z \]
\[ \arg \{ V^-(z) \} = +\beta z \]
\[ \arg \{ \Gamma(z) \} = +2\beta z \]

Now, contrast this with the complex current, voltage, impedance functions:
\[ V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \]
\[ I(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}{Z_0} \]
\[ Z(z) = Z_0 \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} \]

With magnitude:
\[ |V(z)| = |V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}| = ?? \]
\[ |I(z)| = \frac{|V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}|}{Z_0} = ?? \]
\[ |Z(z)| = Z_0 \frac{|V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}|}{|V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}|} = ?? \]

and phase:
\[ \text{arg} \{V(z)\} = \text{arg} \{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\} = ?? \]
\[ \text{arg} \{I(z)\} = \text{arg} \{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\} = ?? \]
\[ \text{arg} \{Z(z)\} = \text{arg} \{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\} - \text{arg} \{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\} = ?? \]
A: Yes it is! However, this does \textbf{not} mean that we \textit{never} determine \(V(z)\), \(I(z)\), or \(Z(z)\); these quantities are still \textbf{fundamental} and very important—particularly at each \textbf{end} of the transmission line!

Q: It appears to me that when attempting to describe the activity along a transmission line—as a function of position \(z\)—it is much \textbf{easier} and more \textbf{straightforward} to use the wave representation.

\textit{Is my insightful conclusion \textbf{correct} (nyuck, nyuck, nyuck)?}