## 2.4-The Smith Chart

## Reading Assignment: pp. 64-73

The Smith Chart $\rightarrow$ An icon of microwave engineering!

The Smith Chart provides:


1) A graphical method to solve many transmission line problems.
2) A visual indication of microwave device performance.

The most important fact about the Smith Chart is:
$\rightarrow$ It exists on the complex $\Gamma$ plane.

## ho: The Complex Г PLane

Q: But how is the complex I plane useful?
A: We can easily plot and determine values of $\Gamma(z)$

## HO: TRANSFORMATIONS ON THE COMPLEX $\Gamma$ PLANE

Q: But transformations of $\Gamma$ are relatively easytransformations of line impedance $Z$ is the difficult one.

A: We can likewise map line impedance onto the complex $\Gamma$ plane!

HO: MAPPING Z TO Г<br>HO: THE SMITH CHART<br>HO: SMITH CHART GEOGRAPHY

## HO: THE OUTER SCALE

The Smith Chart allows us to solve many important transmission line problems!

## HO: $Z_{\text {IN }}$ CALCULATIONS USING THE SMITH CHART

EXAMPLE: THE INPUT IMPEDANCE OF A SHORTED
TRANSMISSION LINE

EXAMPLE: DETERMINING THE LOAD IMPEDANCE OF A TRANSMISSION LINE

## EXAMPLE: DETERMINING THE LENGTH OF A TRANSMISSION

 LINEAn alternative to impedance $Z$, is its inverse-admittance $Y$.

HO: IMPEDANCE AND ADMITTANCE

Expressing a load or line impedance in terms of its admittance is sometimes helpful. Additionally, we can easily map admittance onto the Smith Chart.

HO: ADMITTANCE AND THE SMITH CHART
EXample: Admittance Calculations with the Smith Chart

## The Complex $\Gamma$ Plane

Resistance $R$ is a real value, thus we can indicate specific resistor values as points on the real line:


Likewise, since impedance $Z$ is a complex value, we can indicate specific impedance values as point on a two dimensional complex plane:


Note each dimension is defined by a single real line: the horizontal line (axis) indicating the real component of $Z$ (i.e., $\operatorname{Re}\{Z\}$ ), and the vertical line (axis) indicating the imaginary component of impedance $Z$ (i.e., $\operatorname{Im}\{Z\}$ ). The intersection of these two lines is the point denoting the impedance $Z=0$.

* Note then that a vertical line is formed by the locus of all points (impedances) whose resistive (i.e., real) component is equal to, say, 75.
* Likewise, a horizontal line is formed by the locus of all points (impedances) whose reactive (i.e., imaginary) component is equal to -30.


If we assume that the real component of every impedance is positive, then we find that only the right side of the plane will be useful for plotting impedance $Z$-points on the left side indicate impedances with negative resistances!
(R)

Moreover, we find that common impedances such as $Z=\infty$ (an open circuit!) cannot be plotted, as their points appear an infinite distance from the origin.


Q: Yikes! The complex Z plane does not appear to be a very helpful. Is there some graphical tool that is more useful?

A: Yes! Recall that impedance $Z$ and reflection coefficient $\Gamma$ are equivalent complex values-if you know one, you know the other.

We can therefore define a complex $\Gamma$ plane in the same manner that we defined a complex impedance plane. We will find that there are many advantages to plotting on the complex $\Gamma$ plane, as opposed to the complex $Z$ plane!


Note that the horizontal axis indicates the real component of $\Gamma(\operatorname{Re}\{\Gamma\})$, while the vertical axis indicates the imaginary component of $\Gamma(\operatorname{Im}\{\Gamma\})$.

We could plot points and lines on this plane exactly as before:


However, we will find that the utility of the complex $\Gamma$ pane as a graphical tool becomes apparent only when we represent a complex reflection coefficient in terms of its magnitude $(|\Gamma|)$ and phase ( $\theta_{\Gamma}$ ):

$$
\Gamma=|\Gamma| e^{j \theta_{\Gamma}}
$$

In other words, we express $\Gamma$ using polar coordinates:


Note then that a circle is formed by the locus of all points whose magnitude $|\Gamma|$ equal to, say, 0.7 . Likewise, a radial line is formed by the locus of all points whose phase $\theta_{\Gamma}$ is equal to $135^{\circ}$.


Perhaps the most important aspect of the complex $\Gamma$ plane is its validity region. Recall for the complex $Z$ plane that this validity region was the right-half plane, where $\operatorname{Re}\{Z\}>0$ (i.e., positive resistance).

The problem was that this validity region was unbounded and infinite in extent, such that many important impedances (e.g., open-circuits) could not be plotted.

Q: What is the validity region for the complex $\Gamma$ plane?
A: Recall that we found that for $\operatorname{Re}\{Z\}>0$ (i.e., positive resistance), the magnitude of the reflection coefficient was limited:

$$
0<|\Gamma|<1
$$

Therefore, the validity region for the complex $\Gamma$ plane consists of all points inside the circle $|\Gamma|=1$--a finite and bounded area!


Note that we can plot all valid impedances (i.e., $R>0$ ) within this finite region!


## Transformations on the Complex I Plane

The usefulness of the complex $\Gamma$ plane is apparent when we consider again the terminated, lossless transmission line:


Recall that the reflection coefficient function for any location $z$ along the transmission line can be expressed as (since $z_{L}=0$ ):

$$
\begin{aligned}
\Gamma(z) & =\Gamma_{L} e^{j 2 \beta z} \\
& =\left|\Gamma_{L}\right| e^{j\left(\theta_{r}+2 \beta z\right)}
\end{aligned}
$$

And thus, as we would expect:

$$
\Gamma(z=0)=\Gamma_{L} \quad \text { and } \quad \Gamma(z=-\ell)=\Gamma_{L} e^{-j 2 \beta \ell}=\Gamma_{i n}
$$

Recall this result "says" that adding a transmission line of length $\ell$ to a load results in a phase shift in $\theta_{\Gamma}$ by $-2 \beta \ell$ radians, while the magnitude $|\Gamma|$ remains unchanged.

## Q: Magnitude $|\Gamma|$ and phase $\theta_{\Gamma}$--aren't those the values used when plotting on the complex $\Gamma$ plane?

## A: Precisely! In fact, plotting the

 transformation of $\Gamma_{L}$ to $\Gamma_{\text {in }}$ along a transmission line length $\ell$ has an interesting graphical interpretation. Let's parametrically plot $\Gamma(z)$ from $z=z_{L}$ (i.e., $z=0$ ) to $z=z_{L}-\ell$ (i.e., $z=-\ell$ ):$\operatorname{Im}\{\Gamma\}$


Since adding a length of transmission line to a load $\Gamma_{L}$ modifies the phase $\theta_{\Gamma}$ but not the magnitude $\left|\Gamma_{L}\right|$, we trace a circular arc as we parametrically plot $\Gamma(z)$ ! This arc has a radius $\left|\Gamma_{L}\right|$ and an arc angle $2 \beta \ell$ radians.

With this knowledge, we can easily solve many interesting transmission line problems graphically-using the complex $\Gamma$ plane! For example, say we wish to determine $\Gamma_{\text {in }}$ for a transmission line length $\ell=\lambda / 8$ and terminated with a short circuit.


The reflection coefficient of a short circuit is $\Gamma_{L}=-1=1 e^{j \pi}$, and therefore we begin at that point on the complex $\Gamma$ plane. We then move along a circular arc $-2 \beta \ell=-2(\pi / 4)=-\pi / 2$ radians (i.e., rotate clockwise $90^{\circ}$ ).


When we stop, we find we are at the point for $\Gamma_{\text {in }}$; in this case $\Gamma_{i n}=1 e^{j \pi / 2}$ (i.e., magnitude is one, phase is $90^{\circ}$ ).

Now, let's repeat this same problem, only with a new transmission line length of $\ell=\lambda / 4$. Now we rotate clockwise $2 \beta \ell=\pi$ radians ( $180^{\circ}$ ).


For this case, the input reflection coefficient is $\Gamma_{\text {in }}=1 e^{j 0}=1$ : the reflection coefficient of an open circuit!

Our short-circuit load has been transformed into an open circuit with a quarter-wavelength transmission line!

But, you knew this would happen-right?


Recall that a quarter-wave transmission line was one of the special cases we considered earlier. Recall we found that the input impedance was proportional to the inverse of the load impedance.

Thus, a quarter-wave transmission line transforms a short into an open. Conversely, a quarter-wave transmission can also transform an open into a short:


Finally, let's again consider the problem where $\Gamma_{L}=-1$ (i.e., short), only this time with a transmission line length $\ell=\lambda / 2$ ( $a$ half wavelength!). We rotate clockwise $2 \beta \ell=2 \pi$ radians ( $360^{\circ}$ ).

Hey look! We came clear around to where we started!


Now, let's consider the opposite problem. Say we know that the input impedance at the beginning of a transmission line with length $\ell=\lambda / 8$ is:

$$
\Gamma_{i n}=0.5 e^{j 60^{\circ}}
$$

Q: What is the reflection coefficient of the load?
A: In this case, we begin at $\Gamma_{\text {in }}$ and rotate COUNTERCLOCKWISE along a circular arc (radius 0.5 ) $2 \beta \ell=\pi / 2$ radians (i.e., $60^{\circ}$ ). Essentially, we are removing the phase shift associated with the transmission line!


The reflection coefficient of the load is therefore:

$$
\Gamma_{L}=0.5 e^{j 150^{\circ}}
$$

## Mapping Z to $\Gamma$

Recall that line impedance and reflection coefficient are equivalent-either one can be expressed in terms of the other:

$$
\Gamma(z)=\frac{Z(z)-Z_{0}}{Z(z)+Z_{0}} \quad \text { and } \quad Z(z)=Z_{0}\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)
$$

Note this relationship also depends on the characteristic impedance $Z_{0}$ of the transmission line. To make this relationship more direct, we first define a normalized impedance value $z^{\prime}$ (an impedance coefficient!):

$$
z^{\prime}(z)=\frac{Z(z)}{Z_{0}}=\frac{R(z)}{Z_{0}}+j \frac{X(z)}{Z_{0}}=r(z)+j x(z)
$$

Using this definition, we find:

$$
\begin{aligned}
\Gamma(z) & =\frac{Z(z)-Z_{0}}{Z(z)+Z_{0}} \\
& =\frac{Z(z) / Z_{0}-1}{Z(z) / Z_{0}+1} \\
& =\frac{z^{\prime}(z)-1}{z^{\prime}(z)+1}
\end{aligned}
$$

Thus, we can express $\Gamma(z)$ explicitly in terms of normalized impedance $z^{\prime}$--and vice versa!

$$
\Gamma(z)=\frac{z^{\prime}(z)-1}{z^{\prime}(z)+1} \quad z^{\prime}(z)=\frac{1+\Gamma(z)}{1-\Gamma(z)}
$$

The equations above describe a mapping between coefficients $z^{\prime}$ and $\Gamma$. This means that each and every normalized impedance value likewise corresponds to one specific point on the complex $\Gamma$ plane!

For example, say we wish to mark or somehow indicate the values of normalized impedance $z^{\prime}$ that correspond to the various points on the complex $\Gamma$ plane.

Some values we already know specifically:

| case | $Z$ | $z^{\prime}$ | $\Gamma$ |
| :---: | :---: | :---: | :---: |
| 1 | $\infty$ | $\infty$ | 1 |
| 2 | 0 | 0 | -1 |
| 3 | $Z_{0}$ | 1 | 0 |
| 4 | $j Z_{0}$ | $j$ | $j$ |
| 5 | $-j Z_{0}$ | $-j$ | $-j$ |

Therefore, we find that these five normalized impedances map onto five specific points on the complex $\Gamma$ plane:


Or, the five complex $\Gamma$ map onto five points on the normalized impedance plane:

$$
\begin{aligned}
& \text { Rebid } \\
& \text { Region }
\end{aligned}
$$

Now, the preceding provided examples of the mapping of points between the complex (normalized) impedance plane, and the complex $\Gamma$ plane.

We can likewise map whole contours (i.e., sets of points) between these two complex planes. We shall first look at two familiar cases.

$$
Z=R
$$

In other words, the case where impedance is purely real, with no reactive component (i.e., $X=0$ ).

Meaning that normalized impedance is:

$$
z^{\prime}=r+j 0 \quad(i . e ., x=0)
$$

where we recall that $r=R / Z_{0}$.

Remember, this real-valued impedance results in a real-valued reflection coefficient:

$$
\Gamma=\frac{r-1}{r+1}
$$

I.E.,:

$$
\Gamma_{r} \doteq \operatorname{Re}\{\Gamma\}=\frac{r-1}{r+1}
$$

$$
\Gamma_{i} \doteq \operatorname{Im}\{\Gamma\}=0
$$

Thus, we can determine a mapping between two contours-one contour $(x=0)$ on the normalized impedance plane, the other ( $\Gamma_{i}=0$ ) on the complex $\Gamma$ plane:

$$
x=0 \quad \Leftrightarrow \quad \Gamma_{i}=0
$$




$$
Z=j X
$$

In other words, the case where impedance is purely imaginary, with no resistive component (i.e., $R=0$ ).

Meaning that normalized impedance is:

$$
z^{\prime}=0+j x \quad(i . e ., r=0)
$$

where we recall that $x=X / Z_{0}$.

Remember, this imaginary impedance results in a reflection coefficient with unity magnitude:

$$
|\Gamma|=1
$$

Thus, we can determine a mapping between two contours-one contour ( $r=0$ ) on the normalized impedance plane, the other $(|\Gamma|=1)$ on the complex $\Gamma$ plane:

$$
r=0 \quad \Leftrightarrow \quad|\Gamma|=1
$$



Invalid
Region



A: Actually, not only are mappings of more general impedance contours (such as $r=0.5$ and $x=-1.5$ ) onto the complex $\Gamma$ plane possible, these mappings have already been achievedthanks to Dr. Smith and his famous chart!

## The Smith Chart

Say we wish to map a line on the normalized complex impedance plane onto the complex $\Gamma$ plane.

For example, we could map the vertical line $r=2\left(\operatorname{Re}\left\{z^{\prime}\right\}=2\right)$ or the horizontal line $x=-1\left(\operatorname{Im}\left\{z^{\prime}\right\}=-1\right)$.


Recall we know how to map the vertical line $r=0$; it simply maps to the circle $|\Gamma|=1$ on the complex $\Gamma$ plane.

Likewise, we know how to map the horizontal line $x=0$; it simply maps to the line $\Gamma_{i}=0$ on the complex $\Gamma$ plane.

But for the examples given above, the mapping is not so straight forward. The contours will in general be functions of both $\Gamma_{r}$ and $\Gamma_{i}$ (e.g., $\Gamma_{r}^{2}+\Gamma_{i}^{2}=0.5$ ), and thus the mapping cannot be stated with simple functions such as $|\Gamma|=1$ or $\Gamma_{i}=0$.

As a matter of fact, a vertical line on the normalized impedance plane of the form:

$$
r=c_{r},
$$

where $c_{r}$ is some constant (e.g. $r=2$ or $r=0.5$ ), is mapped onto the complex $\Gamma$ plane as:

$$
\left(\Gamma_{r}-\frac{c_{r}}{1+c_{r}}\right)^{2}+\Gamma_{i}^{2}=\left(\frac{1}{1+c_{r}}\right)^{2}
$$

Note this equation is of the same form as that of a circle:

$$
\left(x-x_{c}\right)^{2}+\left(y-y_{c}\right)^{2}=a^{2}
$$

where:

$$
a=\text { the radius of the circle }
$$

$P_{c}\left(x=x_{c}, y=y_{c}\right) \Rightarrow$ point located at the center of the circle

Thus, the vertical line $r=c_{r}$ maps into a circle on the complex $\Gamma$ plane!

By inspection, it is apparent that the center of this circle is located at this point on the complex $\Gamma$ plane:

$$
P_{c}\left(\Gamma_{r}=\frac{c_{r}}{1+c_{r}}, \Gamma_{i}=0\right)
$$

In other words, the center of this circle always lies somewhere along the $\Gamma_{i}=0$ line.

Likewise, by inspection, we find the radius of this circle is:

$$
a=\frac{1}{1+c_{r}}
$$

We perform a few of these mappings and see where these circles lie on the complex $\Gamma$ plane:


We see that as the constant $c_{r}$ increases, the radius of the circle decreases, and its center moves to the right.

## Note:

1. If $c_{r}>0$ then the circle lies entirely within the circle $|\Gamma|=1$.
2. If $c_{r}<0$ then the circle lies entirely outside the circle $|\Gamma|=1$.
3. If $c_{r}=0$ (i.e., a reactive impedance), the circle lies on circle $|\Gamma|=1$.
4. If $c_{r}=\infty$, then the radius of the circle is zero, and its center is at the point $\Gamma_{r}=1, \Gamma_{i}=0$ (i.e., $\Gamma=1 e^{j 0}$ ). In other words, the entire vertical line $r=\infty$ on the normalized impedance plane is mapped onto just a single point on the complex $\Gamma$ plane!

But of course, this makes sense! If $r=\infty$, the impedance is infinite (an open circuit), regardless of what the value of the reactive component $x$ is.

Now, let's turn our attention to the mapping of horizontal lines in the normalized impedance plane, i.e., lines of the form:

$$
x=c_{i}
$$

where $c_{i}$ is some constant (e.g. $x=-2$ or $x=0.5$ ).

We can show that this horizontal line in the normalized impedance plane is mapped onto the complex $\Gamma$ plane as:

$$
\left(\Gamma_{r}-1\right)^{2}+\left(\Gamma_{i}-\frac{1}{c_{i}}\right)^{2}=\frac{1}{c_{i}^{2}}
$$

Note this equation is also that of a circle! Thus, the horizontal line $x=c_{i}$ maps into a circle on the complex $\Gamma$ plane!

By inspection, we find that the center of this circle lies at the point:

$$
P_{c}\left(\Gamma_{r}=1, \Gamma_{i}=\frac{1}{c_{i}}\right)
$$

in other words, the center of this circle always lies somewhere along the vertical $\Gamma_{r}=1$ line.

Likewise, by inspection, the radius of this circle is:

$$
a=\frac{1}{\left|c_{i}\right|}
$$

We perform a few of these mappings and see where these circles lie on the complex $\Gamma$ plane:


We see that as the magnitude of constant $c_{i}$ increases, the radius of the circle decreases, and its center moves toward the point ( $\left.\Gamma_{r}=1, \Gamma_{i}=0\right)$.

Note:

1. If $c_{i}>0$ (i.e., reactance is inductive) then the circle lies entirely in the upper half of the complex $\Gamma$ plane (i.e., where $\Gamma_{i}>0$ )-the upper half-plane is known as the inductive region.
2. If $c_{i}<0$ (i.e., reactance is capacitive) then the circle lies entirely in the lower half of the complex $\Gamma$ plane (i.e., where $\Gamma_{i}<0$ )-the lower half-plane is known as the capacitive region.
3. If $c_{i}=0$ (i.e., a purely resistive impedance), the circle has an infinite radius, such that it lies entirely on the line $\Gamma_{i}=0$.
4. If $c_{i}= \pm \infty$, then the radius of the circle is zero, and its center is at the point $\Gamma_{r}=1, \Gamma_{i}=0$ (i.e., $\Gamma=1 e^{j 0}$ ). In other words, the entire vertical line $x=\infty$ or $x=-\infty$ on the normalized impedance plane is mapped onto just a single point on the complex $\Gamma$ plane!

But of course, this makes sense! If $x=\infty$, the impedance is infinite (an open circuit), regardless of what the value of the resistive component $r$ is.
5. Note also that much of the circle formed by mapping $x=c_{i}$ onto the complex $\Gamma$ plane lies outside the circle $|\Gamma|=1$.

This makes sense! The portions of the circles laying outside $|\Gamma|=1$ circle correspond to impedances where the real (resistive) part is negative (i.e., $r<0$ ).

Thus, we typically can completely ignore the portions of the circles that lie outside the $|\Gamma|=1$ circle !

Mapping many lines of the form $r=c_{r}$ and $x=c_{i}$ onto circles on the complex $\Gamma$ plane results in tool called the Smith Chart.


Note the Smith Chart is simply the vertical lines $r=c_{r}$ and horizontal lines $x=c_{i}$ of the normalized impedance plane, mapped onto the two types of circles on the complex $\Gamma$ plane.

Note for the normalized impedance plane, a vertical line $r=c_{r}$ and a horizontal line $x=c_{i}$ are always perpendicular to each other when they intersect. We say these lines form a rectilinear grid.

However, a similar thing is true for the Smith Chart! When a mapped circle $r=c_{r}$ intersects a mapped circle $x=c_{i}$, the two circles are perpendicular at that intersection point. We say these circles form a curvilinear grid.
In fact, the Smith Chart is formed by distorting the rectilinear grid of the normalized impedance plane into the curvilinear grid of the Smith Chart!
I.E.,


Distorting this rectilinear grid:
inear grid:


And then distorting some more-we have the curvilinear grid of the Smith Chart!


## Smith Chart Geography

We have located specific points on the complex impedance plane, such as a short circuit or a matched load.

We've also identified contours, such as $r=1$ or $x=-2$.


We can likewise identify whole regions (!) of the complex impedance plane, providing a bit of a geography lesson of the complex impedance plane.

For example, we can divide the complex impedance plane into four regions based on normalized resistance value $r$ :


Just like points and contours, these regions of the complex impedance plane can be mapped onto the complex gamma plane!


Instead of dividing the complex impedance plane into regions based on normalized resistance $r$, we could divide it based on normalized reactance $x$.


These four regions can likewise be mapped onto the complex gamma plane:


Note the four resistance regions and the four reactance regions combine to from 16 separate regions on the complex impedance and complex gamma planes!

Eight of these sixteen regions lie in the valid region (i.e., $r>0$ ), while the other eight lie entirely in the invalid region.

Make sure you can locate the eight impedance regions on a Smith Chart-this understanding of Smith Chart geography will help you understand your design and analysis results!


## The Outer Scale

Note that around the outside of the Smith Chart there is a scale indicating the phase angle $\theta_{\Gamma}$ (i.e., $\Gamma=|\Gamma| e^{j \theta_{\Gamma}}$ ), from $-180^{\circ}<\theta_{\text {Г }}<180^{\circ}$.

Recall however, for a terminated transmission line, the reflection coefficient function is:

$$
\Gamma(\boldsymbol{z})=\Gamma_{0} e^{j 2 \beta z}=\left|\Gamma_{0}\right| e^{j 2 \beta z+\theta_{0}}
$$

Thus, the phase of the reflection coefficient function depends on transmission line position $z$ as:

$$
\theta_{\Gamma}(z)=2 \beta z+\theta_{0}=2\left(\frac{2 \pi}{\lambda}\right) z+\theta_{0}=4 \pi\left(\frac{z}{\lambda}\right)+\theta_{0}
$$

As a result, a change in line position $z$ (i.e., $\Delta z$ ) results in a change in reflection coefficient phase $\theta_{\Gamma}$ (i.e., $\Delta \theta_{\Gamma}$ ):

$$
\Delta \theta_{\Gamma}=4 \pi\left(\frac{\Delta z}{\lambda}\right)
$$

For example, a change of position equal to one-quarter wavelength $\Delta \boldsymbol{z}=\lambda / 4$ results in a phase change of $\pi$ radians-we rotate half-way around the complex $\Gamma$ plane (otherwise known as the Smith Chart).

Or, a change of position equal to one-half wavelength $\Delta \boldsymbol{z}=\frac{1}{2}$ results in a phase change of $2 \pi$ radians-we rotate completely around the complex $\Gamma$ plane (otherwise known as the Smith Chart).

The Smith Chart then has a second scale (besides $\theta_{\Gamma}$ ) that surrounds it-one that relates transmission line position in wavelengths (i.e., $\Delta z / \lambda$ ) to the reflection coefficient phase:


Since the phase scale on the Smith Chart extends from $-180^{\circ}<\theta_{\Gamma}<180^{\circ}$ (i.e., $-\pi<\theta_{\Gamma}<\pi$ ), this electrical length scale extends from:

$$
0<z / \lambda<0.5
$$

Note for this mapping the reflection coefficient phase at location $z=0$ is $\theta_{L}=-\pi$. Therefore, $\theta_{0}=-\pi$, and we find that:

$$
\Gamma_{0}=\left|\Gamma_{0}\right| e^{j \theta_{0}}=\left|\Gamma_{0}\right| e^{-j \pi}=-\left|\Gamma_{0}\right|
$$

In other words, $\Gamma_{0}$ is a negative real value.

Q: But, $\Gamma_{0}$ could be anything! What is the likelihood of $\Gamma_{0}$ being a real and negative value? Most of the time this is not the case-this second Smith Chart scale seems to be nearly useless!?

A: Quite the contrary! This electrical length scale is in fact very useful-you just need to understand how to utilize it!


This electrical length scale is very much like the mile markers you see along an interstate highway; although the specific numbers are quite arbitrary, they are still very useful.

Take for example Interstate 70, which stretches across Kansas. The western end of I-70 (at the Colorado border) is denoted as mile 1.


At each mile along I-70 a new marker is placed, such that the eastern end of I-70 (at the Missouri border) is labeled mile 423-
Interstate 70 runs for 423 miles across Kansas!


The location of various towns and burgs along I-70 can thus be specified in terms of these mile markers. For example, along I70 we find:

Oakley at mile marker 76
Russell at mile marker 184
Salina at mile marker 251
Junction City at mile marker 296
Topeka at mile marker 361
Lawrence at mile marker 388


So say you are traveling eastbound ( $\rightarrow$ ) along I70, and you want to know the distance to Topeka. Topeka is at mile marker 361, but this does not of course mean you are 361 miles from Topeka.


Instead, you subtract from 361 the value of the mile marker denoting your position along I-70.


For example, if you find yourself in the lovely borough of Russell (mile marker 184), you have precisely 361$184=177$ miles to go before reaching Topeka!

Q: I'm confused! Say I'm in Lawrence (mile marker 388); using your logic I am a distance of 361-388 =-27 miles from Topeka! How can I be a negative distance from something??

A: The mile markers across Kansas are arranged such that their value increases as we move from west to east across the state. Take the value of the mile marker denoting to where you are traveling, and subtract from it the value of the mile marker where you are.

If this value is positive, then your destination is East of you; if this value is negative, it is West of your current position (hopefully you're in the westbound lane!).

For example, say you're traveling to Salina (mile marker 251). If you are in Oakley (mile marker 76) then:

$$
251-76=175 \rightarrow \text { Salina is } 175 \text { miles East of Oakley }
$$

If, on the other hand, you begin your journey from Junction City (mile marker 296), we find:

251-296 = -45 $\rightarrow$ Salina is 45 miles West of Junction City

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Q: But just what the \&()\#\$@\% does this discussion have to do with SMITH CHARTS !!?!?

A: The electrical length scale $(z / \lambda)$ around the perimeter of the Smith Chart is precisely analogous to mile markers along an interstate!


For example, say you're at some location $z=z_{1}$ along a transmission line. The value of the reflection coefficient function at that point happens to be:

$$
\Gamma\left(z=z_{1}\right)=0.685 e^{-j 65^{5}}
$$

Finding the phase angle of $\theta_{\mathrm{\Gamma}}=-65^{\circ}$ on the outer scale of the Smith Chart, we note that the corresponding electrical length value is:
$0.160 \lambda$
Note this tells us nothing about the location $z=z_{1}$. This does not mean that $z_{1}=0.160 \lambda$, for example!

Now, say we move a short distance $\Delta z$ (i.e., a distance less than $\lambda / 2$ ) along the transmission line, to a new location denoted as $z=z_{2}$.

We find that this new location that the reflection coefficient function has a value of:

$$
\Gamma\left(\boldsymbol{z}=\mathbf{z}_{2}\right)=0.685 e^{+j 74^{4}}
$$

Now finding the phase angle of $\theta_{\Gamma}=+74^{\circ}$ on the outer scale of the Smith Chart, we note that the corresponding electrical length value is:

$$
0.353 \lambda
$$

Note this tells us nothing about the location $z=z_{2}$. This does not mean that $z_{1}=0.353 \lambda$, for example!


Q: So what do the values $0.160 \lambda$ and $0.353 \lambda$ tell us?

A: They allow us to determine the distance between points $z_{2}$ and $z_{1}$ on the transmission line:

$$
\frac{\Delta \boldsymbol{z}}{\lambda}=\frac{z_{2}}{\lambda}-\frac{z_{1}}{\lambda}!!!
$$

Thus, for this example, the distance between locations $z_{2}$ and $z_{1}$ is:

$$
\Delta \boldsymbol{z}=0.353 \lambda-0.160 \lambda=0.193 \lambda
$$

$\rightarrow$ The transmission line location $z_{2}$ is a distance of $0.193 \lambda$ from location $z_{1}$ !


The distance between $z_{3}$ and $z_{1}$ would then turn out to be:

$$
\frac{\Delta z}{\lambda}=0.094-0.160=-0.066
$$

What does the negative value mean??
A: Just like our I-70 mile marker analogy, the sign (plus or minus) indicates the direction of movement from one point to another.

In the first example, we find that $\Delta z>0$, meaning $z_{2}>z_{1}$ :

$$
z_{2}=z_{1}+0.094 \lambda
$$

Clearly, the location $z_{2}$ is further down the transmission line (i.e., closer to the load) than is location $z_{1}$.

For the second example, we find that $\Delta z<0$, meaning $z_{3}<z_{1}$ :

$$
z_{3}=z_{1}-0.066 \lambda
$$

Conversely, in this second example, the location $z_{3}$ is closer to the beginning of the transmission line (i.e., farther from the load) than is location $z_{1}$.

This is completely consistent with what we already know to be true!

In the first case, the positive value $\Delta z=0.193 \lambda$ maps to a phase change of $\Delta \theta_{\Gamma}=74^{\circ}-\left(-65^{\circ}\right)=139^{\circ}$.

In other words, as we move toward the load from location $z_{1}$ to location $z_{2}$, we rotate counter-clockwise around the Smith Chart.

Likewise, the negative value $\Delta \boldsymbol{z}=-0.066 \lambda$ maps to a phase change of $\Delta \theta_{\Gamma}=-112^{\circ}-\left(-65^{\circ}\right)=-47^{\circ}$.

In other words, as we move away from the load (toward the source) from a location $z_{1}$ to location $z_{3}$, we rotate clockwise around the Smith Chart.


Q: I notice that there is a second electrical length scale on the Smith Chart. Its values increase as we move clockwise from an initial value of zero to a maximum value of $0.5 \lambda$.

## What's up with that?

A: This scale uses an alternative mapping between $\theta_{\Gamma}$ and $z / \lambda$ :

$$
\frac{z}{\lambda}=\frac{1}{4}-\frac{\theta_{\Gamma}}{4 \pi} \quad \Leftrightarrow \quad \theta_{\Gamma}=4 \pi\left(\frac{1}{4}-\frac{z}{\lambda}\right)
$$



This scale is analogous to a situation wherein a second set of mile markers were placed along I-70. These mile markers begin at the east side of Kansas (at the Missouri border), and end at the west side of Kansas (at the Colorado border).


Q: What good would this second set do? Would it serve any purpose?

A: Not much really. After all, this second set is redundant-it does not provide any new information that the original set already provides.

> Yet, if we were to place this new set along I-70, we almost certainly would place the original mile markers along the eastbound lanes, and this new set along the westbound lanes.

In this manner, all I-70 motorists (eastbound or westbound) would see an increase in the mile markers as they traverse the Sunflower State.

As a result, a positive distance to their destination indicates to all drivers that their destination is in front of them (in the direction they are driving), while a negative distance indicates to all drivers that their destination is behind the (they better turn around!).

Thus, it could be argued that each set of mile markers is optimized for a specific direction of travel-the original set if you are traveling east, and this second set if you are traveling west.


Similarly, the two electrical length scales on the Smith Chart are meant for two different "directions of travel". If we move down the transmission line toward the load, the value $\Delta z$ will be positive.

Conversely, if we move up the transmission line and away from the load (i.e., "toward the generator"), this second electrical length scale will also provide a positive value of $\Delta z$.

Again, these two electrical length scales are redundant-you will get the correct answer regardless of the scale you use, but be careful to interpret negative signs properly.


Q: Wait! I just used a Smith Chart to analyze a transmission line problem in the manner you have just explained. At one point on my transmission line the phase of the reflection coefficient is $\theta_{\Gamma}=+170^{\circ}$, which is denoted as $0.486 \lambda$ on the "wavelengths toward load" scale.

I then moved a short distance along the line toward the load, and found that the reflection coefficient phase was $\theta_{\Gamma}=-144^{\circ}$, which is denoted as $0.050 \lambda$ on the "wavelengths toward load" scale.

According to your "instruction", the distance between these two points is:

$$
\Delta \boldsymbol{z}=0.050 \lambda-0.486 \lambda=-0.436 \lambda
$$

A large negative value! This says that I moved nearly a half wavelength away from the load, but I know that I moved just a short distance toward the load! What happened?

A: Note the electrical length scales on the Smith Chart begin and end where $\theta_{\Gamma}= \pm \pi$ (by the short circuit!).


In your example, when rotating counter-clockwise around the chart (i.e., moving toward the load) you passed by this transition. This makes the calculation of $\Delta z$ a bit more problematic.


To see why, let's again consider our I-70 analogy. Say we are Lawrence, and wish to drive eastbound on Interstate 70 until we reach Columbia, Missouri.

The mile marker for Lawrence is of course 388, and Columbia Missouri is located at mile marker 126. We might conclude that the distance from Lawrence to Columbia is:

$$
126-388=-262 \text { miles }
$$

Q: Yikes! According to this, Columbia is 262 miles west of Lawrence-should we turn the car around?

A: Columbia, Missouri is most decidedly east of Lawrence, Kansas. The calculation above is incorrect. The problem is that mile markers "reset" once we reach a state border. Once we hit the Missouri-Kansas border, the mile markers reset to zero, and then again increase as we travel eastward.


Thus, to accurately determine the distance between Lawrence and Columbia, we need to break the problem into two steps:

Step 1: Determine the distance between Lawrence (mile marker 388), and the last mile marker before the state line (mile marker 423):

$$
423-388=35 \text { miles }
$$

Step 2: Determine the distance between the first mile marker after the state line (mile marker 0) and Columbia (mile marker 126):

$$
126-0=126 \text { miles }
$$

Thus, the distance between Lawrence and Columbia is the distance between Lawrence and the state line ( 35 miles), plus the distance from the state line to Columbia (126 miles):

$$
35+126=161 \text { miles }
$$

Columbia, Missouri is 161 miles east of Lawrence, Kansas!
Now back to the Smith Chart problem; as we rotate counterclockwise around the Smith Chart, the "wavelengths toward load" scale increases in value, until it reaches a maximum value of $0.5 \lambda$ (at $\theta_{\Gamma}= \pm \pi$ ).

At that point, the scale "resets" to its minimum value of zero. We have metaphorically "crossed the state line" of this scale.

Thus, to accurately determine the electrical length moved along a transmission line, we must divide the problem into two steps:

Step 1: Determine the electrical length from the initial point to the "end" of the scale at $0.5 \lambda$.

Step 2: Determine the electrical distance from the "beginning" of the scale (i.e., 0 ) and the second location on the transmission line.

Add the results of steps 1 and 2, and you have your answer!

For example, let's look at the case that originally gave us the erroneous result. The distance from the initial location to the end of the scale is:

$$
0.500 \lambda-0.486 \lambda=+0.014 \lambda
$$

And the distance from the beginning of the scale to the second point is:

$$
0.050 \lambda-0.000 \lambda=+0.050 \lambda
$$

Thus the distance between the two points is:

$$
0.014 \lambda+0.050 \lambda=+0.064 \lambda
$$

The second point is just a little closer to the load than the first!


## Zin Calculations using the Smith Chart



The normalized input impedance $z_{i n}^{\prime}$ of a transmission line length $\ell$, when terminated in normalized load $z_{L}^{\prime}$, can be determined as:

$$
\begin{aligned}
z_{i n}^{\prime} & =\frac{Z_{i n}}{Z_{0}} \\
& =\frac{1}{Z_{0}} Z_{0}\left(\frac{Z_{L}+j Z_{0} \tan \beta \ell}{Z_{0}+j Z_{L} \tan \beta \ell}\right) \\
& =\frac{Z_{L} / Z_{0}+j \tan \beta \ell}{1+j Z_{L} / Z_{0} \tan \beta \ell} \\
& =\frac{z_{L}^{\prime}+j \tan \beta \ell}{1+j z_{L}^{\prime} \tan \beta \ell}
\end{aligned}
$$

Q: Evaluating this unattractive expression looks not the least bit pleasant. Isn't there a less disagreeable method to determine $z_{\text {in }}^{\prime}$ ?

A: Yes there is! Instead, we could determine this normalized input impedance by following these three steps:

1. Convert $z_{L}^{\prime}$ to $\Gamma_{L}$, using the equation:

$$
\begin{aligned}
\Gamma_{L} & =\frac{Z_{L}-Z_{0}}{Z_{L}+Z_{0}} \\
& =\frac{Z_{L} / Z_{0}-1}{Z_{L} / Z_{0}+1} \\
& =\frac{Z_{L}^{\prime}-1}{z_{L}^{\prime}+1}
\end{aligned}
$$

2. Convert $\Gamma_{L}$ to $\Gamma_{i n}$, using the equation:

$$
\Gamma_{i n}=\Gamma_{L} e^{-j 2 \beta \ell}
$$

3. Convert $\Gamma_{\text {in }}$ to $z_{i n}^{\prime}$, using the equation:

$$
Z_{i n}^{\prime}=\frac{Z_{i n}}{Z_{0}}=\frac{1+\Gamma_{i n}}{1-\Gamma_{i n}}
$$

Q: But performing these three calculations would be even more difficult than the single step you described earlier. What short of dimwit would ever use (or recommend) this approach?

A: The benefit in this last approach is that each of the three steps can be executed using a Smith Chart-no complex calculations are required!

1. Convert $z_{L}^{\prime}$ to $\Gamma_{L}$

Find the point $z_{L}^{\prime}$ from the impedance mappings on your Smith Chart. Place you pencil at that point-you have now located the correct $\Gamma_{L}$ on your complex $\Gamma$ plane!

For example, say $z_{L}^{\prime}=0.6-j 1.4$. We find on the Smith Chart the circle for $r=0.6$ and the circle for $x=-1.4$. The intersection of these two circles is the point on the complex $\Gamma$ plane corresponding to normalized impedance $z_{L}^{\prime}=0.6-j 1.4$.

This point is a distance of 0.685 units from the origin, and is located at angle of -65 degrees. Thus the value of $\Gamma_{L}$ is:

$$
\Gamma_{L}=0.685 e^{-j 65^{\circ}}
$$

## 2. Convert $\Gamma_{L}$ to $\Gamma_{\text {in }}$

Since we have correctly located the point $\Gamma_{L}$ on the complex $\Gamma$ plane, we merely need to rotate that point clockwise around a circle $(|\Gamma|=0.685)$ by an angle $2 \beta \ell$.

When we stop, we are located at the point on the complex $\Gamma$ plane where $\Gamma=\Gamma_{i n}$ !

For example, if the length of the transmission line terminated in $z_{L}^{\prime}=0.6-j 1.4$ is $\ell=0.307 \lambda$, we should rotate around the Smith Chart a total of $2 \beta \ell=1.228 \pi$ radians, or $221^{\circ}$. We are now at the point on the complex $\Gamma$ plane:

$$
\Gamma=0.685 e^{+j 74^{i}}
$$

This is the value of $\Gamma_{i n}$ !
3. Convert $\Gamma_{\text {in }}$ to $z_{\text {in }}^{\prime}$

When you get finished rotating, and your pencil is located at the point $\Gamma=\Gamma_{\text {in }}$, simply lift your pencil and determine the values $r$ and $x$ to which the point corresponds!

For example, we can determine directly from the Smith Chart that the point $\Gamma_{\text {in }}=0.685 e^{+j 744}$ is located at the intersection of circles $r=0.5$ and $x=1.2$. In other words:

$$
z_{i n}^{\prime}=0.5+j 1.2
$$

## Step 1





## Example: The Input

## Impedance of a Shorted

## Transmission Line

Let's determine the input impedance of a transmission line that is terminated in a short circuit, and whose length is:
a) $\ell=\lambda / 8=0.125 \lambda \Rightarrow 2 \beta \ell=90^{\circ}$
b) $\ell=3 \lambda / 8=0.375 \lambda \Rightarrow 2 \beta \ell=270^{\circ}$

a) $\ell=\lambda / 8=0.125 \lambda \Rightarrow 2 \beta \ell=90^{\circ}$

Rotate clockwise $90^{\circ}$ from $\Gamma=-1.0=e^{j 180^{\circ}}$ and find $z_{\text {in }}^{\prime}=j$.

b) $\ell=3 \lambda / 8=0.375 \lambda \Rightarrow 2 \beta \ell=270^{\circ}$

Rotate clockwise $270^{\circ}$ from $\Gamma=-1.0=e^{j 180^{\circ}}$ and find $z_{\text {in }}^{\prime}=-j$.


## Example: Determining the

## Load Impedance of a

## Transmission Line

Say that we know that the input impedance of a transmission line length $\ell=0.134 \lambda$ is:

$$
z_{i n}^{\prime}=1.0+j 1.4
$$

Let's determine the impedance of the load that is terminating this line.


Locate $z_{\text {in }}^{\prime}$ on the Smith Chart, and then rotate counterclockwise (yes, I said counter-clockwise) $2 \beta \ell=96.5^{\circ}$.
Essentially, you are removing the phase shift associated with the transmission line. When you stop, lift your pencil and find $z_{L}^{\prime}$ !


## Example: Determining

## Transmission Line Length

A load terminating at transmission line has a normalized impedance $z_{L}^{\prime}=2.0+j$ 2.0. What should the length $\ell$ of transmission line be in order for its input impedance to be:
a) purely real (i.e., $x_{i n}=0$ )?
b) have a real (resistive) part equal to one (i.e., $r_{\text {in }}=1.0$ )?

## Solution:

a) Find $z_{L}^{\prime}=2.0+j 2.0$ on your Smith Chart, and then rotate clockwise until you "bump into" the contour $x=0$ (recall this is contour lies on the $\Gamma_{r}$ axis!).

When you reach the $x=0$ contour-stop! Lift your pencil and note that the impedance value of this location is purely real (after all, $x=0$ !).

Now, measure the rotation angle that was required to move clockwise from $z_{L}^{\prime}=2.0+j 2.0$ to an impedance on the $x=0$ contour-this angle is equal to $2 \beta \ell$ !

You can now solve for $\ell$, or alternatively use the electrical length scale surrounding the Smith Chart.

## One more important point-there are two possible solutions!

Solution 1:


## Solution 2:


b) Find $z_{L}^{\prime}=2.0+j 2.0$ on your Smith Chart, and then rotate clockwise until you "bump into" the circle $r=1$ (recall this circle intersects the center point or the Smith Chart!).

When you reach the $r=1$ circle-stop! Lift your pencil and note that the impedance value of this location has a real value equal to one (after all, $r=1$ !).

Now, measure the rotation angle that was required to move clockwise from $z_{L}^{\prime}=2.0+j 2.0$ to an impedance on the $r=1$ circle-this angle is equal to $2 \beta \ell$ !

You can now solve for $\ell$, or alternatively use the electrical length scale surrounding the Smith Chart.

Again, we find that there are two solutions!

## Solution 1:



## Solution 2:



Q: Hey! For part b), the solutions resulted in $z_{i n}^{\prime}=1-j 1.6$ and $z_{\text {in }}^{\prime}=1+j 1.6$--the imaginary parts are equal but opposite! Is this just a coincidence?

A: Hardly! Remember, the two impedance solutions must result in the same magnitude for $\Gamma$--for this example we find $|\Gamma(z)|=0.625$.

Thus, for impedances where $r=1$ (i.e., $z^{\prime}=1+j x$ ):

$$
\Gamma=\frac{z^{\prime}-1}{z^{\prime}+1}=\frac{(1+j x)-1}{(1+j x)+1}=\frac{j x}{2+j x}
$$

and therefore:

$$
|\Gamma|^{2}=\frac{|j x|^{2}}{|2+j x|^{2}}=\frac{x^{2}}{4+x^{2}}
$$

Meaning:

$$
x^{2}=\frac{4|\Gamma|^{2}}{1-|\Gamma|^{2}}
$$

of which there are two equal by opposite solutions!

$$
x= \pm \frac{2|\Gamma|}{\sqrt{1-|\Gamma|^{2}}}
$$

Which for this example gives us our solutions $x= \pm 1.6$.

## Impedance \& Admittance

As an alternative to impedance $Z$, we can define a complex parameter called admittance $Y$ :

$$
y=\frac{I}{V}
$$

where $V$ and $I$ are complex voltage and current, respectively.
Clearly, admittance and impedance are not independent parameters, and are in fact simply geometric inverses of each other:

$$
y=\frac{1}{z} \quad z=\frac{1}{y}
$$

Thus, all the impedance parameters that we have studied can be likewise expressed in terms of admittance, e.g.:

$$
y(z)=\frac{1}{Z(z)} \quad y_{L}=\frac{1}{Z_{L}} \quad y_{i n}=\frac{1}{Z_{i n}}
$$

Moreover, we can define the characteristic admittance $y_{0}$ of a transmission line as:

$$
y_{0}=\frac{I^{+}(z)}{V^{+}(z)}
$$

And thus it is similarly evident that characteristic impedance and characteristic admittance are geometric inverses:

$$
y_{0}=\frac{1}{z_{0}} \quad z_{0}=\frac{1}{y_{0}}
$$

As a result, we can define a normalized admittance value $y^{\prime}$ :

$$
y^{\prime}=\frac{y}{y_{0}}
$$

An therefore (not surprisingly) we find:

$$
y^{\prime}=\frac{y}{y_{0}}=\frac{z_{0}}{z}=\frac{1}{z^{\prime}}
$$

Note that we can express normalized impedance and admittance more compactly as:

$$
y^{\prime}=y Z_{0} \quad \text { and } \quad z^{\prime}=Z Y_{0}
$$

Now since admittance is a complex value, it has both a real and imaginary component:

$$
y=G+j B
$$

where:

$$
\begin{aligned}
& \operatorname{Re}\{y\} \doteq G=\text { Conductance } \\
& \operatorname{Im}\{Z\} \doteq B=\text { Susceptance }
\end{aligned}
$$

Now, since $Z=R+j X$, we can state that:

$$
G+j B=\frac{1}{R+j X}
$$



Q: Yes yes, I see, and from this we can conclude:

$$
G=\frac{1}{R} \quad \text { and } \quad B=\frac{-1}{X}
$$

and so forth. Please speed this up and quit wasting my valuable time making such obvious statements!

A: NOOOO! We find that $G \neq 1 / R$ and $B \neq 1 / X$ (generally). Do not make this mistake!

In fact, we find that

$$
\begin{aligned}
G+j B & =\frac{1}{R+j X} \frac{R-j X}{R-j X} \\
& =\frac{R-j X}{R^{2}+X^{2}} \\
& =\frac{R}{R^{2}+X^{2}}-j \frac{X}{R^{2}+X^{2}}
\end{aligned}
$$

Thus, equating the real and imaginary parts we find:

$$
G=\frac{R}{R^{2}+X^{2}} \quad \text { and } \quad B=\frac{-X}{R^{2}+X^{2}}
$$

Note then that IF $X=0$ (ie., $Z=R$ ), we get, as expected:

$$
G=\frac{1}{R} \quad \text { and } \quad B=0
$$

And that IF $R=0$ (ie., $Z=R$ ), we get, as expected:

$$
G=0 \quad \text { and } \quad B=\frac{-1}{x}
$$

I wish I had a nickel for every time my software
has crashed-oh wait, I do!

## Admittance and the Smith Chart

Just like the complex impedance plane, we can plot points and contours on the complex admittance plane:


Q: Can we also map these points and contours onto the complex $\Gamma$ plane?
A: You bet! Let's first rewrite the refection coefficient function in terms of line admittance $Y(z)$ :

$$
\Gamma(z)=\frac{y_{0}-y(z)}{y_{0}+y(z)}
$$

## Rotation around the Smith Chart

Thus,

$$
\Gamma_{L}=\frac{y_{0}-y_{L}}{y_{0}+y_{L}} \quad \text { and } \quad \Gamma_{i n}=\frac{y_{0}-y_{i n}}{y_{0}+y_{i n}}
$$

We can therefore likewise express $\Gamma$ in terms of normalized admittance:

$$
\Gamma=\frac{y_{0}-y}{y_{0}+y}=\frac{1-y / y_{0}}{1+y / y_{0}}=\frac{1-y^{\prime}}{1+y^{\prime}}
$$

Note this can likewise be expressed as:

$$
\Gamma=\frac{1-y^{\prime}}{1+y^{\prime}}=-\frac{y^{\prime}-1}{y^{\prime}+1}=e^{j \pi} \frac{y^{\prime}-1}{y^{\prime}+1}
$$

Contrast this to the mapping between normalized impedance and $\Gamma$ :

$$
\Gamma=\frac{z^{\prime}-1}{z^{\prime}+1}
$$

The difference between the two is simply the factor $e^{j \pi}$-a rotation of $180^{\circ}$ around the Smith Chart!.

## An example

For example, let's pick some load at random; $z^{\prime}=1+j$, for instance. We know where this point is mapped onto the complex $\Gamma$ plane; we can locate it on our Smith Chart.

Now let's consider a different load, and express it in terms of its normalized admittance-an admittance that has the same numerical value as the impedance of the first load (i.e., $y^{\prime}=1+j$ ).

Q: Where would this admittance value map onto the complex $\Gamma$ plane?

A: Start at the location $z^{\prime}=1+j$ on the Smith Chart, and then rotate around the center $180^{\circ}$. You are now at the proper location on the complex $\Gamma$ plane for the admittance $y^{\prime}=1+j$ !


We of course could just directly calculate $\Gamma$ from the equation above, and then plot that point on the $\Gamma$ plane.

Note the reflection coefficient for $z^{\prime}=1+j$ is:

$$
\Gamma=\frac{z^{\prime}-1}{z^{\prime}+1}=\frac{1+j-1}{1+j+1}=\frac{j}{2+j}
$$

while the reflection coefficient for $y^{\prime}=1+j$ is:

$$
\Gamma=\frac{1-y^{\prime}}{1+y^{\prime}}=\frac{1-(1+j)}{1+(1+j)}=\frac{-j}{2+j}
$$

Note the two results have equal magnitude, but are separated in phase by $180^{\circ}\left(-1=e^{j \pi}\right)$. This means that the two loads occupy points on the complex $\Gamma$ plane that are a $180^{\circ}$ rotation from each other!

Moreover, this is a true statement not just for the point we randomly picked, but is true for any and all values of $z^{\prime}$ and $y^{\prime}$ mapped onto the complex $\Gamma$ plane, provided that $z^{\prime}=y^{\prime}$.

## Another example

For example, the $g=2$ circle mapped on the complex plane can be determined by rotating the $r=2$ circle $180^{\circ}$ around the complex $\Gamma$ plane, and the $b=-1$ contour can be found by rotating the $x=-1$ contour $180^{\circ}$ around the complex $\Gamma$ plane.

## The Admittance Smith Chart

Thus, rotating all the resistance circles and reactance contours of the Smith Chart 180 around the complex $\Gamma$ plane provides us a mapping of complex admittance onto the complex $\Gamma$ plane:

Note that circles and contours have been rotated with respect to the complex $\Gamma$ plane-the complex $\Gamma$ plane remains unchanged!


## We're not surprised!

This result should not surprise us. Recall the case where a transmission line of length $\ell=\lambda / 4$ is terminated with a load of impedance $z_{L}^{\prime}$ (or equivalently, an admittance $y_{L}^{\prime}$ ). The input impedance (admittance) for this case is:

$$
Z_{\text {in }}=\frac{Z_{0}^{2}}{Z_{L}} \Rightarrow \frac{Z_{\text {in }}}{Z_{0}}=\frac{Z_{0}}{Z_{L}} \Rightarrow z_{\text {in }}^{\prime}=\frac{1}{Z_{L}^{\prime}}=y_{L}^{\prime}
$$

In other words, when $\ell=\lambda / 4$, the input impedance is numerically equal to the load admittance-and vice versa!

But note that if $\ell=\lambda / 4$, then $2 \beta \ell=\pi$--a rotation around the Smith Chart of $180^{\circ}$ !

## Example: Admittance

 Calculations with the Smith ChartSay we wish to determine the normalized admittance $y_{1}^{\prime}$ of the network below:


First, we need to determine the normalized input admittance of the transmission line:


## There are two ways to determine this value!

## Method 1

First, we express the load $z_{L}=1.6+j 2.6$ in terms of its admittance $y_{L}^{\prime}=1 / z_{L}$. We can calculate this complex value-or we can use a Smith Chart!


The Smith Chart above shows both the impedance mapping (red) and admittance mapping (blue). Thus, we can locate the impedance $z_{L}=1.6+j 2.6$ on the impedance (red) mapping, and then determine the value of that same $\Gamma_{L}$ point using the admittance (blue) mapping.

From the chart above, we find this admittance value is approximately $y_{L}=0.17-j 0.28$.

Now, you may have noticed that the Smith Chart above, with both impedance and admittance mappings, is very busy and complicated. Unless the two mappings are printed in different colors, this Smith Chart can be very confusing to use!

But remember, the two mappings are precisely identical-they're just rotated $180^{\circ}$ with respect to each other. Thus, we can alternatively determine $y_{\llcorner }$by again first locating $z_{L}=1.6+j 2.6$ on the impedance mapping :

$$
z_{L}=1.6+j 2.6
$$

Then, we can rotate the entire Smith Chart $180^{\circ}$--while keeping the point $\Gamma_{L}$ location on the complex $\Gamma$ plane fixed.


Thus, use the admittance mapping at that point to determine the admittance value of $\Gamma_{L}$.

Note that rotating the entire Smith Chart, while keeping the point $\Gamma_{L}$ fixed on the complex $\Gamma$ plane, is a difficult maneuver to successfully-as well as accurately-execute.

But, realize that rotating the entire Smith Chart $180^{\circ}$ with respect to point $\Gamma_{L}$ is equivalent to rotating $180^{\circ}$ the point $\Gamma_{L}$ with respect to the entire Smith Chart!

This maneuver (rotating the point $\Gamma_{L}$ ) is much simpler, and the typical method for determining admittance.



Transforming the load admittance to the beginning of the transmission line, we have determined that $y_{i n}^{\prime}=0.7-j 1.7$.

## Method 2

Alternatively, we could have first transformed impedance $z_{L}^{\prime}$ to the end of the line (finding $z_{\text {in }}^{\prime}$ ), and then determined the value of $y_{\text {in }}^{\prime}$ from the admittance mapping (i.e., rotate $180^{\circ}$ around the Smith Chart).


The input impedance is determined after rotating clockwise $2 \beta \ell$, and is $z_{\text {in }}^{\prime}=0.2+j 0.5$.

Now, we can rotate this point $180^{\circ}$ to determine the input admittance value $y_{\text {in }}^{\prime}$ :


The result is the same as with the earlier method-$y_{\text {in }}^{\prime}=0.7-j 1.7$.

Hopefully it is evident that the two methods are equivalent. In method 1 we first rotate $180^{\circ}$, and then rotate $2 \beta \ell$. In the second method we first rotate $2 \beta \ell$, and then rotate $180^{\circ}$--the result is thus the same!

Now, the remaining equivalent circuit is:

Determining $y_{1}^{\prime}$ is just basic circuit theory. We first express $z_{2}^{\prime}$ in terms of its admittance $y_{2}^{\prime}=1 / z_{2}^{\prime}$.

Note that we could do this using a calculator, but could likewise use a Smith Chart (locate $z_{2}^{\prime}$ and then rotate $180^{\circ}$ ) to accomplish this calculation! Either way, we find that $y_{2}^{\prime}=0.3+j 0.3$.

Thus, $y_{1}^{\prime}$ is simply:

$$
\begin{aligned}
y_{1}^{\prime} & =y_{2}^{\prime}+y_{i n}^{\prime} \\
& =(0.3+j 0.3)+(0.7-j 1.7) \\
& =1.0-j 1.4
\end{aligned}
$$

