EECS 723-Microwave Engineering

Teacher: *"Bart, do you even **know** your multiplication tables?"*

Bart: *" Well, I know **of** them".*

Like Bart and his multiplication tables, many electrical engineers know **of** the concepts of microwave engineering.

Concepts such as characteristic impedance, scattering parameters, Smith Charts and the like are familiar, but often we find that a **complete**, **thorough** and **unambiguous** understanding of these concepts can be somewhat lacking.

Thus, the goals of this class are for **you** to:

1. Obtain a complete, thorough, and unambiguous understanding of the fundamental concepts on microwave engineering.

2. Apply these concepts to the **design** and **analysis** of useful microwave devices.

2.1 -The Lumped Element Circuit Model for Transmission Lines

Reading Assignment: *pp. 1-5, 48-51*

The most important fact about microwave devices is that they are connected together using transmission lines.

Q: *So just what is a **transmission line**?*

A: A passive, linear, two port device that allows bounded E. M. energy to flow from one device to another.

→ Sort of an "electromagnetic pipe" !

Q: *Oh, so it's simply a conducting **wire**, right?*

A: NO! At high frequencies, things get much more complicated!

HO: THE TELEGRAPHERS EQUATIONS

HO: TIME-HARMONIC SOLUTIONS FOR TRANSMISSION LINES

Q: *So, what complex functions $I(z)$ and $V(z)$ **do** satisfy both telegrapher equations?*

A: The solutions to the transmission line **wave equations**!

HO: The Transmission Line Wave Equations

Q: *Are the solutions for I(z) and V(z) completely* ***independent****, or are they* ***related*** *in any way ?*

A: The two solutions are related by the transmission line **characteristic impedance**.

HO: The Transmission Line Characteristic Impedance

Q: *So what is the significance of the complex constant γ? What does it tell us?*

A: It describes the **propagation** of each **wave** along the transmission line.

HO: The Complex Propagation Constant

Q: *Now,* ***you*** *said earlier that* ***characteristic*** *impedance Z_0 is a* ***complex*** *value. But I recall engineers referring to a transmission line as simply a "50 Ohm line", or a "300 Ohm line". But these are* ***real*** *values; are they* ***not*** *referring to characteristic impedance Z_0 ??*

A: These real values are in fact some **standard** Z_0 values. They are **real** values because the transmission line is **lossless** (or nearly so!).

HO: The Lossless Transmission Line

Q: *Is **characteristic** impedance Z_0 the same as the concept of impedance I learned about in **circuits** class?*

A: **NO!** The Z_0 is a **wave** impedance. However, we can also define **line impedance**, which **is** the same as that used in circuits.

HO: LINE IMPEDANCE

Q: *These **wave** functions $V^+(z)$ and $V^-(z)$ seem to be important. How are **they** related?*

A: They are in fact **very** important! They are related by a function called the **reflection coefficient**.

HO: THE REFLECTION COEFFICIENT

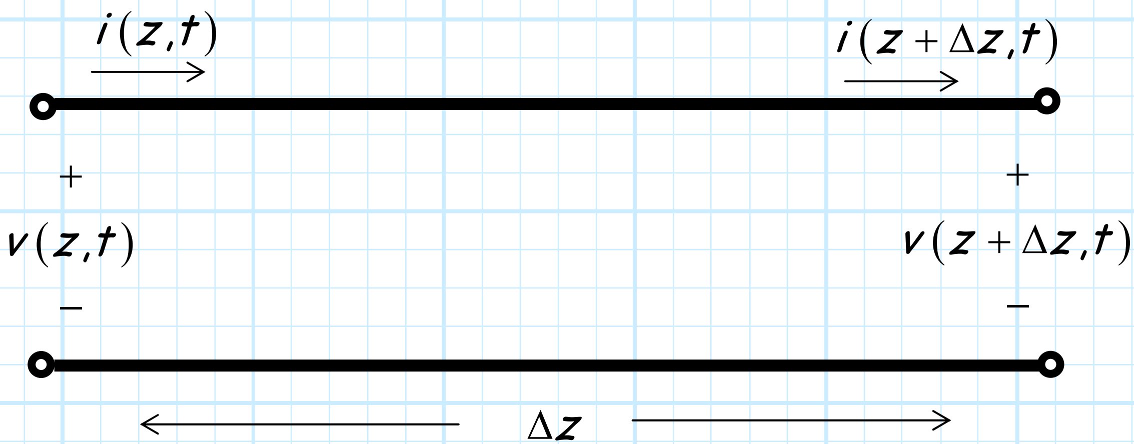
Q: *Does this mean I can describe transmission line activity in terms of (complex) voltage, current, and impedance, **or alternatively** in terms of an incident wave, reflected wave, and reflection coefficient?*

A: Absolutely! A microwave engineer has a **choice** to make when describing transmission line activity.

HO: V, I, Z OR V^+, V^-, Γ ?

The Telegrapher Equations

Consider a section of "wire":



Where:

$$i(z,t) \neq i(z+\Delta z,t)$$

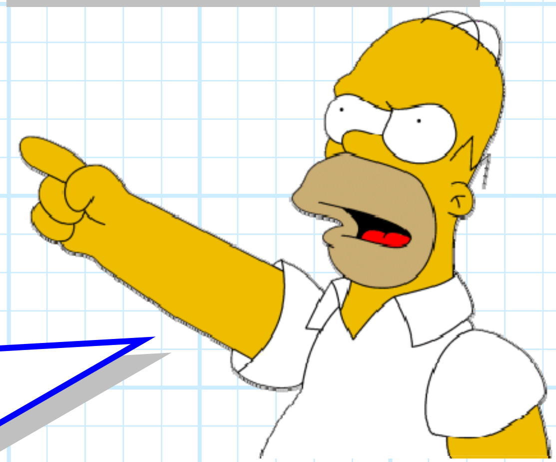
$$v(z,t) \neq v(z+\Delta z,t)$$

Q: *No way! Kirchoff's Laws tells me that:*

$$i(z,t) = i(z+\Delta z,t)$$

$$v(z,t) = v(z+\Delta z,t)$$

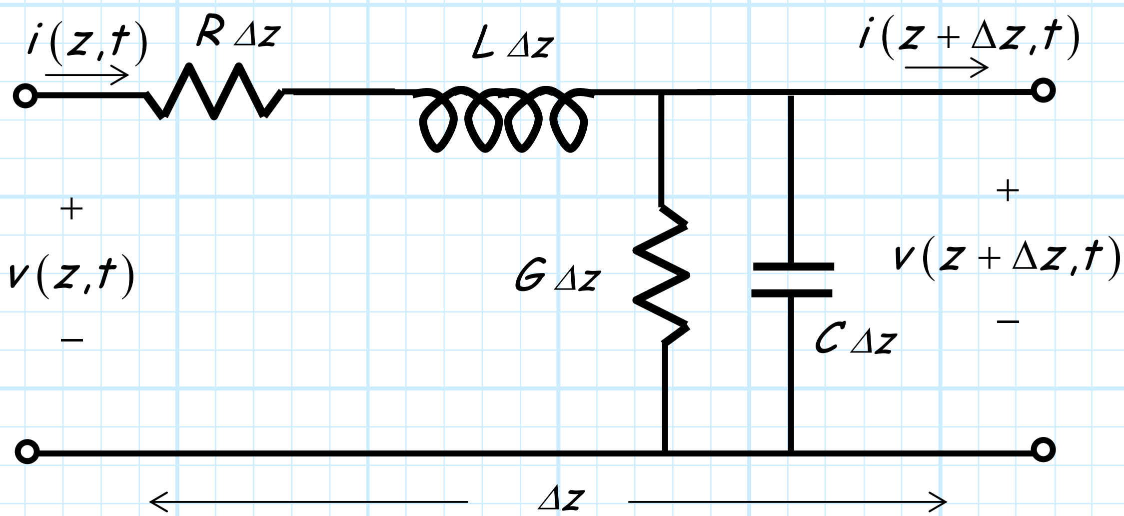
*How can the voltage/current at the **end** of the line (at $z+\Delta z$) be **different** than the voltage/current at the **beginning** of the line (at z)??*



A: Way. The structure above actually exhibits some non-zero value of **inductance**, **capacitance**, **conductance**, and **admittance**!

An Accurate Model

A more **accurate** transmission line model is:



Where:

R = resistance/unit length
L = inductance/unit length
C = capacitance/unit length
G = conductance/unit length

∴ resistance of wire length Δz is $R\Delta z$

Now evaluating **KVL**, we find:

$$v(z+\Delta z,t)-v(z,t)=-R\Delta z\, i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}\neq 0$$

and from **KCL**:

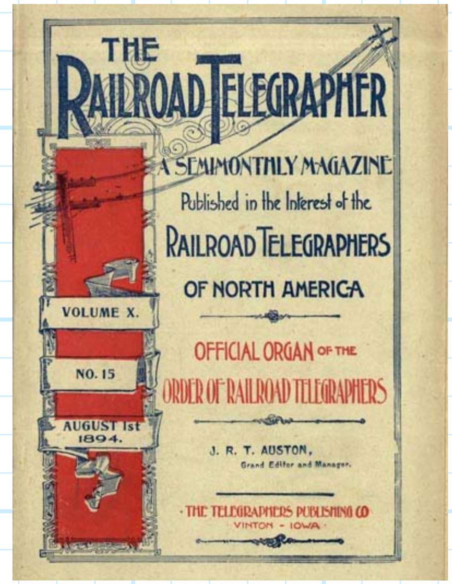
$$i(z+\Delta z,t)-i(z,t)=-G\Delta z\, v(z,t)-C\Delta z\frac{\partial v(z,t)}{\partial t}\neq 0$$

The Telegrapher's Equations

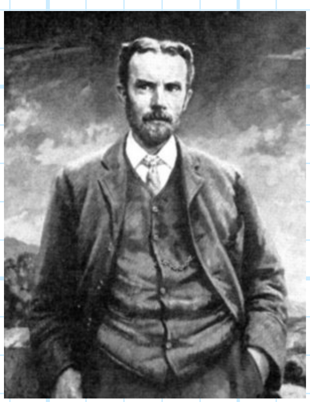
Dividing these equations by Δz, and then taking the **limit as** $\Delta z \rightarrow 0$, we find a set of **differential equations** that describe the voltage $v(z,t)$ and current $i(z,t)$ along a transmission line:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L \frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C \frac{\partial v(z,t)}{\partial t}$$



These equations are known as the **telegrapher's equations**.



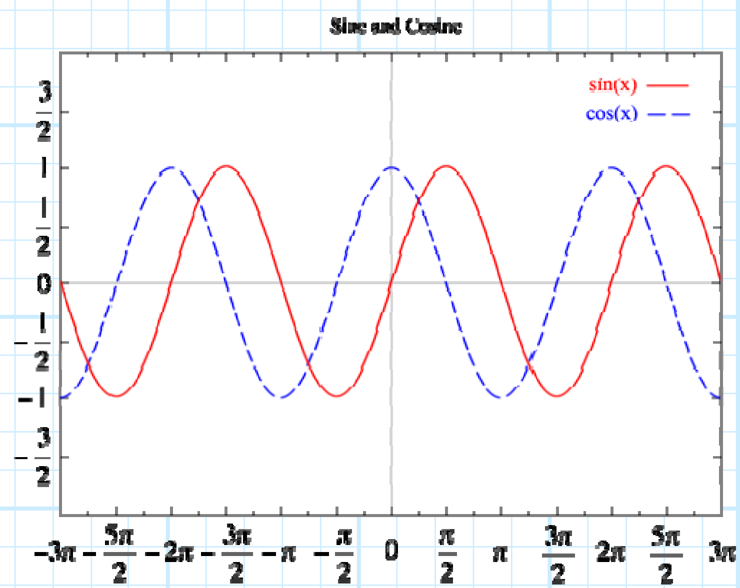
Derived by **Oliver Heavyside**, the telegrapher's equations are essentially the Maxwell's equations of transmission lines.

Although **mathematically** the functions $v(z,t)$ and current $i(z,t)$ can take any form, they can **physically** exist **only if** they satisfy the both of the differential equations shown above!

Time-Harmonic Solutions for Transmission Lines

There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations!

However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).



Q: *Why on earth would we assume a **sinusoidal** function of time?*

*Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!

Eigen Functions

Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

→ If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

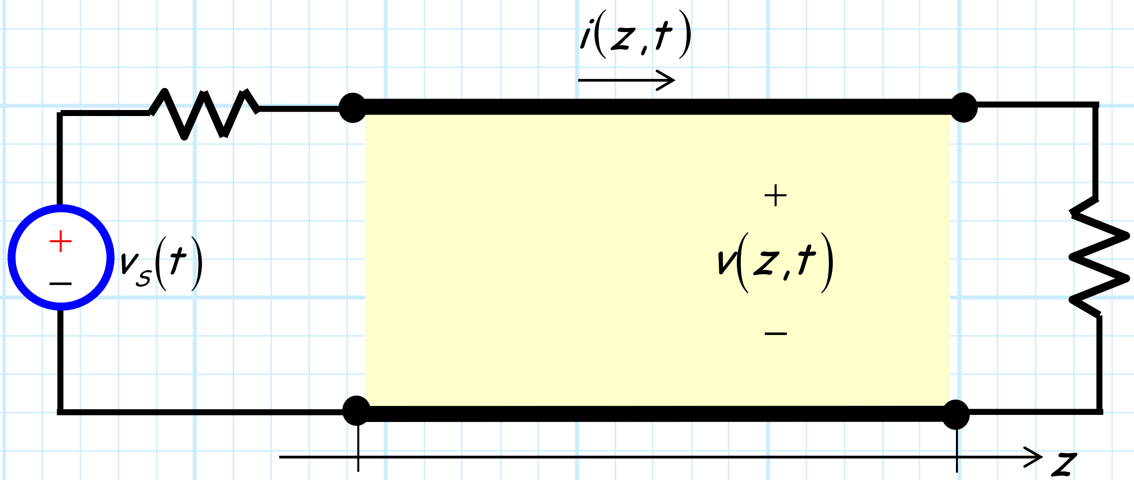
A: Not quite **exactly** the same.

Although at every point within the circuit the voltage will be **precisely** sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be **different** at each and every point within the circuit.

Eigen Functions and Transmission Lines

Thus, the voltage along a transmission line—**when** excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos(\omega t + \varphi(z))$$



In other words, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

For a given frequency ω, the **two functions** $v(z)$ and $\varphi(z)$ (functions of position z only!) **completely** describe the oscillating voltage at each and **every** point along the transmission line.

A Complex Representation of $v(z,t)$

Q: *I just thought of something!*

*Our sinusoidal oscillations are described by a magnitude ($v(z)$) and a phase ($\varphi(z)$)—but a complex value is **also** defined by its magnitude and phase (i.e., $c = |c| e^{j\varphi_c}$).*

*Is there a **connection** between our oscillations and a **complex value**?*

A: Absolutely! A connection made by **Euler's Identity**

$$e^{j\psi} = \cos\psi + j\sin\psi$$

From this it is apparent that:

$$\text{Re}\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the real **voltage on a transmission line** can be expressed as:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right) = \text{Re}\left\{v(z)\, e^{j(\omega t + \varphi(z))}\right\} = \text{Re}\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\}$$

The Complex Function V(z)

It is apparent that we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

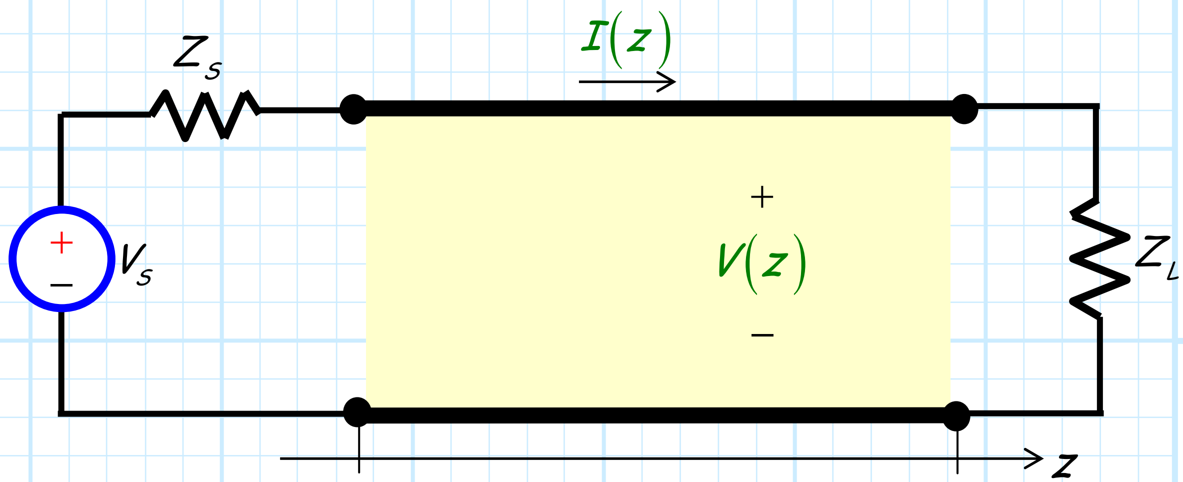
$$V(z) = v(z)\, e^{-j\varphi(z)}$$

So that:

$$v(z,t) = v(z)\cos(\omega t + \varphi(z)) = \text{Re}\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\} = \text{Re}\left\{V(z)\, e^{j\omega t}\right\}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = |V(z)|$$



and the **phase** of the complex function is the **relative phase** of the sinusoid :

$$\varphi(z) = \arg\{V(z)\}$$

All we need to determine is V(z)

Note then that only **unknown** is the **complex** function $V(z)$.

Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} = v(z)\cos\left(\omega t + \varphi(z)\right)$$

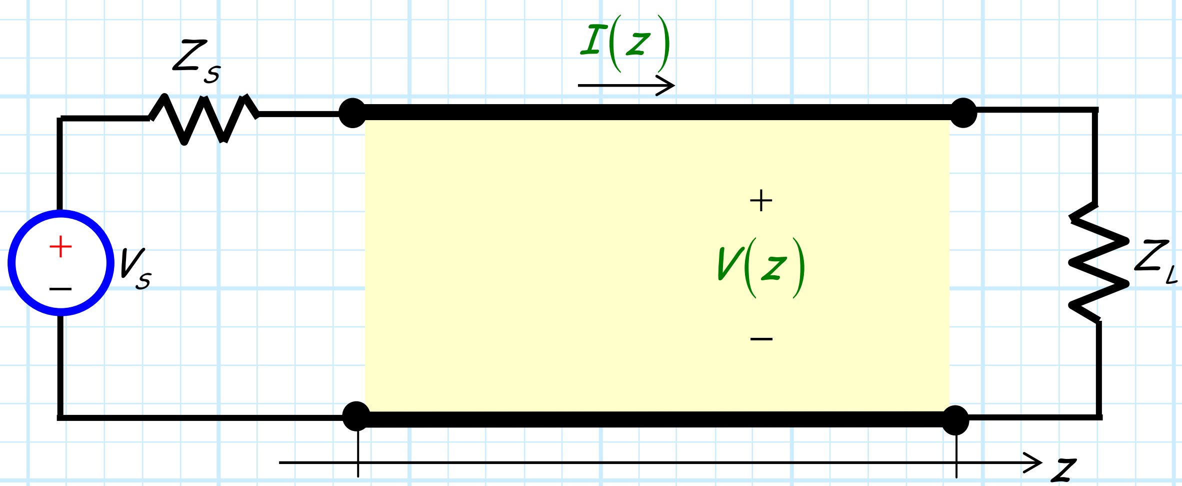
Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$!

Make this make sense to you

Microwave engineers almost **always** describe the activity of a transmission line (if excited by time harmonic sources) in terms of **complex functions of position** z — and **only** in terms of complex functions of position z !!

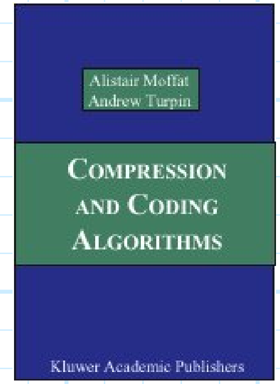
As a result, it is **unfathomably important** that you understand what these complex functions **mean**.

You **must understand** what these complex functions are telling you about the currents, voltages, etc. along a transmission line.



The Complex Function $V(z)$ and You

Perhaps it's helpful to think about these functions as sort of a **compression algorithm**, with the important information **"embedded"** in the complex values.



To recover the information, we simply take the magnitude and phase of these complex values.

$$V(z)$$

$$v(z) = |V(z)|$$

$$\varphi(z) = \arg\{V(z)\}$$

$$v(z,t) = v(z)\cos(\omega t + \varphi(z))$$

Note that the complex function $V(z)$ is a function of position z **only**!

Why we Love our Eigen Functions

Q: *Hey wait a minute! What happened to the time-harmonic function* $e^{j\omega t}$ *??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as the excitation source) then this must be time function at **all** transmission line locations z.

The only **unknown** is the **complex** function $V(z)$!

Once we determine $V(z)$, we can always (**if** we so desire) "recover" the **real** function $v(z,t)$ as:

$$\text{Re}\left\{V(z)\,e^{j\omega t}\right\} = v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$!!!

Quiz !!

See if **you** can determine what these complex values tell you about the **voltage** at different points z along a transmission line:

$V(z=0)=3$ → $v(z=0,t)= \quad \cos(\omega t + \quad)$

$V(z=1)=j$ → $v(z=1,t)= \quad \cos(\omega t + \quad)$

$V(z=2)=e^{j\pi/4}$ → $v(z=2,t)= \quad \cos(\omega t + \quad)$

$V(z=3)=-2$ → $v(z=3,t)= \quad \cos(\omega t + \quad)$

$V(z=4)=\sqrt{2}+j\sqrt{2}$ → $v(z=4,t)= \quad \cos(\omega t + \quad)$

$V(z=5)=3e^{-j\pi/4}$ → $v(z=5,t)= \quad \cos(\omega t + \quad)$

The Transmission Line Wave Equations

So let's assume that $v(z,t)$ and $i(z,t)$ each have the **time-harmonic** form:

$$v(z,t) = \text{Re}\left\{V(z)\, e^{j\omega t}\right\} \qquad \text{and} \qquad i(z,t) = \text{Re}\left\{I(z)\, e^{j\omega t}\right\}$$

The **time**-derivative of these **eigen** functions are easily determined. E.G., :

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega\, V(z)\, e^{j\omega t}\right\}$$

From this we can show that the **telegrapher equations** relate $I(z)$ and $V(z)$ as:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z) \qquad \frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

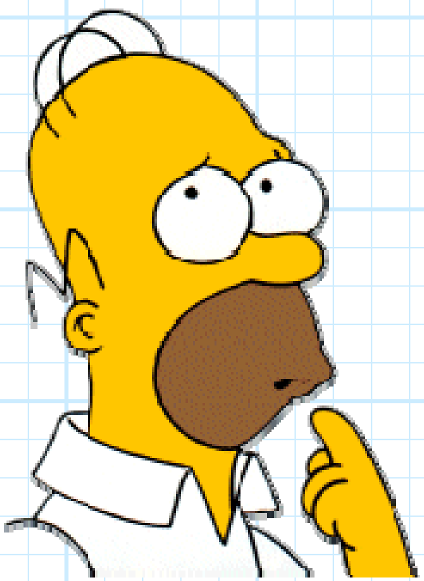
These are the **complex form** of the **telegrapher equations**.

What's your quest?

Note that these complex differential equations are **not** a function of **time** t !

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$



* The functions $I(z)$ and $V(z)$ are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position z.

* **Remember**, not just **any** function $I(z)$ and $V(z)$ can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.

Our quest, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

The Transmission Line Wave Equations

Q: *So, what functions $I(z)$ and $V(z)$* ***do*** *satisfy both telegrapher's equations??*

A: The complex telegrapher's equations are a pair of **coupled** differential equations.

With a little mathematical elbow grease, we can **decouple** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z)$$

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)$$

where

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

These equations are known as the transmission line **wave equations**. Since they each involve only **one** unknown function they are **easily** solved!

The (one and only) solution to the Wave Equations

The **solutions** to these wave equations are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants.**

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

The wave interpretation

We can **alternatively** write these solutions as:

$$V(z) = V^{+}(z) + V^{-}(z) \qquad I(z) = I^{+}(z) + I^{-}(z)$$

where:

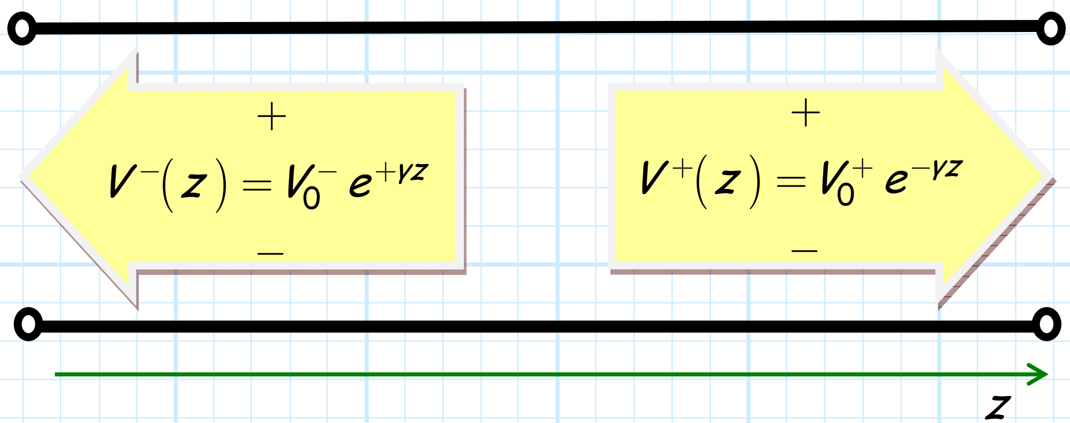
$$V^{+}(z) \doteq V_0^{+}\, e^{-\gamma z} \qquad V^{-}(z) \doteq V_0^{-}\, e^{+\gamma z}$$

$$I^{+}(z) \doteq I_0^{+}\, e^{-\gamma z} \qquad I^{-}(z) \doteq I_0^{-}\, e^{+\gamma z}$$

Q: *Just what do the two functions $V^{+}(z)$ and $V^{-}(z)$ **tell** us? Do they have any **physical meaning?***

A: An **incredibly important** physical meaning!

Function $V^{+}(z)$ describes a **wave propagating** in the direction of **increasing** z, and $V^{-}(z)$ describes a **wave** in the **opposite** direction.



Complex amplitudes

Q: *So just what **are** the complex values* $V_0^+, V_0^-, I_0^+, I_0^-$ *?*

A: They are called the **complex amplitudes** of each propagating **wave**.

Q: *Do they have any **physical meaning**?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point where $z=0$. We find that the **complex value** of the wave at that point is:

$$\begin{aligned} V^+(z=0) &= V_0^+ \, e^{-\gamma(z=0)} \\ &= V_0^+ \, e^{-(0)} \\ &= V_0^+ (1) \\ &= V_0^+ \end{aligned}$$

likewise:

$$V_0^- = V^-(z=0)$$

$$I_0^+ = I^+(z=0)$$

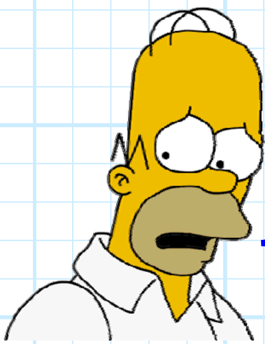
$$I_0^- = I^-(z=0)$$

So, the complex **wave amplitude** V_0^+ is simply the **complex** value of the wave function $V^+(z=0)$ at the point $z=0$ on the transmission line (that's what the **subscript** $_0$ means—the value at $z=0$)!

Determining the 4 complex wave amplitudes

Again, the **four** complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at **any and all** points on the transmission line!

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z), I^+(z), V^-(z), I^-(z)$.



Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*

A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active **sources** and/or passive **loads**)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!

The Characteristic Impedance of a Transmission Line

So, from the **telegrapher's** differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens**!

$$\frac{d\,V(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L)\, I(z)$$

Therefore, rearranging, current $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

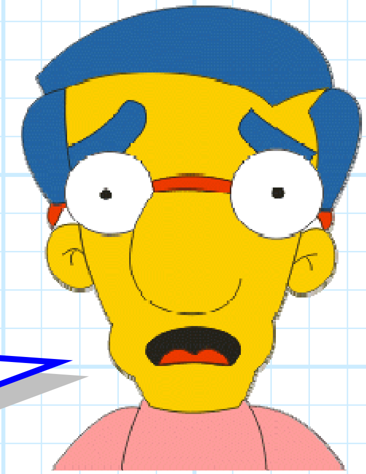
I thought we knew this?!

Q: ***But wait***! *I thought we already knew current* $I(z)$.

Isn't it:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for* $I(z)$ *be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \qquad \text{and} \qquad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

A startling conclusion

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{+\gamma}{R + j\omega L}\right) V^+(z) \quad \text{and} \quad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We find that:

$$\frac{\gamma}{R + j\omega L} = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{R + j\omega L} = \sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^+(z)}{I^+(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Characteristic Impedance

Q: *What's so startling about **this** conclusion?*

A: Note that although each propagating wave is a **function** of transmission line **position** z (e.g., $V^{+}(z)$ and $I^{+}(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's **connected** to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the **transmission line only** (i.e., R, L, G, C).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance Z_0**.

$$Z_0 \doteq \frac{V^{+}(z)}{I^{+}(z)} = \frac{-V_0^{-}(z)}{I_0^{-}(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

An alternative transmission line description

We can therefore describe the **current and voltage** along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} \, e^{-\gamma z} - \frac{V_0^-}{Z_0} \, e^{+\gamma z}$$

or equivalently:

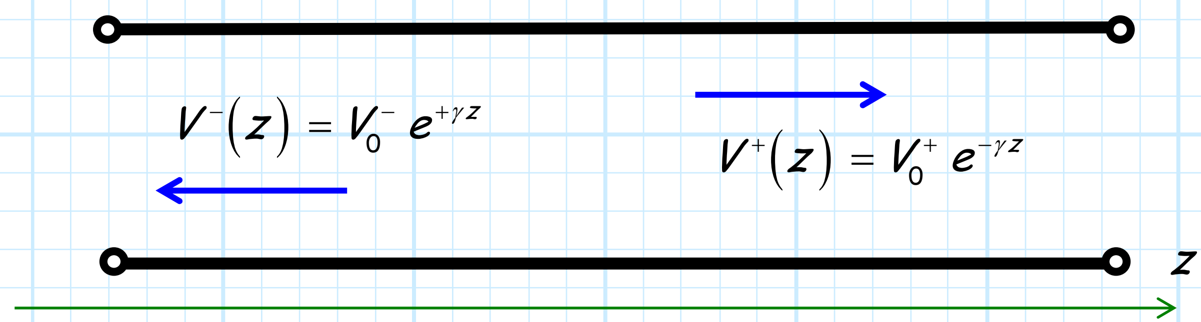
$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

→ Note that instead of characterizing a transmission line with **real** parameters R, G, L, and C, we can (and typically do!) describe a transmission line using **complex** parameters Z_0 and γ.

Complex Propagation Constant γ

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as **wave** functions:



where γ is a **complex constant** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \doteq \alpha + j\beta$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$V^+(z) = V_0^+ \, e^{-\gamma z} = V_0^+ \, e^{-(\alpha + j\beta)z} = V_0^+ \, e^{-\alpha z} \, e^{-j\beta z}$$

The value α

Q: *What **are** these constants α and β? What do they **physically** represent?*

A: Remember, a complex value can be expressed in terms of its **magnitude** and **phase**.

For example:

$$V_0^+ = \left|V_0^+\right| e^{j\varphi_0^+}$$

Likewise:

$$V^+(z) = \left|V^+(z)\right| e^{j\varphi^+(z)}$$

And since:

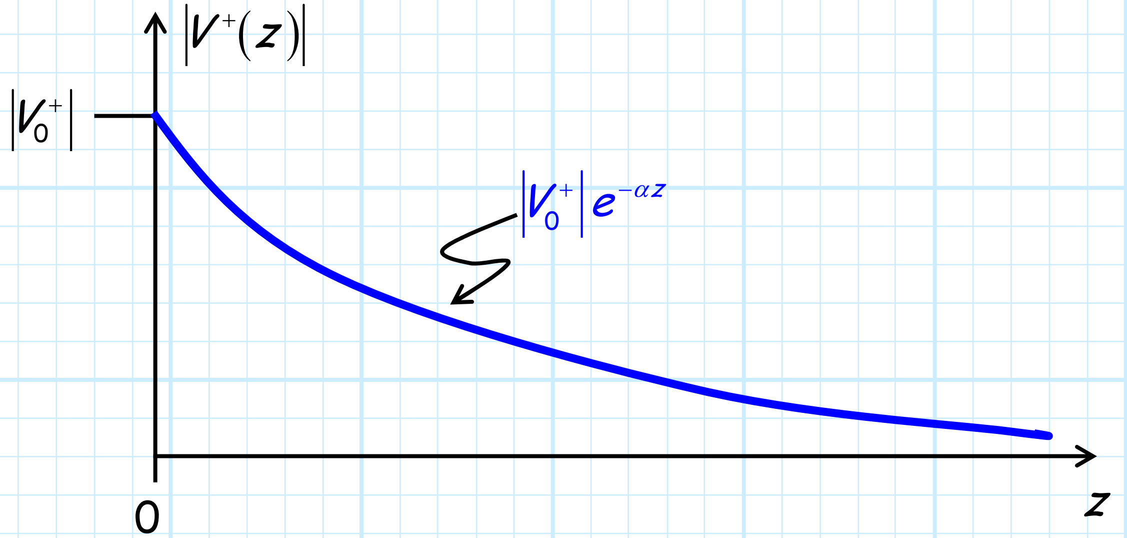
$$\begin{aligned} V^+(z) &= V_0^+ \, e^{-\alpha z} \, e^{-j\beta z} \\ &= \left|V_0^+\right| e^{j\varphi_0^+} \, e^{-\alpha z} \, e^{-j\beta z} \\ &= \left|V_0^+\right| e^{-\alpha z} \, e^{j\left(\varphi_0^+ - \beta z\right)} \end{aligned}$$

we find:

$$\left|V^+(z)\right| = \left|V_0^+\right| e^{-\alpha z} \qquad\qquad \varphi^+(z) = \varphi_0^+ - \beta z$$

The value α specifies attenuation

It is thus evident that $e^{-\alpha z}$ **alone** determines the **magnitude** of wave $V^+(z) = V_0^+ e^{-\gamma z}$ as a function of position z.



Therefore, α expresses the **attenuation** of the signal due to the **loss** in the transmission line.

The **larger** the value of α, the **greater** the exponential attenuation.

Q: *So just **why** does the wave attenuate as it propagates down the transmission line?*

A:

The value β

Q: *So what **is** the constant β? What does **it** physically mean?*

A: Recall the function:

$$\varphi^{+}(z) = \varphi_0^{+} - \beta z$$

represents the relative **phase** of wave $V^{+}(z)$; a **function** of transmission line **position** z.

Since phase φ is expressed in **radians**, and z is distance (in meters), the value β must have **units** of:

$$\beta = \frac{\varphi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

Thus, if the value β is **small**, we will need to move a **significant distance** Δz down the transmission line in order to observe a change in the relative phase of the oscillation.

Conversely, if the value β is **large**, a significant change in relative phase can be observed if traveling a **short** distance $\Delta z_{2\pi}$ down the transmission line.

The Wavelength λ

Q: *How far must we move along a transmission line in order to observe a change in relative phase of 2π radians?*

A: We can easily determine this distance ($\Delta z_{2\pi}$, say) from the transmission line characteristic β.

$$2\pi = \varphi(z + \Delta z_{2\pi}) - \varphi(z) = \beta \, \Delta z_{2\pi}$$

or, rearranging:

$$\Delta z_{2\pi} = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\Delta z_{2\pi}}$$

The **distance** $\Delta z_{2\pi}$ over which the relative phase changes by 2π **radians**, is more specifically known as the **wavelength** λ of the propagating wave (i.e., $\lambda \doteq \Delta z_{2\pi}$):

$$\lambda = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\lambda}$$

β is Spatial Frequency

The value β is thus essentially a **spatial frequency**, in the same way that ω is a **temporal** frequency:

$$\omega = \frac{2\pi}{T}$$

Note T is the **time** required for the phase of the oscillating signal to change by a value of 2π radians, i.e.:

$$\omega T = 2\pi$$

And the **period** of a sinewave, and related to its **frequency** in Hertz (cycles/second) as:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Compare these results to:

$$\beta = \frac{2\pi}{\lambda} \qquad 2\pi = \beta\lambda \qquad \lambda = \frac{2\pi}{\beta}$$

Propagation Velocity

Q: *So, just how **fast** does this wave propagate down a transmission line?*

A: We describe wave velocity in terms of its **phase velocity**—in other words, how **fast** does a specific value of absolute phase φ seem to **propagate** down the transmission line.

It can be shown that this velocity is:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

From this we can conclude:

$$\boxed{v_p = f\lambda}$$

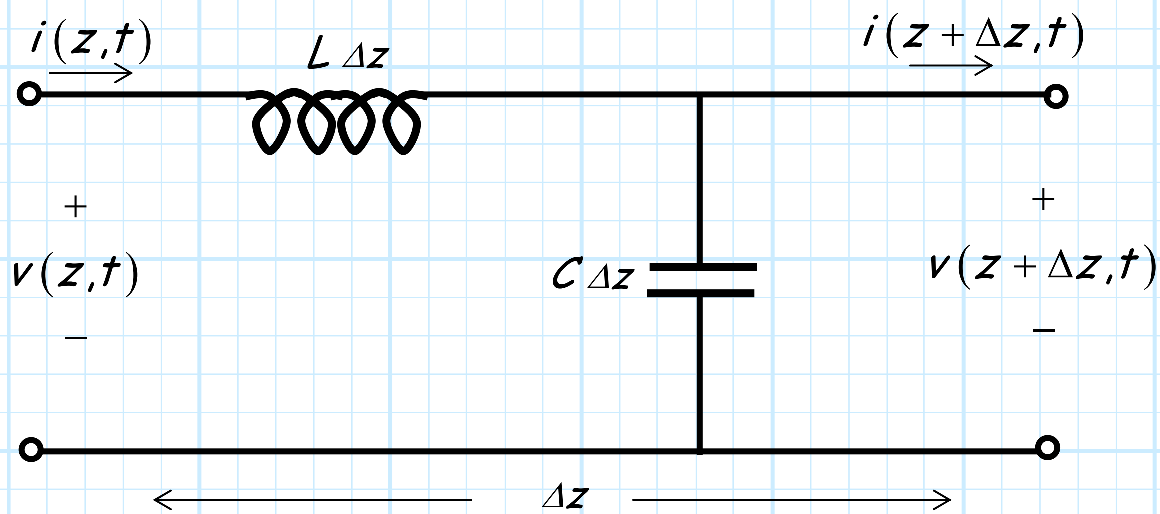
as well as:

$$\boxed{\beta = \frac{\omega}{v_p}}$$

The Lossless Transmission Line

Say a transmission line is **lossless** (i.e., $R = G = 0$).

Thus, this lossless transmission line is a **purely reactive** two port device—it exhibits only **capacitance** and **inductance**!!!



As a result, the transmission line equations are then **significantly** simplified!

The characteristic impedance of the lossless transmission line

For example, the **characteristic impedance** of a lossless lines simply becomes:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

Ironically, the characteristic **impedance** of a **lossless** (i.e., purely reactive) transmission line is—purely **real**!

The propagation constant

Moreover, the **propagation constant** of a lossless line is purely **imagingary**:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(j\omega L)(j\omega C)} = \sqrt{-\omega^2 LC} = j\,\omega\sqrt{LC}$$

In other words, for a **lossless** transmission line:

$$\alpha = 0 \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

Note that since $\alpha = 0$, **neither** propagating wave is **attenuated** as they travel down the line—a wave at the **end** of the line is as large as it was at the **beginning**!

→ And this **makes sense**!

Wave attenuation occurs when **energy is extracted** from the propagating wave and turned into **heat**.

This can **only** occur if resistance and/or conductance are present in the line.

If $R = G = 0$, then **no attenuation** occurs—that why we call the line **lossless**.

Velocity and Wavelength

The **complex functions** describing the magnitude and phase of the **voltage/current** at every location z along a transmission line are for a **lossless** line are:

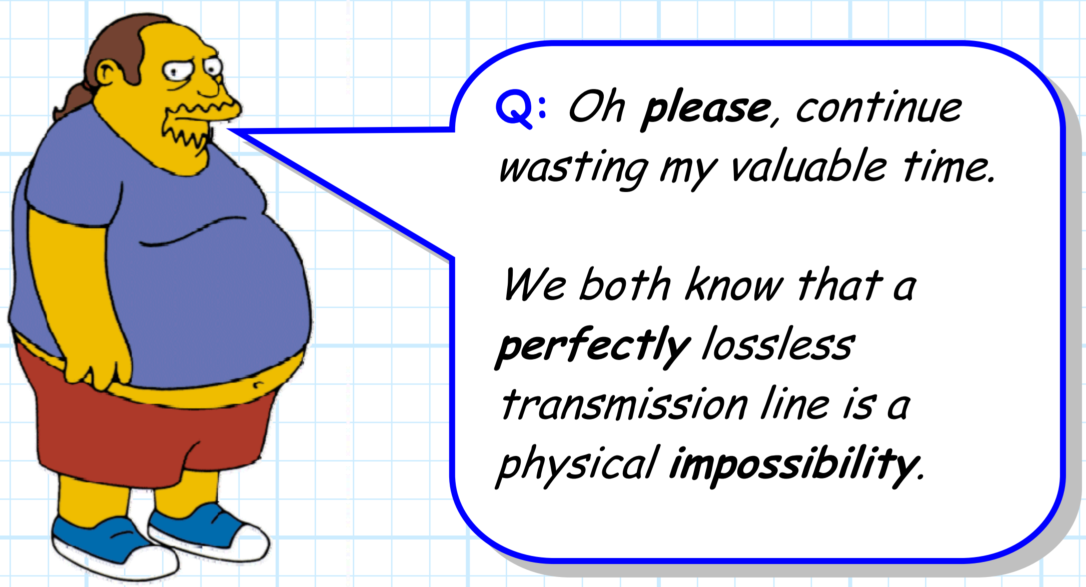
$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

We can now **explicitly** write the **wavelength** and propagation **velocity** of the two transmission line waves in terms of transmission line parameters L and C:

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}} \qquad\qquad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

The low-loss approximation



A: True! However, a **low-loss** line **is** possible—in fact, it is **typical**!

If $R \ll \omega L$ and $G \ll \omega C$, we find that the lossless transmission line equations are excellent **approximations**!

Unless **otherwise** indicated, **we will use the lossless equations** to **approximate** the behavior of a **low-loss** transmission line.

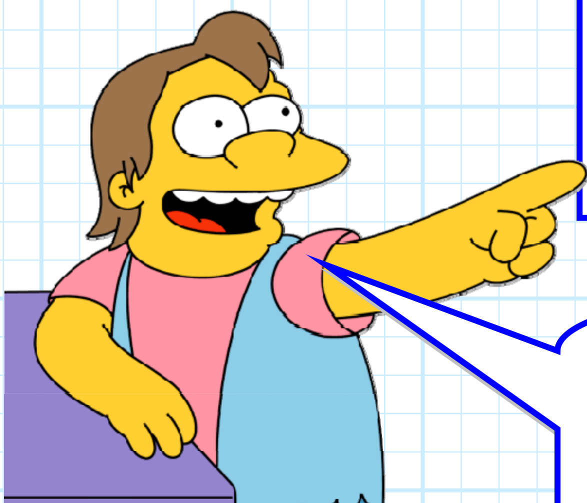
The lone **exception** is when determining the attenuation of a **long** transmission line. For that case we will use the approximation:

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right)$$

where $Z_0 = \sqrt{L/C}$.

Line Impedance

Now let's define **line impedance** $Z(z)$, a **complex** function which is simply the ratio of the complex line **voltage** and complex line **current**:



$$Z(z) = \frac{V(z)}{I(z)}$$

Q: *Hey! I know what this is!*

*The ratio of the voltage to current is simply the **characteristic impedance** Z_0, right ???*

A: **NO!** The line impedance $Z(z)$ is (generally speaking) **NOT** the transmission line **characteristic** impedance Z_0 !!!

→ It is **unfathomably important** that you understand this!!!! ←

Why Line Impedance is not Z_0

To see why **line** impedance $Z(z)$ is different than **characteristic** impedance Z_0, recall that:

$$V(z) = V^+(z) + V^-(z) \qquad \text{and that} \qquad I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

Therefore, line impedance is:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)}\right) \neq Z_0$$

Or, **more specifically**, we can write:

$$Z(z) = Z_0\left(\frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}\right)$$

What then is Z_0 ?

Q: *I'm confused! Isn't:*

$$V^{+}(z)/I^{+}(z)=Z_0\ ???$$

A: Yes! That **is** true! The ratio of the voltage to current for **each** of the two propagating waves **is** $\pm Z_0$.

However, the ratio of the **sum** of the two voltages, to the **sum** of the two currents, is **not** equal to Z_0 (generally speaking)!

→ This is actually confirmed by the expression of $Z(z)$ above.

Say that $V^{-}(z)=0$, so that only **one** wave ($V^{+}(z)$) is propagating on the line.

In this case, the ratio of the **total** voltage to the total current is simply the ratio of the voltage and current of the **one** remaining wave—the **characteristic impedance** Z_0!

$$Z(z)=\frac{V(z)}{I(z)}=Z_0\left(\frac{V^{+}(z)}{V^{+}(z)}\right)=\frac{V^{+}(z)}{I^{+}(z)}=Z_0 \quad \text{(when } V^{-}(z)=0\text{)}$$

Let's Summarize!!

Q: *So, it appears to me that characteristic impedance Z_0 is a* ***transmission line parameter****, depending* ***only*** *on the transmission line values L and C.*

Whereas ***line impedance*** *is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$—values that depend* ***not only*** *on the transmission line, but also on the two things* ***attached*** *to either* ***end*** *of the transmission line!*

Right !?

A: **Exactly!**

Moreover, note that characteristic impedance Z_0 is simply a **number**, whereas line impedance $Z(z)$ is a **function** of position (z) on the transmission line.

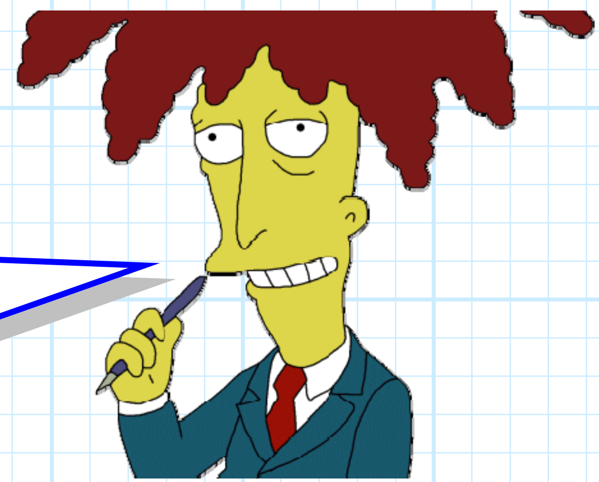
The Reflection Coefficient

So, we know that the transmission line **voltage** $V(z)$ and the transmission line **current** $I(z)$ can be related by the **line impedance** $Z(z)$:

$$V(z) = Z(z)\, I(z) \qquad \text{or equivalently} \qquad I(z) = \frac{V(z)}{Z(z)}$$

Q: *Piece of cake! I fully understand the concepts of **voltage**, **current** and **impedance** from my **circuits** classes.*

*Let's move on to something more important (or, at the very least, more **interesting**).*



Expressing the "activity" on a transmission line in terms of **voltage**, **current** and **impedance** is of course **perfectly** valid.

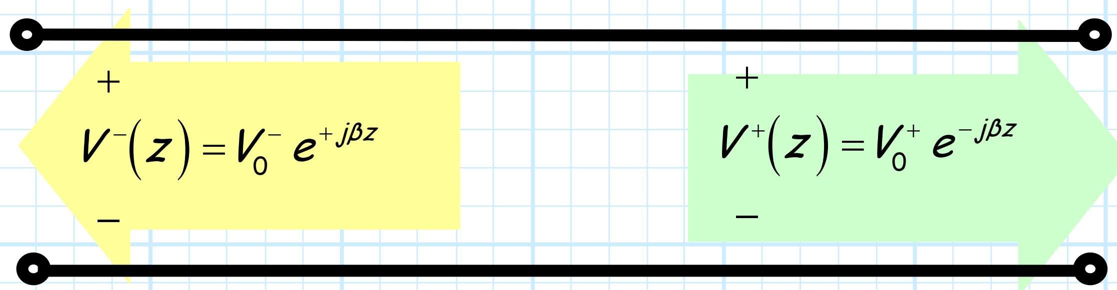
→ However, there is an **alternative** (and much simpler!) way to describe transmission line activity !!!!

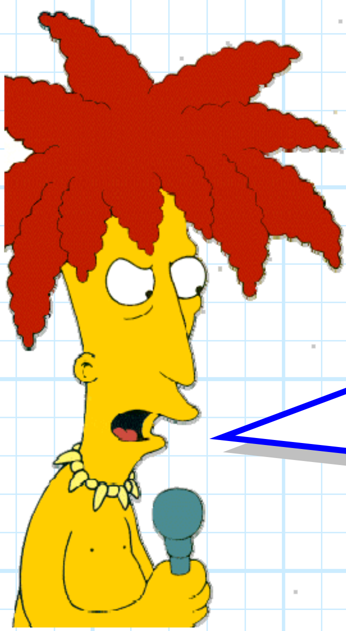
Wave Functions $V^+(z)$ and $V^-(z)$ Describe All!

Look closely at the expressions for **voltage**, **current**, and **impedance**:

$$V(z) = V^+(z) + V^-(z) \qquad I(z) = \frac{V^+(z) - V^-(z)}{Z_0} \qquad Z(z) = Z_0\left(\frac{V^+(z)+V^-(z)}{V^+(z)-V^-(z)}\right)$$

It is evident that we can **alternatively** express all "activity" on the transmission line in terms of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.





Q: *I know $V(z)$ and $I(z)$ are **related** by **line impedance** $Z(z)$:*

$$Z(z) = \frac{V(z)}{I(z)}$$

But how are $V^+(z)$ and $V^-(z)$ related?

The Reflection Coefficient Function

A: Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$—as the **ratio** of the two quantities:

$$\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)} \quad \Rightarrow \quad V^-(z) = \Gamma(z)\,V^+(z)$$

More **specifically**, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z}} = \frac{V_0^-}{V_0^+} \, e^{+j2\beta z}$$

Note then, the value of the reflection coefficient at ***z* = 0** is:

$$\Gamma(z=0) = \frac{V^-(z=0)}{V_0^+(z=0)} \, e^{+j2\beta(0)} = \frac{V_0^-}{V_0^+}$$

The Value Γ_0

We define this value as Γ_0, where:

$$\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}$$

Note then that we can **alternatively** write $\Gamma(z)$ as:

$$\Gamma(z) = \Gamma_0 \; e^{+j2\beta z}$$

So we have **two different,** but equivalent ways, to describe transmission line activity!

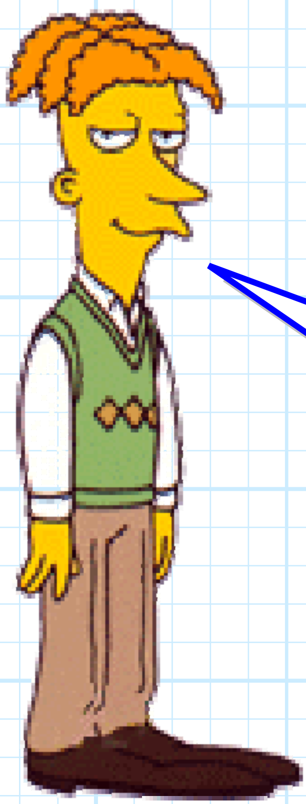
We **can** use (total) **voltage** and **current**, related by **line impedance**:

$$Z(z) = \frac{V(z)}{I(z)} \quad \therefore \quad V(z) = Z(z)\, I(z)$$

Or, ...

The Wave Description of Transmission Line Activity

.......we can use the two propagating **voltage waves**, related by the **reflection coefficient**:



$$\Gamma(z) = \frac{V^{-}(z)}{V^{+}(z)} \quad \therefore \quad V^{-}(z) = \Gamma(z)\, V^{+}(z)$$

These are **equivalent** relationships—we can use **either** when describing a transmission line.

*Based on your **circuits** experience, you might well be **tempted** to always use the **first** relationship.*

*However, we will find it useful (as well as simple) indeed to describe activity on a transmission line in terms of the **second** relationship—in terms of the **two** propagating transmission line **waves**!*

V,I,Z or V^+,V^-,Γ ?

Q: *How do I choose **which** relationship to use when describing/analyzing transmission line activity? What if I make the **wrong** choice? How will I know **if** my analysis is correct?*

A: Remember, the two relationships are **equivalent**.

There is **no** explicitly wrong or right choice—**both** will provide you with precisely the **same** correct answer!

For example, we know that the total voltage and current can be determined from knowledge wave representation:

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V^+(z)\big(1+\Gamma(z)\big) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V^+(z) - V^-(z)}{Z_0} \\ &= \frac{V^+(z)\big(1-\Gamma(z)\big)}{Z_0} \end{aligned}$$

A direct mapping from Z to Γ

More importantly, we find that **line impedance** $Z(z) = V(z)/I(z)$ can be expressed as:

$$
\begin{aligned}
Z(z) &= Z_0 \, \frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \\
&= Z_0 \left(\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)
\end{aligned}
$$

→ **Look** what happened—the line impedance can be **completely** and unambiguously expressed in terms of **reflection coefficient** $\Gamma(z)$!

And a mapping from Γ to Z

With a little **algebra**, we find likewise that the wave functions can be determined from $V(z), I(z)$ and $Z(z)$:

$$V^{+}(z) = \frac{V(z) + I(z)Z_0}{2} = \frac{V(z)}{Z(z)}\left(\frac{Z(z) + Z_0}{2}\right)$$

$$V^{-}(z) = \frac{V(z) - I(z)Z_0}{2} = \frac{V(z)}{Z(z)}\left(\frac{Z(z) - Z_0}{2}\right)$$

From this result we easily find that the reflection coefficient $\Gamma(z)$ can **likewise** be written directly in terms of **line impedance**:

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

The two representations are equivalent!

Thus, the values $\Gamma(z)$ and $Z(z)$ are **equivalent** parameters—if we know **one**, then we can directly determine the **other**—each is dependent on transmission line parameters (*L,C,R,G*) **only**!

Q: *So, if they are equivalent, why wouldn't I* ***always*** *use the current, voltage, line impedance representation?*

After all, I am more ***familiar*** *and more confident those quantities.*

The ***wave*** *representation sort of* ***scares*** *me!*

A: Perhaps I can **convince** you of the value of the **wave** representation.

Remember, the time-harmonic solution to the telegraphers equation simply boils down to **two complex constants**—V_0^+ and V_0^-.

Once these complex values have been determined, we can describe **completely** the activity **all** points along our transmission line.

Look how simple this is!

For the **wave representation** we find:

$$V^+(z) = V_0^+ \, e^{-j\beta z} \qquad V^-(z) = V_0^+ \, e^{+j\beta z} \qquad \Gamma(z) = \frac{V_0^-}{V_0^+} e^{+j2\beta z}$$

Note that the **magnitudes** of the complex functions are in fact **constants** (with respect to position z):

$$|V^+(z)| = |V_0^+| \qquad |V^-(z)| = |V_0^+| \qquad |\Gamma(z)| = \left|\frac{V_0^-}{V_0^+}\right|$$

While the **relative phase** of these complex functions are expressed as a **simple** linear relationship with respect to z:

$$\arg\{V^+(z)\} = -\beta z \qquad \arg\{V^-(z)\} = +\beta z \qquad \arg\{\Gamma(z)\} = +2\beta z$$

Yuck!

Now, **contrast** this with the complex **current**, **voltage**, **impedance** functions:

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

$$I(z) = \frac{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}}{Z_0}$$

$$Z(z) = Z_0 \frac{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}}$$

With magnitudes:

$$|V(z)| = \left| V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z} \right| = ??$$

$$|I(z)| = \frac{\left| V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z} \right|}{Z_0} = ??$$

$$|Z(z)| = Z_0 \frac{\left| V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z} \right|}{\left| V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z} \right|} = ??$$

V^+, V^-, Γ is much simpler

And likewise phase:

$$\arg\{V(z)\} = \arg\{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\} = ??$$

$$\arg\{I(z)\} = \arg\{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\} = ??$$

$$\begin{aligned}\arg\{Z(z)\} &= \arg\{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}\} \\ &\quad -\arg\{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\} \\ &= ??\end{aligned}$$

Q: *It appears to me that when attempting to describe the activity along a transmission line—as a function of **position** z—it is much **easier** and more **straightforward** to use the **wave** representation(nyuck, nyuck, nyuck).*

A: That's right! However, this does **not** mean that we **never** determine $V(z)$, $I(z)$, or $Z(z)$; these quantities are still **fundamental** and very important—particularly at each **end** of the transmission line!