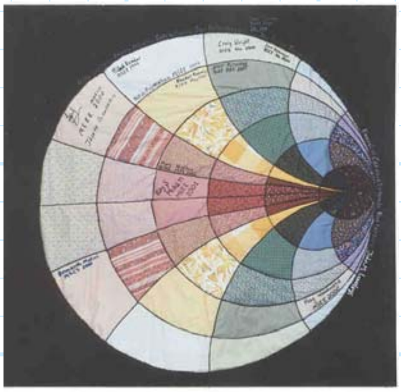
2.4 - The Smith Chart

Reading Assignment: ***pp. 64-73***



The Smith Chart → An **icon** of microwave engineering!

The Smith Chart provides:

1) A **graphical** method to solve many transmission line problems.

2) A **visual** indication of microwave device performance.

The **most** important fact about the Smith Chart is:

→ It exists on the complex Γ plane.

HO: THE COMPLEX Γ PLANE

Q: *But how is the complex Γ plane **useful**?*

A: We can easily plot and determine values of $\Gamma(z)$

HO: TRANSFORMATIONS ON THE COMPLEX Γ PLANE

Q: *But transformations of Γ are relatively easy—transformations of line impedance Z is the **difficult** one.*

A: We can likewise map **line impedance** onto the complex Γ plane!

HO: Mapping Z to Γ

HO: The Smith Chart

HO: Smith Chart Geography

HO: The Outer Scale

The Smith Chart allows us to **solve** many important transmission line problems!

HO: Z_{IN} Calculations using the Smith Chart

Example: The Input Impedance of a Shorted Transmission Line

Example: Determining the Load Impedance of a Transmission Line

Example: Determining the Length of a Transmission Line

An alternative to impedance Z, is its inverse—**admittance** Y.

HO: Impedance and Admittance

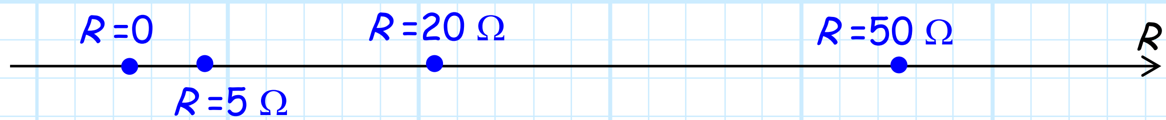
Expressing a load or line impedance in terms of its admittance is sometimes helpful. Additionally, we can easily map admittance onto the Smith Chart.

HO: ADMITTANCE AND THE SMITH CHART

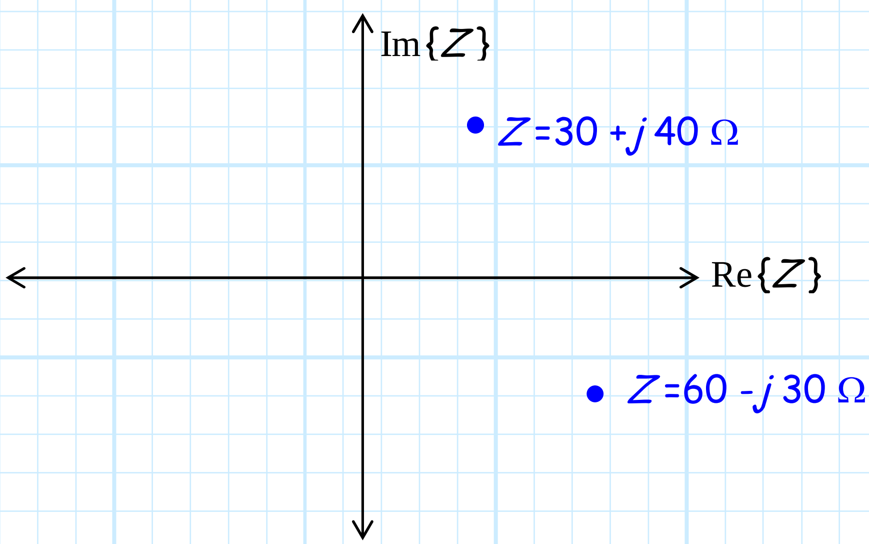
EXAMPLE: ADMITTANCE CALCULATIONS WITH THE SMITH CHART

The Complex Γ Plane

Resistance R is a **real** value, thus we can indicate specific resistor values as points on the **real line**:



Likewise, since impedance Z is a **complex** value, we can indicate specific impedance values as point on a two dimensional **complex impedance plane** :



Note each dimension is defined by a single real line:

* The **horizontal** line (axis) indicating the **real** component of Z (i.e., $\text{Re}\{Z\}$).

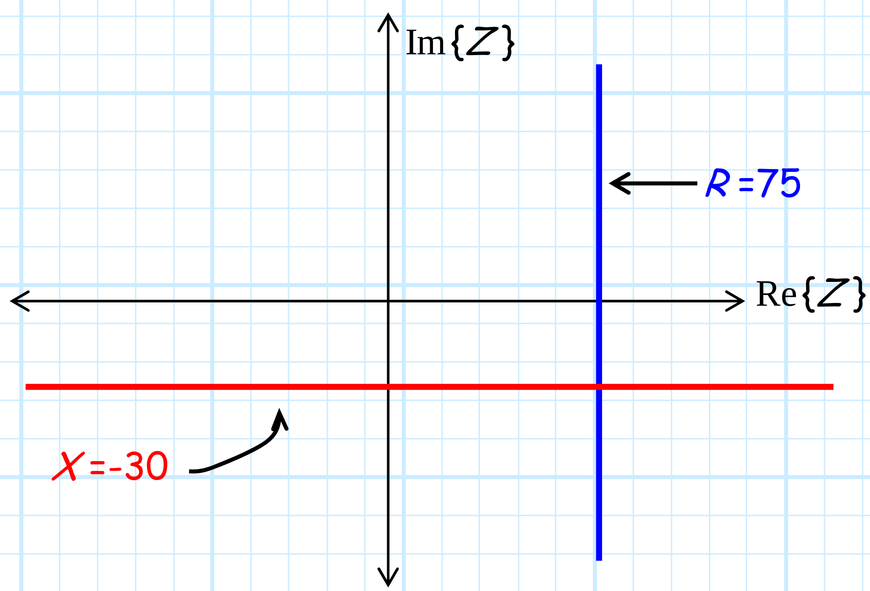
* The **vertical** line (axis) indicating the **imaginary** component of impedance Z (i.e., $\text{Im}\{Z\}$).

The **intersection** of these two lines is the point denoting the impedance $\boldsymbol{Z} = \mathbf{0}$.

Lines and Curves on the Complex Z Plane

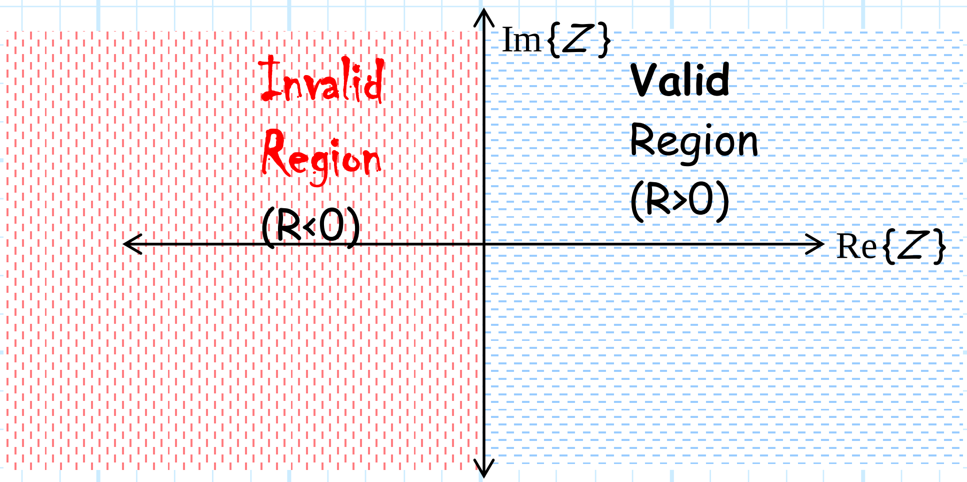
* Note then that a **vertical line** is formed by the locus of **all** points (impedances) whose **resistive** (i.e., real) component is equal to, say, 75.

* Likewise, a **horizontal line** is formed by the locus of **all** points (impedances) whose **reactive** (i.e., imaginary) component is equal to -30.

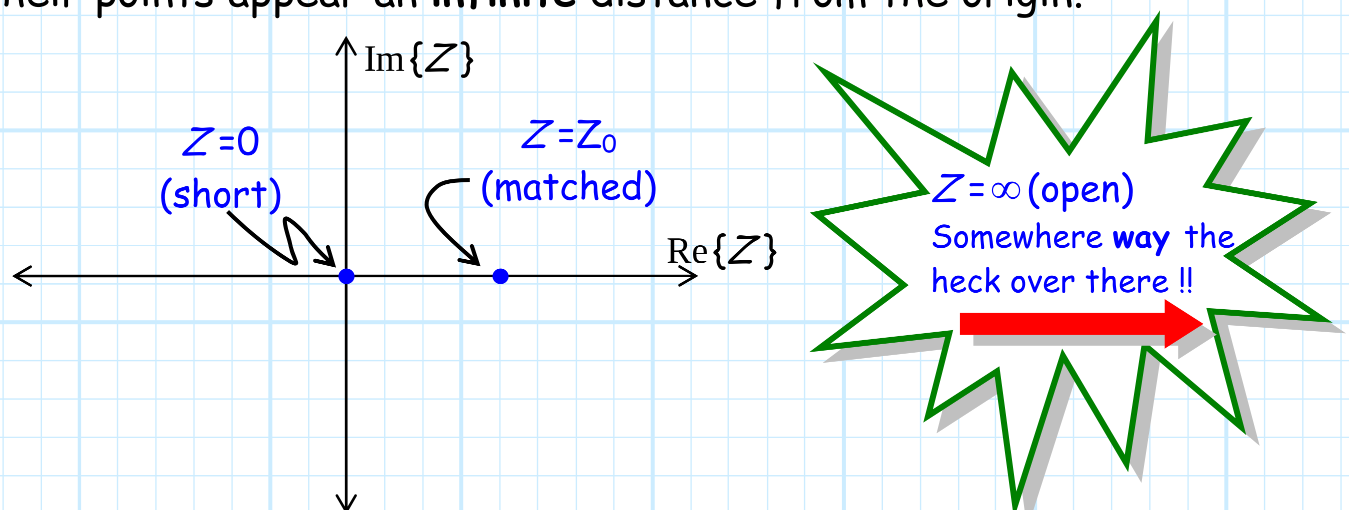


The Validity Region of the Complex Z Plane

If we assume that the **real** component of **every** impedance is **positive**, then we find that **only the right side** of the plane will be useful for plotting impedance Z—points on the left side indicate impedances with **negative** resistances!



Moreover, we find that common impedances such as $Z=\infty$ (an open circuit!) **cannot** be plotted, as their points appear an **infinite** distance from the origin.

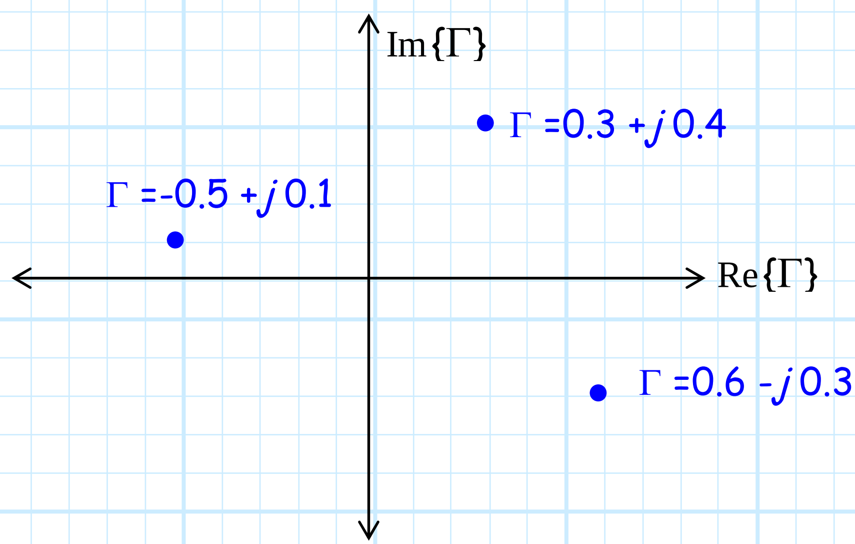


The Complex Γ Plane

Q: *Yikes! The complex Z plane does **not** appear to be a very helpful. Is there some graphical tool that **is** more useful?*

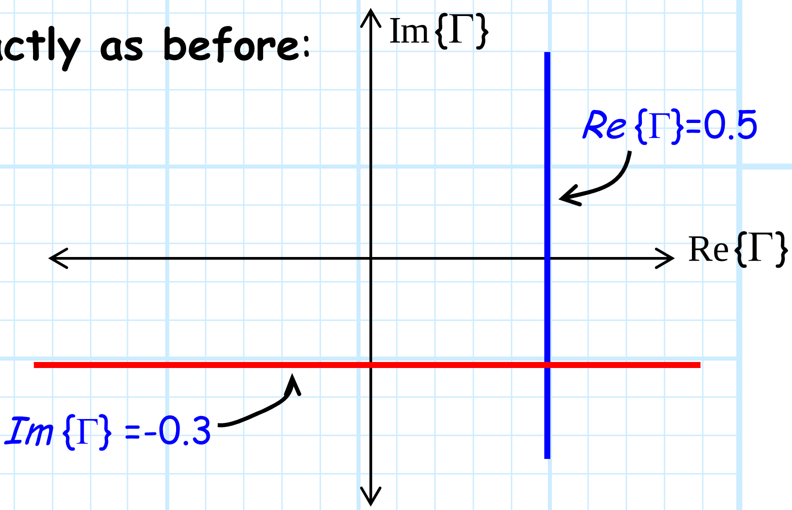
A: Yes! Recall that impedance Z and reflection coefficient Γ are **equivalent complex values**—if you know **one**, you know the **other**.

We can therefore define a **complex Γ plane** in the same manner that we defined a complex impedance plane. We will find that there are **many** advantages to plotting on the complex Γ plane, as opposed to the complex Z plane!



Lines and Curves on the Complex Γ Plane

We **can** plot points and lines on this complex Γ plane **exactly as before**:

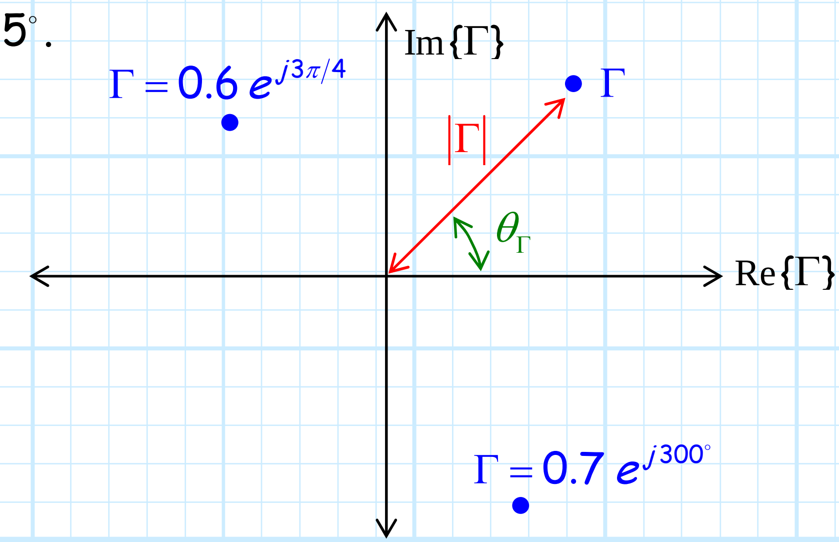


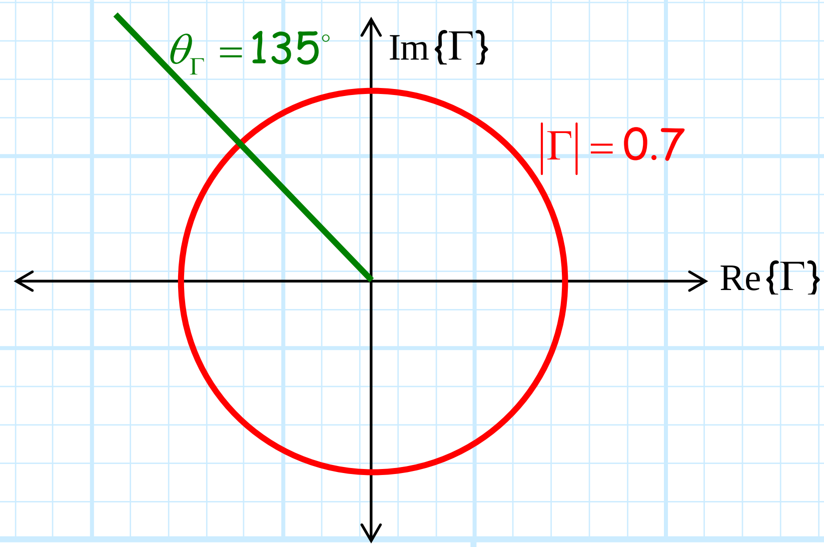
However, we will find that the utility of the complex Γ pane as a graphical tool becomes apparent **only** when we represent a **complex** reflection coefficient in terms of its **magnitude** ($|\Gamma|$) and **phase** (θ_Γ):

$$\Gamma = |\Gamma| e^{j\theta_\Gamma}$$

In other words, we express Γ using **polar coordinates**.

Note then that a **circle** is formed by the locus of all points whose **magnitude** $|\Gamma|$ equal to, say, 0.7. Likewise, a **radial line** is formed by the locus of all points whose **phase** θ_Γ is equal to 135°.





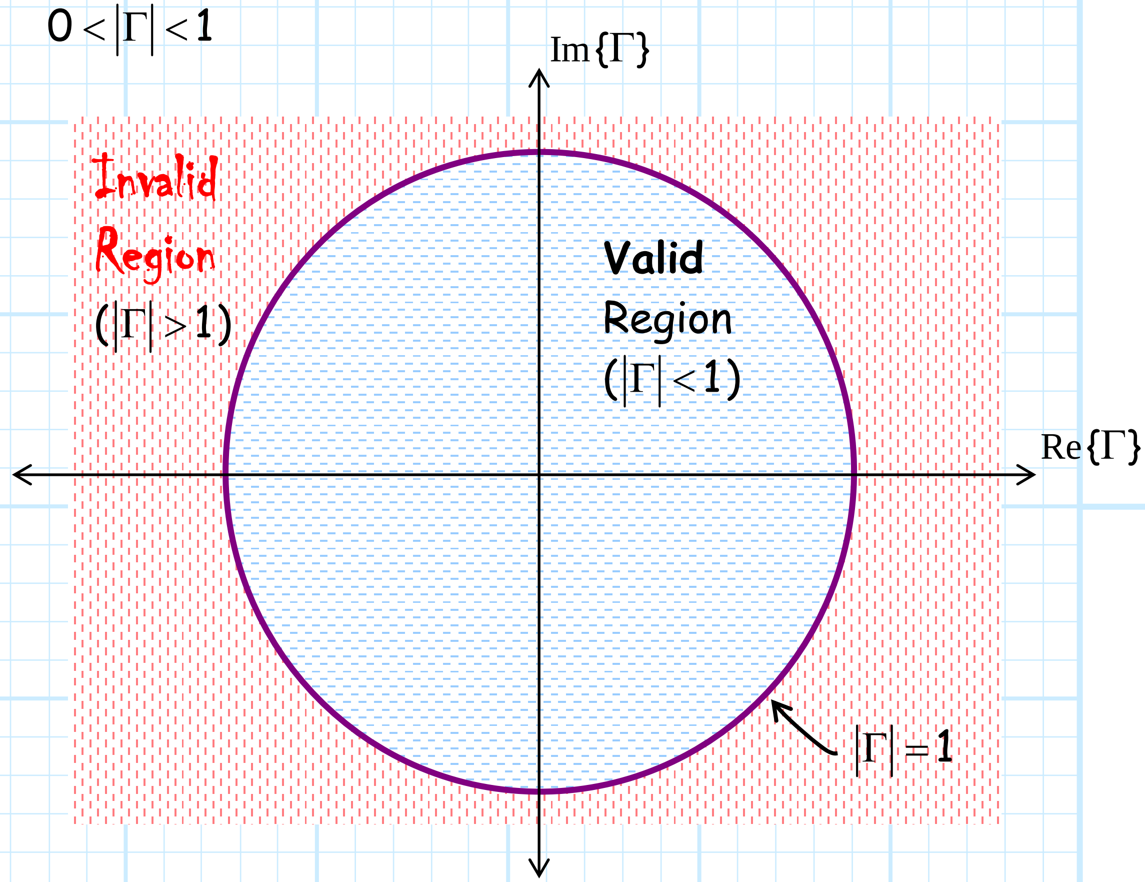
The Validity Region of the Complex Γ Plane

Perhaps the most important aspect of the complex Γ plane is its **validity region**. Recall for the complex Z plane that this validity region was **unbounded** and **infinite** in extent, such that many important impedances (e.g., open-circuits) could **not** be plotted.

Q: *What is the validity region for the complex Γ plane?*

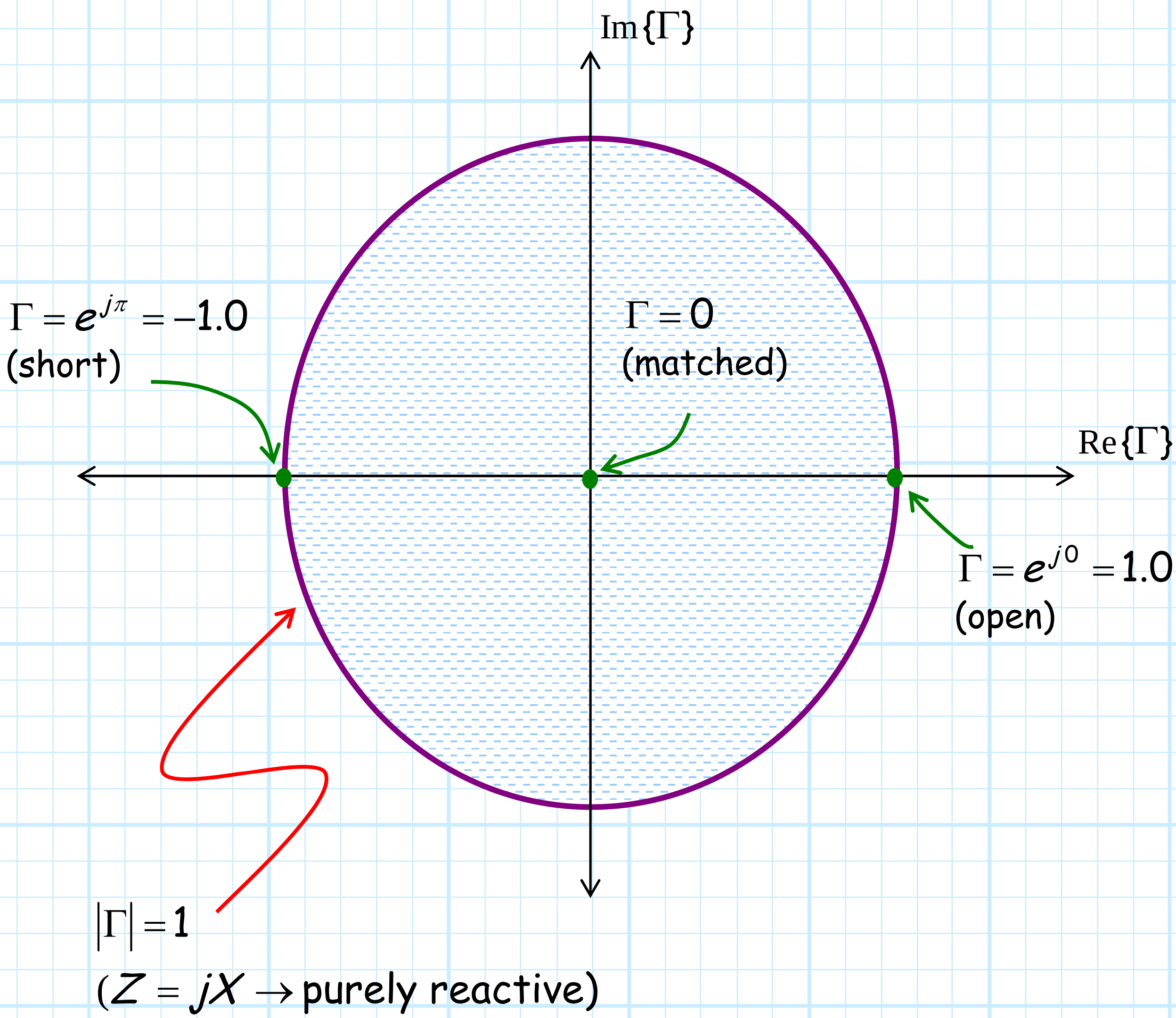
A: Recall that we found that for $\text{Re}\{Z\} > 0$ (i.e., positive resistance), the **magnitude** of the reflection coefficient was **limited**:

$$0 < |\Gamma| < 1$$



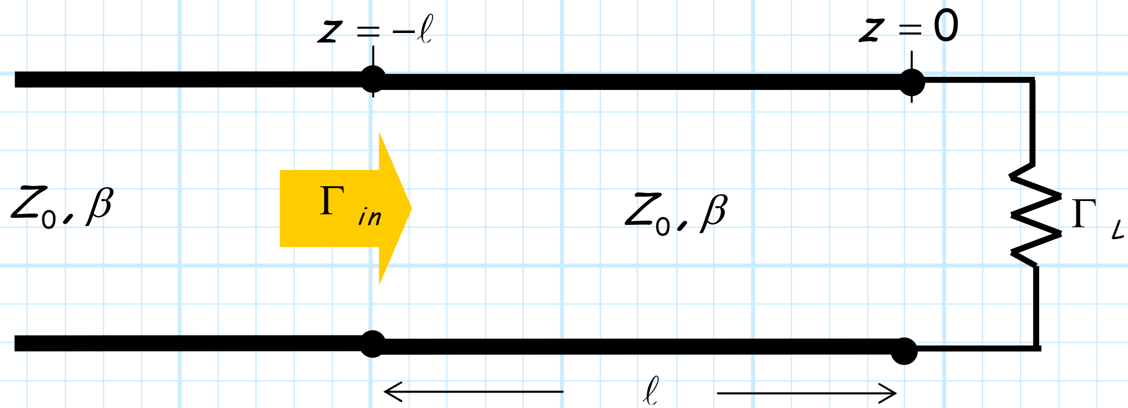
Therefore, the **validity region** for the complex Γ plane consists of all points **inside the circle** $|\Gamma| = 1$--a finite **and** bounded area!

Note that we can plot **all** valid impedances (i.e., $R>0$) within this **finite** validity region!



Transformations on the Complex Γ Plane

The usefulness of the complex Γ plane is apparent when we consider again the **terminated, lossless transmission line**:



Recall that the reflection coefficient function for **any** location z along the transmission line can be expressed as (since $z_L = 0$):

$$\Gamma(z) = \Gamma_L \, e^{j2\beta z} = |\Gamma_L| \, e^{j(\theta_\Gamma + 2\beta z)}$$

And thus, as we would **expect**:

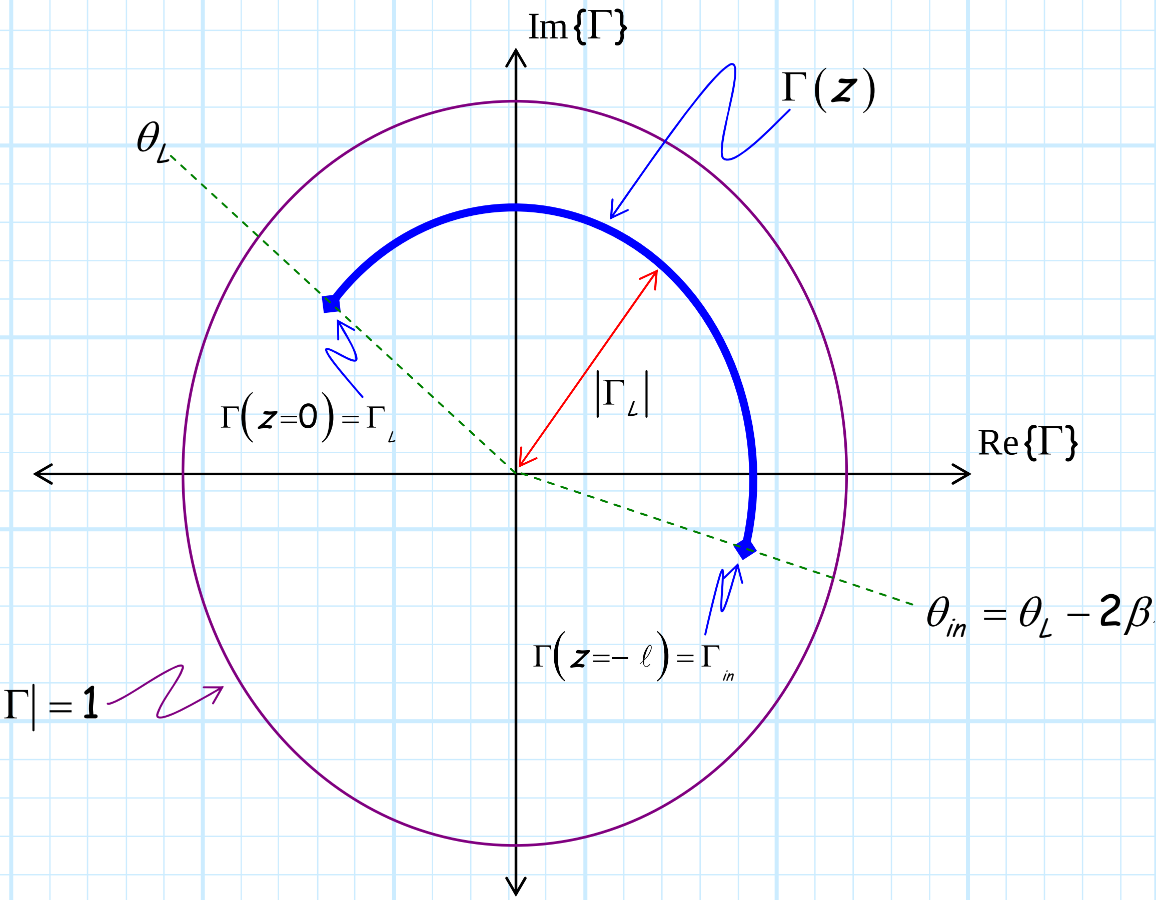
$$\Gamma(z=0) = \Gamma_L \quad \text{and} \quad \Gamma(z=-\ell) = \Gamma_L e^{-j2\beta\ell} = \Gamma_{in}$$

Transforming Γ_L to Γ_{in}

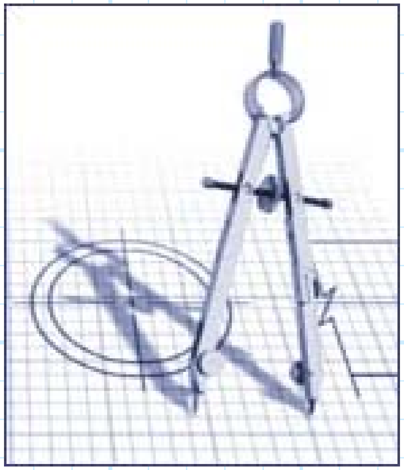
Recall this result "says" that adding a transmission line of length ℓ to a load results in a **phase shift** in θ_Γ by $-2\beta\ell$ radians, while the **magnitude** $|\Gamma|$ remains **unchanged**.

Q: *Magnitude $|\Gamma|$ and phase θ_Γ --aren't those the values used when **plotting** on the complex Γ plane?*

A: Precisely! In fact, **plotting** the transformation of Γ_L to Γ_{in} along a transmission line length ℓ has an interesting **graphical** interpretation. Let's **parametrically** plot $\Gamma(z)$ from $z = z_L$ (i.e., $z = 0$) to $z = z_L - \ell$ (i.e., $z = -\ell$):



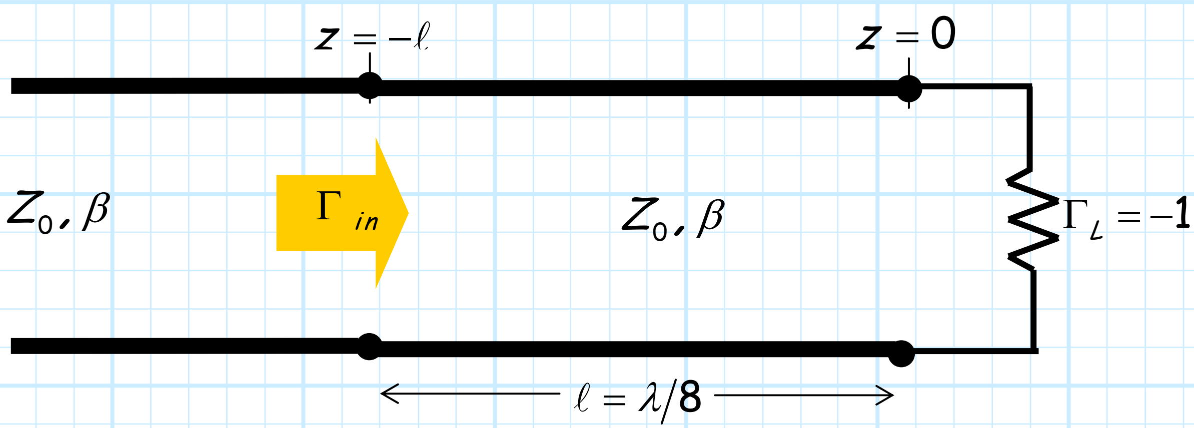
Graphically Transforming Γ_L to Γ_{in}



Since adding a length of transmission line to a load Γ_L **modifies** the **phase** θ_Γ but **not** the **magnitude** $|\Gamma_L|$, we trace a **circular arc** as we parametrically plot $\Gamma(z)$! This arc has a **radius** $|\Gamma_L|$ and an **arc angle** $2\beta\ell$ radians.

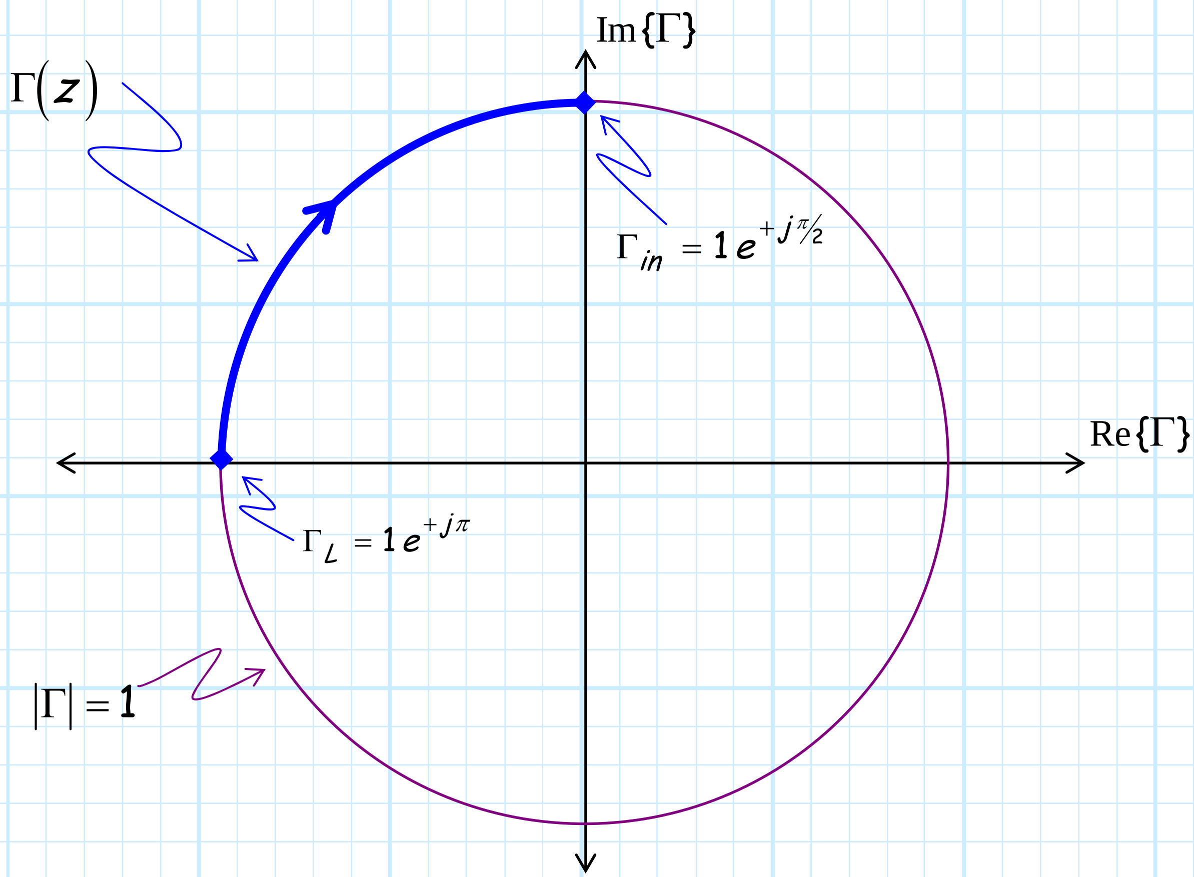
With this knowledge, we can **easily** solve many interesting transmission line problems **graphically**—using the complex Γ plane!

For **example**, say we wish to determine Γ_{in} for a transmission line length $\ell = \lambda/8$ and terminated with a **short** circuit.



Example: Graphically Transforming Γ_L to Γ_{in}

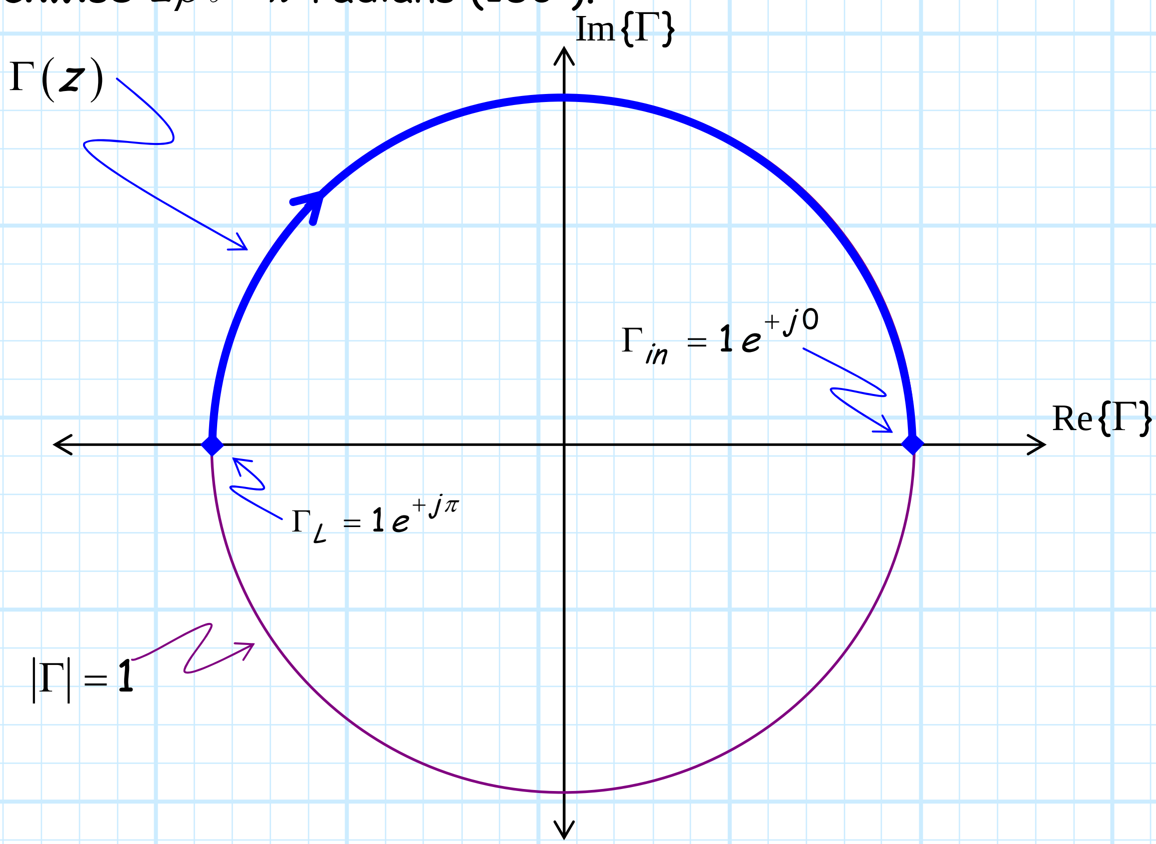
The reflection coefficient of a **short** circuit is $\Gamma_L = -1 = 1\,e^{j\pi}$, and therefore we **begin** at that point on the complex Γ plane. We then move along a **circular arc** $-2\beta\ell = -2(\pi/4) = -\pi/2$ radians (i.e., rotate **clockwise** 90°).



When we **stop**, we find we are at the point for Γ_{in}; in this case $\Gamma_{in} = 1\,e^{j\pi/2}$ (i.e., magnitude is **one**, phase is **90°**).

Example: Now with $\ell = \lambda/4$

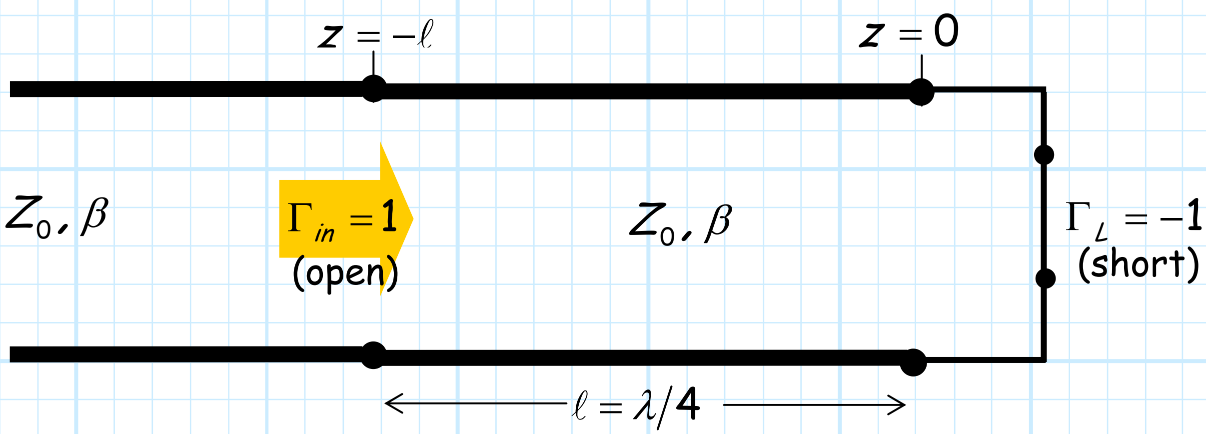
Now, let's **repeat** this same problem, only with a **new** transmission line **length** of $\ell = \lambda/4$. Now we rotate **clockwise** $2\beta\ell = \pi$ radians (180°).



For this case, the **input** reflection coefficient is $\Gamma_{in} = 1e^{j0} = 1$: the reflection coefficient of an **open circuit**!

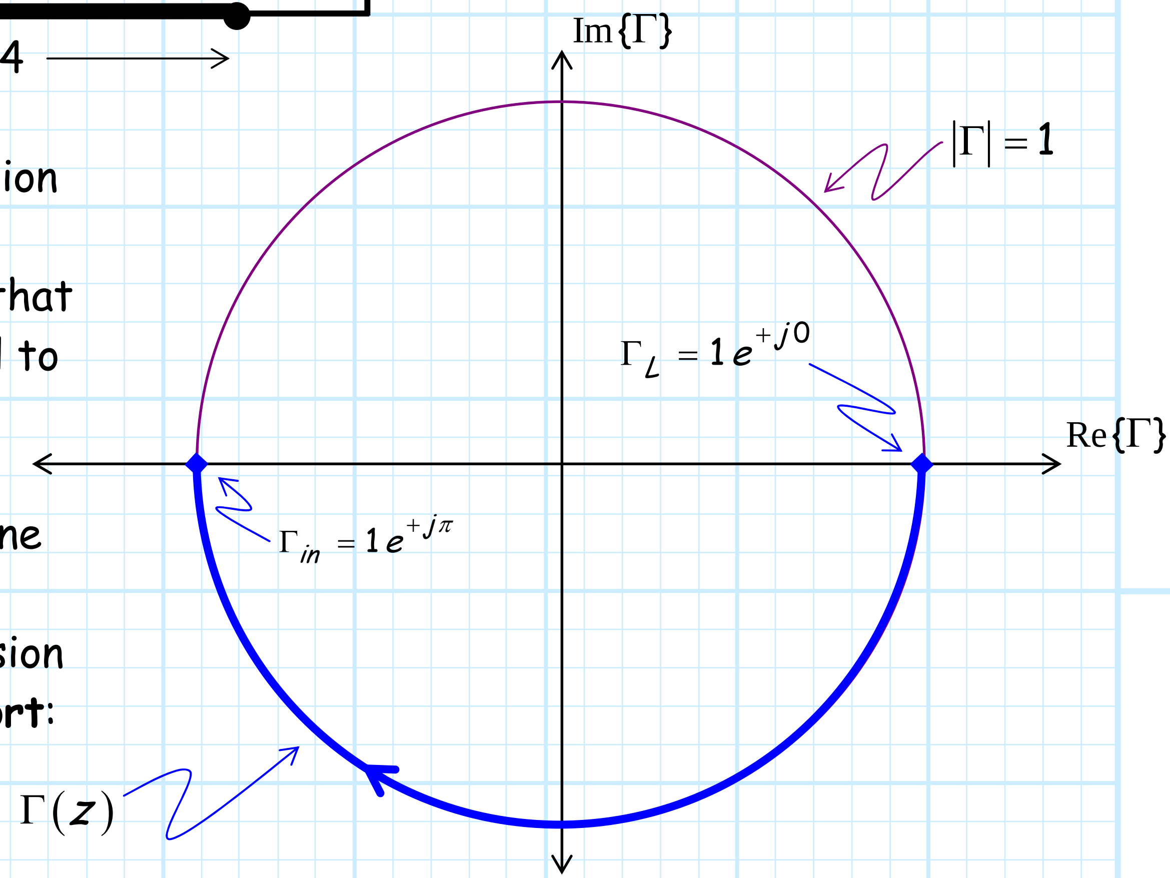
Our **short**-circuit load has been transformed into an **open** circuit with a **quarter-wavelength** transmission line!

You're not surprised—are you?



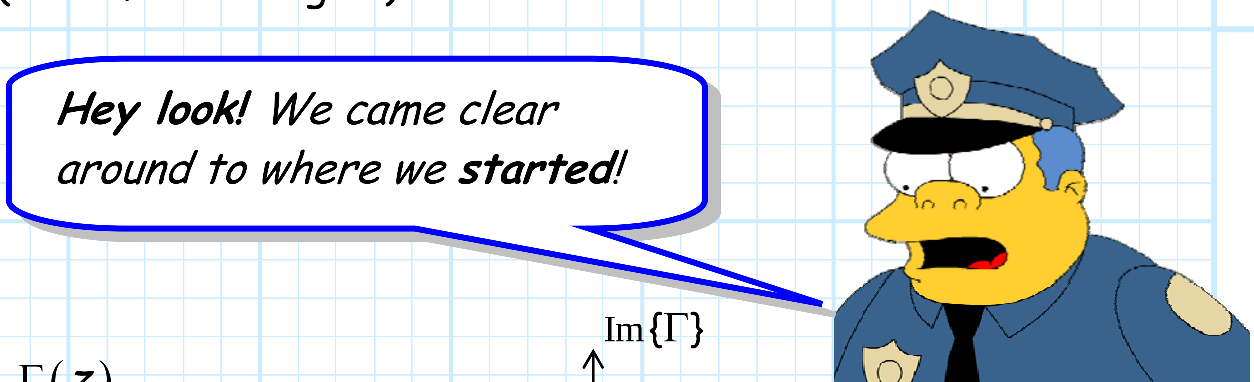
Recall that a **quarter**-wave transmission line was one of the **special cases** we considered earlier. Recall we found that the input impedance was proportional to the **inverse** of the load impedance.

Thus, a **quarter-wave** transmission line transforms a **short** into an **open**. Conversely, a quarter-wave transmission can also transform an **open** into a **short**:



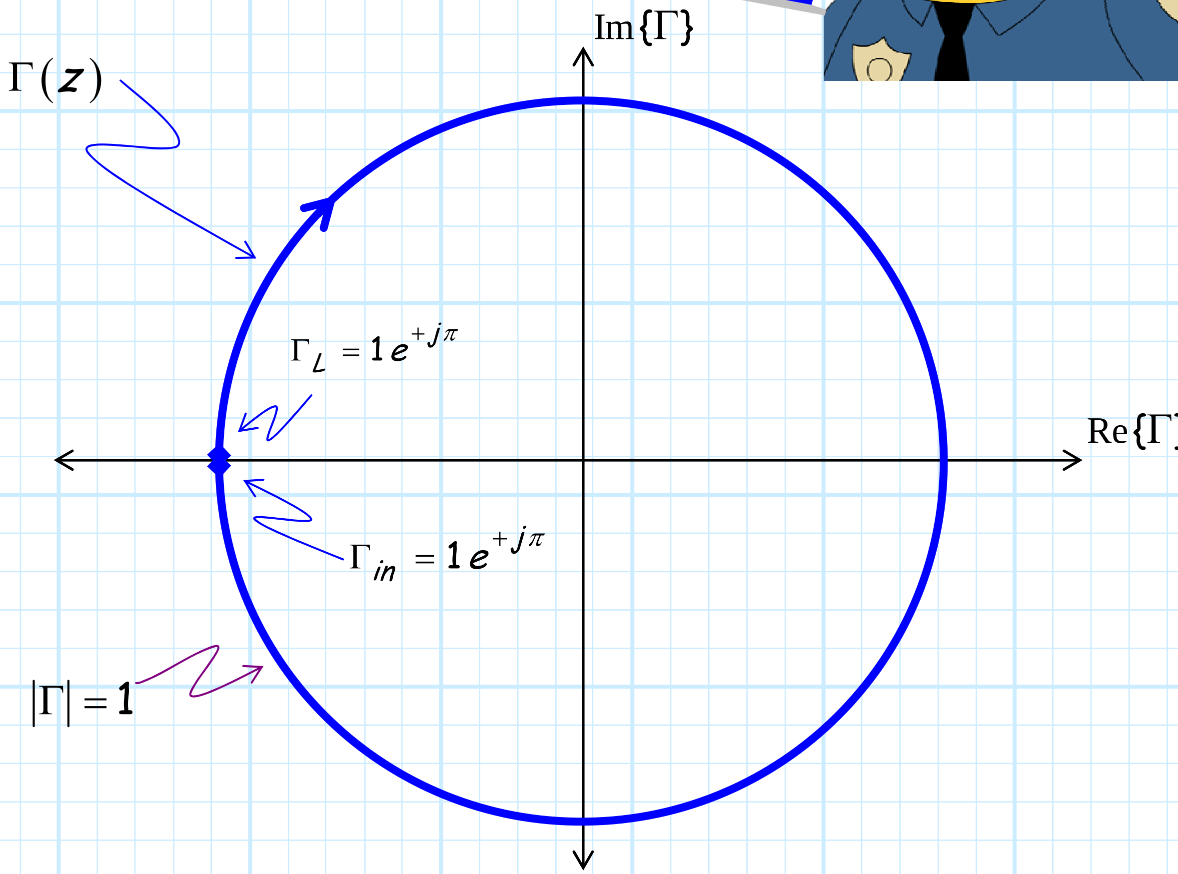
Example: Now with $\ell = \lambda/2$

Finally, let's **again** consider the problem where $\Gamma_L = -1$ (i.e., short), only this time with a transmission line length $\ell = \lambda/2$ (a **half** wavelength!). We rotate **clockwise** $2\beta\ell = 2\pi$ radians (360°).



Thus, we find that $\Gamma_{in} = \Gamma_L$ **if** $\ell = \lambda/2$--but you knew **this** too!

Recall that the **half**-wavelength transmission line is likewise a **special case**, where we found that $Z_{in} = Z_L$. This result, of course, likewise means that $\Gamma_{in} = \Gamma_L$.



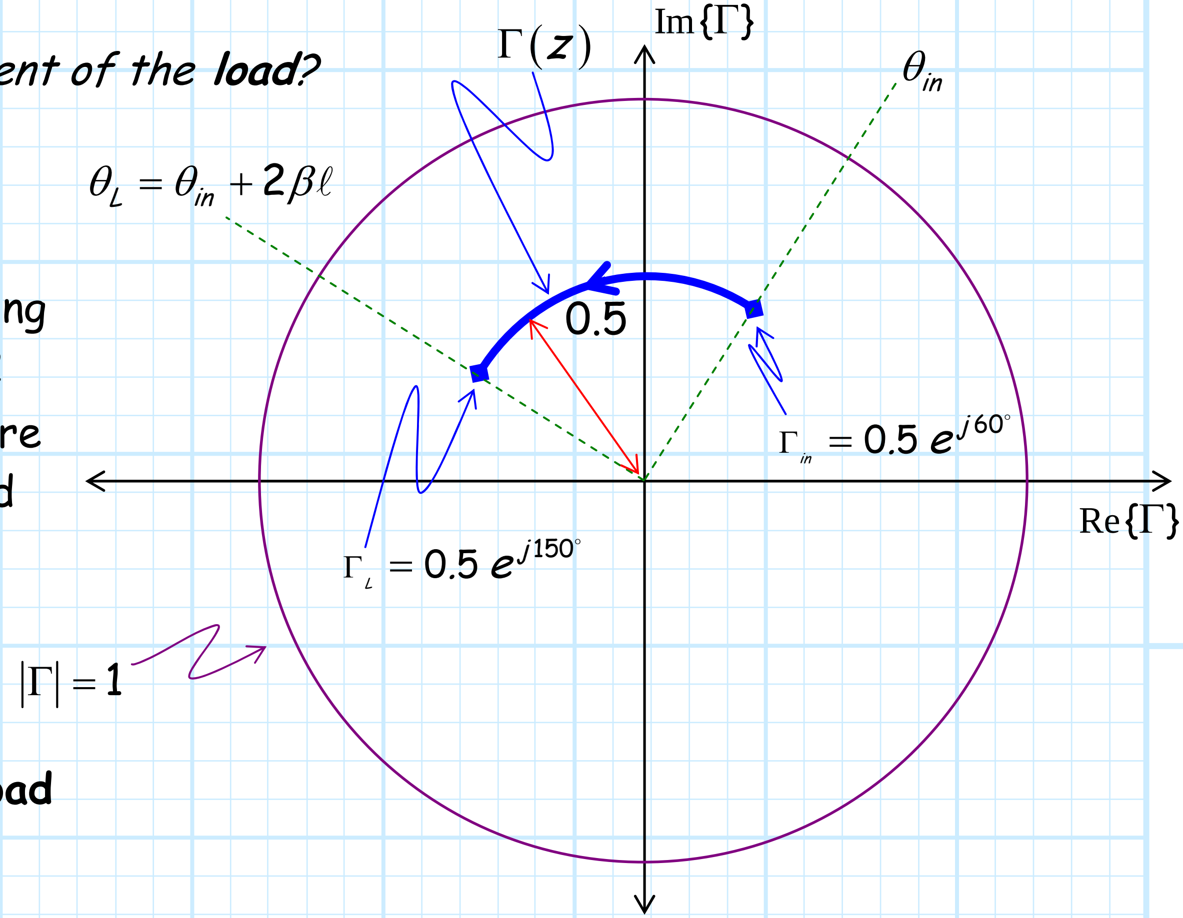
Example: Now transform Γ_{in} to Γ_L

Now, let's consider the **opposite** problem. Say we know that the **input** impedance at the **beginning** of a transmission line with length $\ell = \lambda/8$ is:

$$\Gamma_{in} = 0.5\, e^{j60^\circ}$$

Q: *What is the reflection coefficient of the **load**?*

A: In this case, we begin at Γ_{in} and rotate **COUNTER-CLOCKWISE** along a circular arc (radius 0.5) $2\beta\ell = \pi/2$ radians (i.e., 60°). Essentially, we are **removing** the phase shift associated with the transmission line!



The reflection coefficient of the **load** is therefore:

$$\Gamma_L = 0.5\, e^{j150^\circ}$$

Mapping Z to Γ

Recall that line impedance and reflection coefficient are **equivalent**—either one can be expressed in terms of the other:

$$\Gamma(z)=\frac{Z(z)-Z_0}{Z(z)+Z_0} \qquad \text{and} \qquad Z(z)=Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

Note this relationship also depends on the **characteristic impedance** Z_0 of the transmission line. To make this relationship **more direct**, we first define a **normalized** impedance value z' (an impedance coefficient!):

$$z'(z)=\frac{Z(z)}{Z_0}=\frac{R(z)}{Z_0}+j\,\frac{X(z)}{Z_0}=r(z)+j\,x(z)$$

Using this definition, we find:

$$\Gamma(z)=\frac{Z(z)-Z_0}{Z(z)+Z_0}=\frac{Z(z)/Z_0-1}{Z(z)/Z_0+1}=\frac{z'(z)-1}{z'(z)+1}$$

Normalized Impedance

Thus, we can express $\Gamma(z)$ explicitly in terms of **normalized impedance** z'--and vice versa!

$$\Gamma(z)=\frac{z'(z)-1}{z'(z)+1} \qquad z'(z)=\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

The equations above describe a **mapping** between coefficients z' and Γ. This means that each and every normalized **impedance** value likewise corresponds to one specific point on the **complex** Γ **plane**!

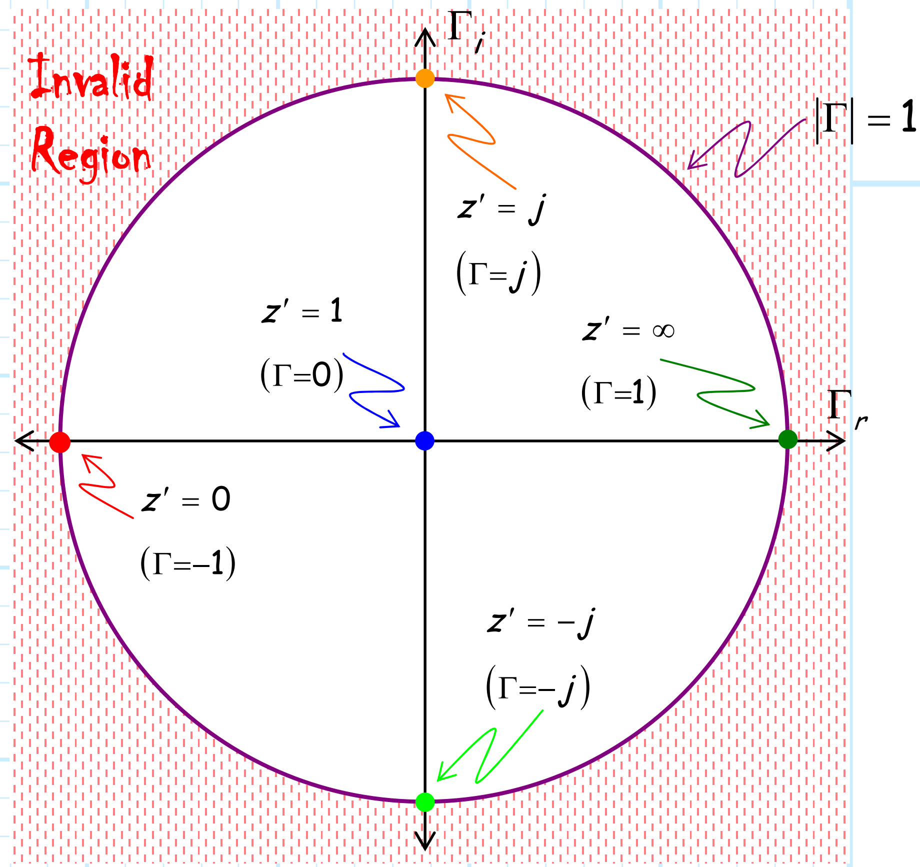
For example, say we wish to mark or somehow indicate the values of normalized **impedance** z' that correspond to the various points on the **complex** Γ **plane**.

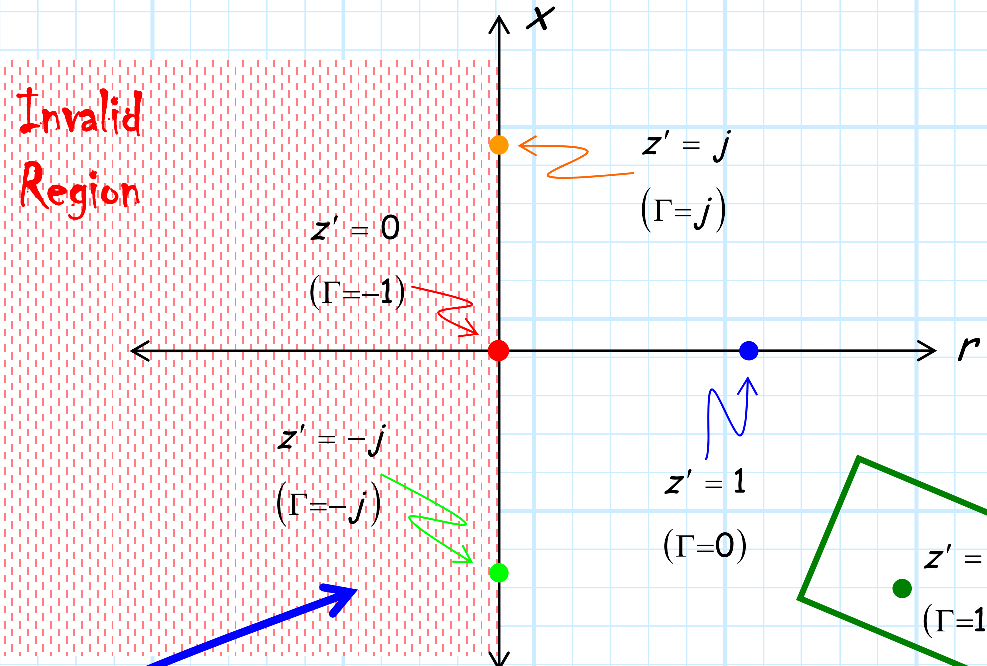
Some values we already know **specifically** →

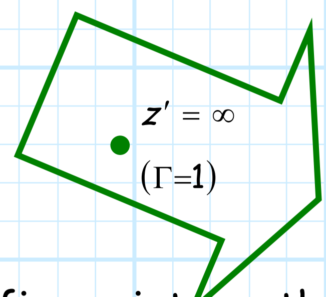
case	Z	z'	Γ
1	∞	∞	1
2	0	0	-1
3	Z_0	1	0
4	jZ_0	j	j
5	$-jZ_0$	$-j$	$-j$

Mapping points on both the Γ and Z planes

Therefore, we find that these five normalized impedances map onto five specific points on the **complex Γ plane** →







Or, the five complex Γ map onto five points on the **normalized impedance plane**.

Mapping contours on both the Γ and Z planes

Now, the preceding provided examples of the mapping of **points** between the complex (normalized) impedance plane, and the complex Γ plane. We can likewise **map whole contours** (i.e., sets of points) between these two complex planes. We shall first look at two familiar cases.

$$Z = R$$

In other words, the case where impedance is purely **real**, with **no** reactive component (i.e., $X = 0$); meaning that **normalized** impedance is:

$$z' = r + j0 \qquad (i.e., x = 0)$$

where we recall that $r = R/Z_0$.

Remember, this real-valued impedance results in a **real-valued** reflection coefficient:

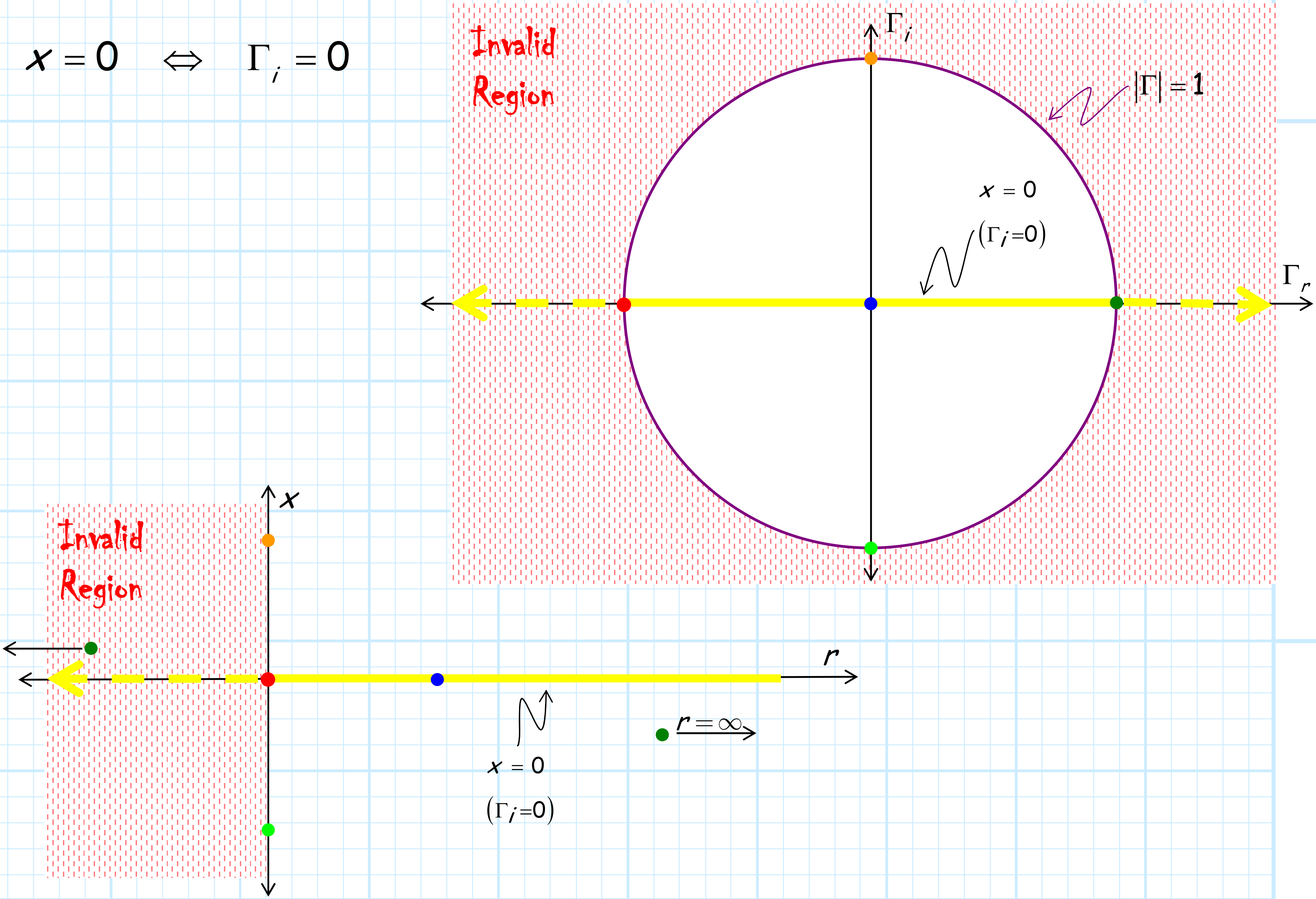
$$\Gamma = \frac{r-1}{r+1}$$

I.E.,:

$$\Gamma_r \doteq Re\{\Gamma\} = \frac{r-1}{r+1} \qquad \Gamma_i \doteq Im\{\Gamma\} = 0$$

Thus, we can determine a mapping between two **contours**—one contour ($x=0$) on the normalized **impedance** plane, the other ($\Gamma_i = 0$) on the complex Γ **plane**:

$$x = 0 \quad \Leftrightarrow \quad \Gamma_i = 0$$



$$Z = jX$$

In other words, the case where impedance is **purely imaginary**, with **no** resistive component (i.e., $R = 0$).

Meaning that normalized impedance is:

$$z' = 0 + jx \qquad (i.e., r = 0)$$

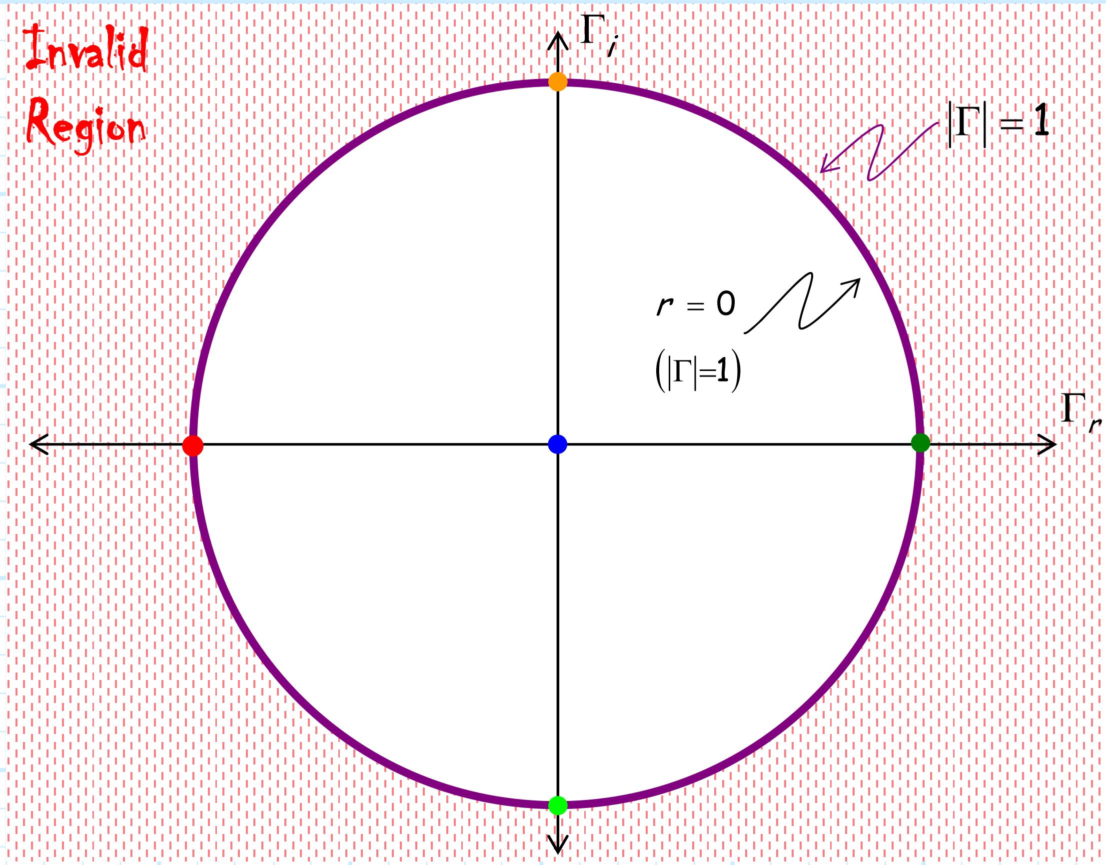
where we recall that $x = X/Z_0$.

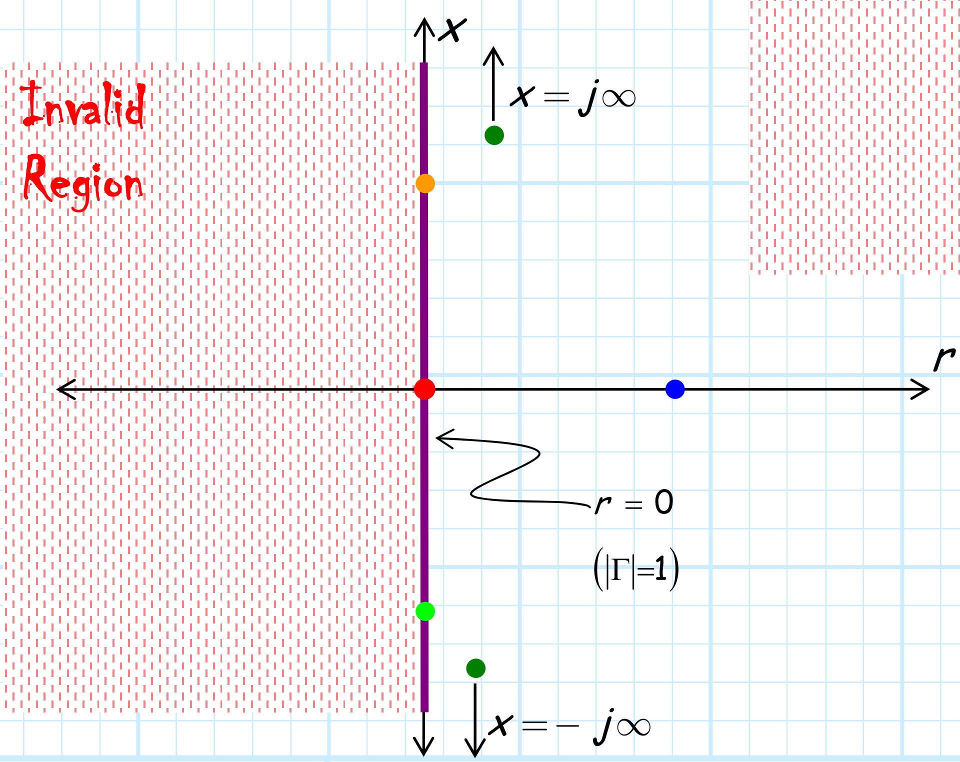
Remember, this **imaginary** impedance results in a reflection coefficient with **unity magnitude**:

$$|\Gamma| = 1$$

Thus, we can determine a mapping between two contours—one contour ($r = 0$) on the normalized **impedance plane**, the other ($|\Gamma| = 1$) on the **complex** Γ **plane**:

$$r = 0 \quad \Leftrightarrow \quad |\Gamma| = 1$$





What about r=0.5, or x=-1.5??

Q: *These **two** "mappings" may very well be fascinating in an **academic** sense, but they are **not** particularly relevant, since actual values of impedance generally have **both** a real and imaginary component.*

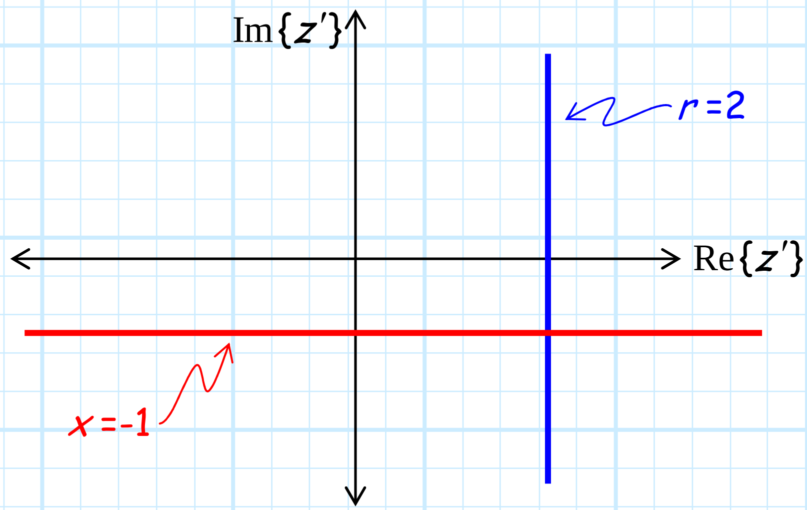
*Sure, mappings of more **general** impedance contours (e.g., $r = 0.5$ or $x = -1.5$) onto the complex Γ **would** be useful—but it seems clear that those mappings are impossible to achieve!?!*

A: Actually, not only are mappings of more general impedance contours (such as $r = 0.5$ and $x = -1.5$) onto the complex Γ plane **possible**, these mappings have **already** been achieved—thanks to **Dr. Smith** and his famous **chart**!

The Smith Chart

Say we wish to map a **line** on the **normalized** complex **impedance** plane **onto** the complex Γ plane.

For example, we could **map** the vertical line r=2 ($\text{Re}\{z'\}=2$) or the horizontal line x=-1 ($\text{Im}\{z'\}=-1$).



Recall r=0 simply maps to the **circle** $|\Gamma|=1$ on the complex Γ plane, and x= 0 simply maps to the **line** $\Gamma_i=0$.

But, for the examples given above, the mapping is **not** so straight forward. The contours will in general be functions of both Γ_r and Γ_i (e.g., $\Gamma_r^2+\Gamma_i^2=0.5$), and thus the mapping **cannot** be stated with **simple** functions such as $|\Gamma|=1$ or $\Gamma_i=0$.

Vertical contours on the complex Z plane map…

As a matter of fact, a **vertical line** on the normalized **impedance** plane of the form:

$$r = c_r \quad ,$$

where c_r is some **constant** (e.g. $r = 2$ or $r = 0.5$), is **mapped** onto the complex Γ plane as:

$$\left(\Gamma_r - \frac{c_r}{1 + c_r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + c_r}\right)^2$$

Note this equation is of the same form as that of a **circle**:

$$(x - x_c)^2 + (y - y_c)^2 = a^2$$

where:

a = the radius of the circle

$P_c(x = x_c, y = y_c) \quad \Rightarrow$ point located at the center of the circle

Thus, the vertical **line** $r = c_r$ maps into a **circle** on the complex Γ plane!

...onto circles on the complex G plane

By inspection, it is apparent that the **center** of this circle is located at this point on the complex Γ plane:

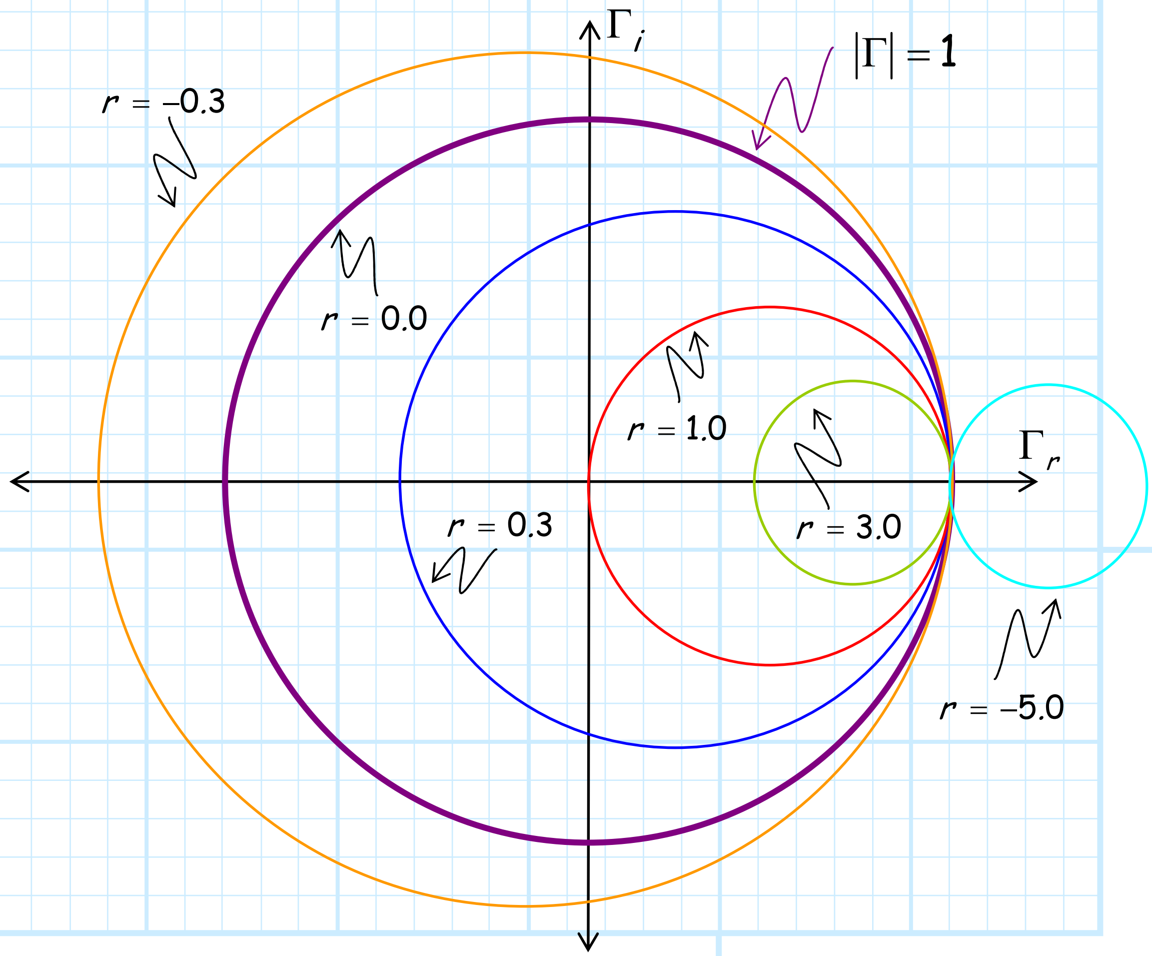
$$P_c\left(\Gamma_r = \frac{c_r}{1+c_r}, \Gamma_i = 0\right)$$

In other words, the center of this circle **always** lies somewhere along the $\Gamma_i = 0$ line.

Likewise, by inspection, we find the **radius** of this circle is:

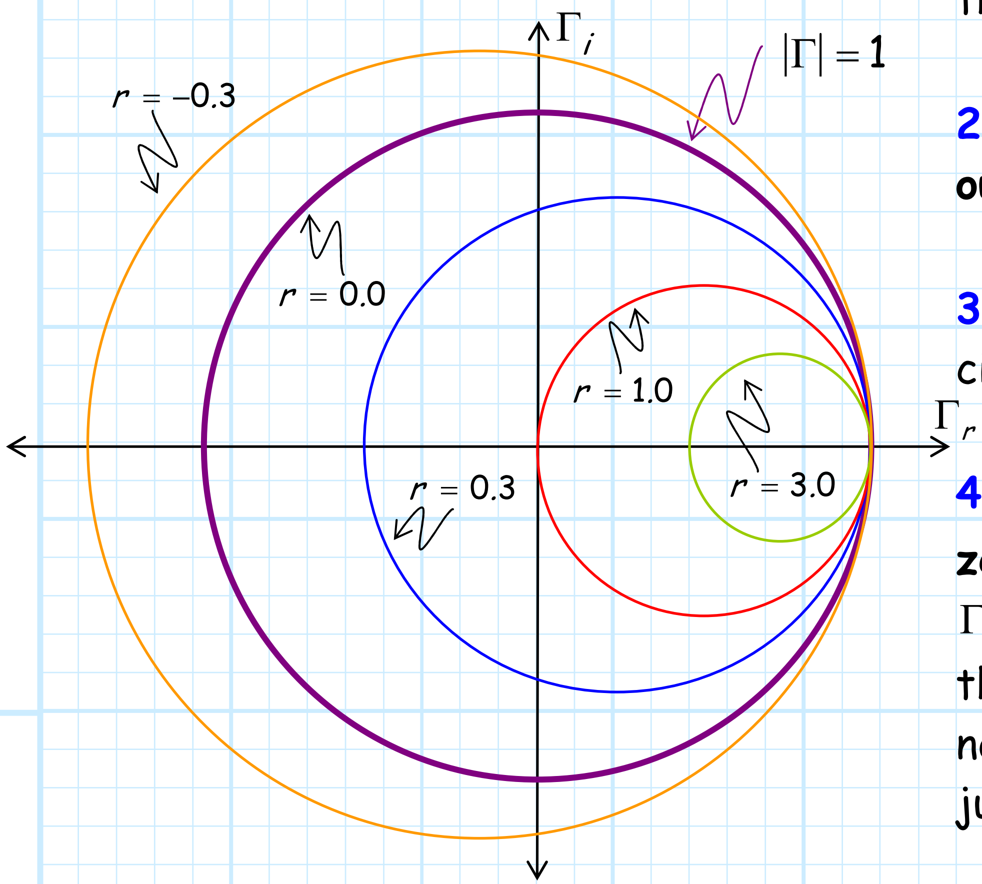
$$a = \frac{1}{1+c_r}$$

We perform a few of these **mappings** and see where these **circles** lie on the complex Γ plane →



Some important stuff to notice

We see that as the constant c_r **increases**, the radius of the circle **decreases**, and its center moves to the **right**. **Note:**



1. If $c_r > 0$ then the circle lies entirely **within** the circle $|\Gamma| = 1$.

2. If $c_r < 0$ then the circle lies entirely **outside** the circle $|\Gamma| = 1$.

3. If $c_r = 0$ (i.e., a reactive impedance), the circle lies **on** circle $|\Gamma| = 1$.

4. If $c_r = \infty$, then the **radius** of the circle is **zero**, and its **center** is at the point $\Gamma_r = 1, \Gamma_i = 0$ (i.e., $\Gamma = 1e^{j0}$). In other words, the **entire** vertical line $r = \infty$ on the normalized **impedance** plane is mapped onto just a **single point** on the complex Γ plane!

But of course, this **makes sense**! If $r = \infty$, the impedance is **infinite** (an open circuit), regardless of what the value of the **reactive** component x is.

Horizontal contours on the complex Z plane map…

Now, let's turn our attention to the mapping of **horizontal lines** in the normalized impedance plane, i.e., lines of the form:

$$x = c_i$$

where c_i is some **constant** (e.g. $x = -2$ or $x = 0.5$).

We can show that this **horizontal** line in the normalized impedance plane is **mapped** onto the **complex** Γ **plane** as:

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{c_i}\right)^2 = \frac{1}{c_i^2}$$

Note this equation is **also** that of a **circle**! Thus, the horizontal line $x = c_i$ maps into a circle on the complex Γ plane!

...onto circles on the complex G plane

By inspection, we find that the **center** of this circle lies at the point:

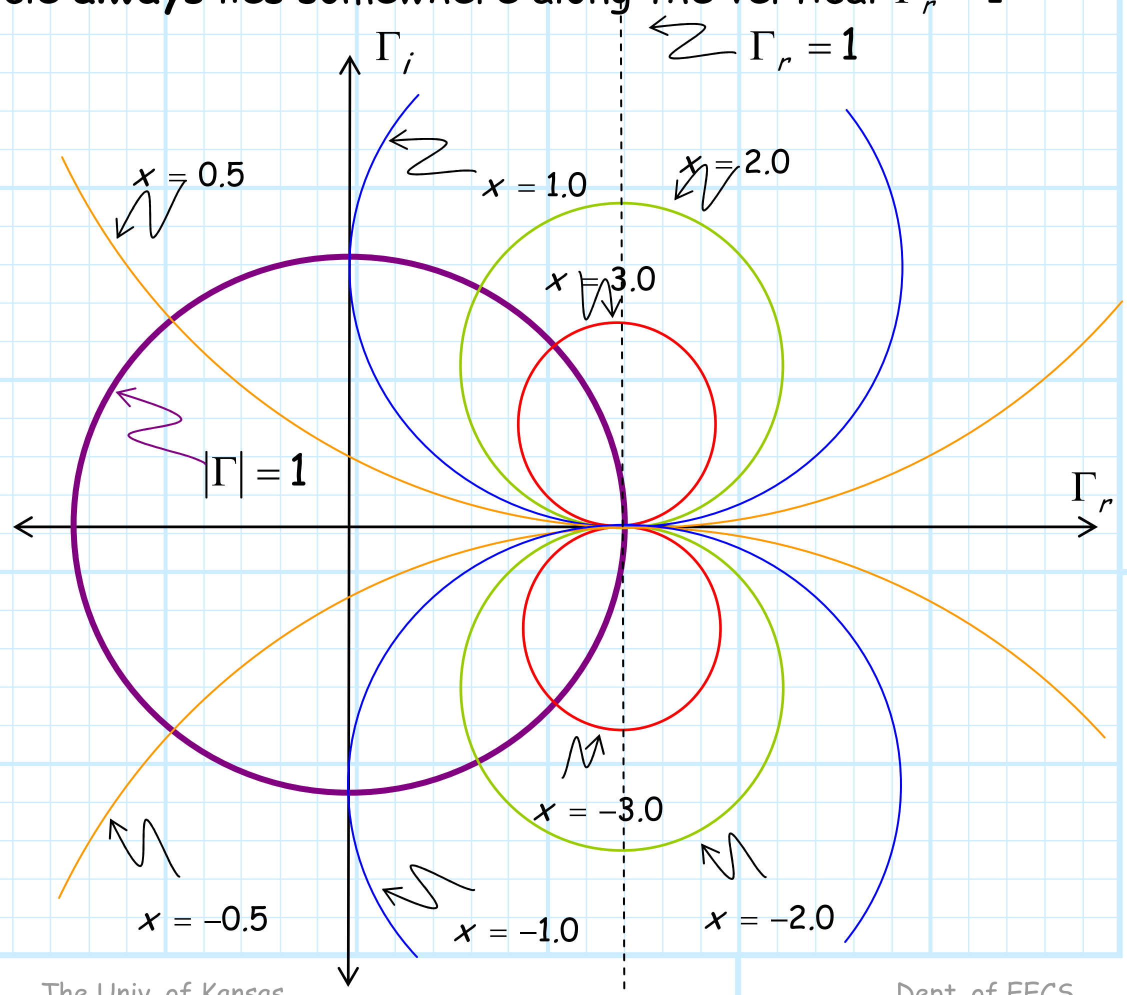
$$P_c\left(\Gamma_r = 1, \Gamma_i = \frac{1}{c_i}\right)$$

in other words, the center of this circle **always** lies somewhere along the vertical $\Gamma_r = 1$ line.

Likewise, by inspection, the **radius** of this circle is:

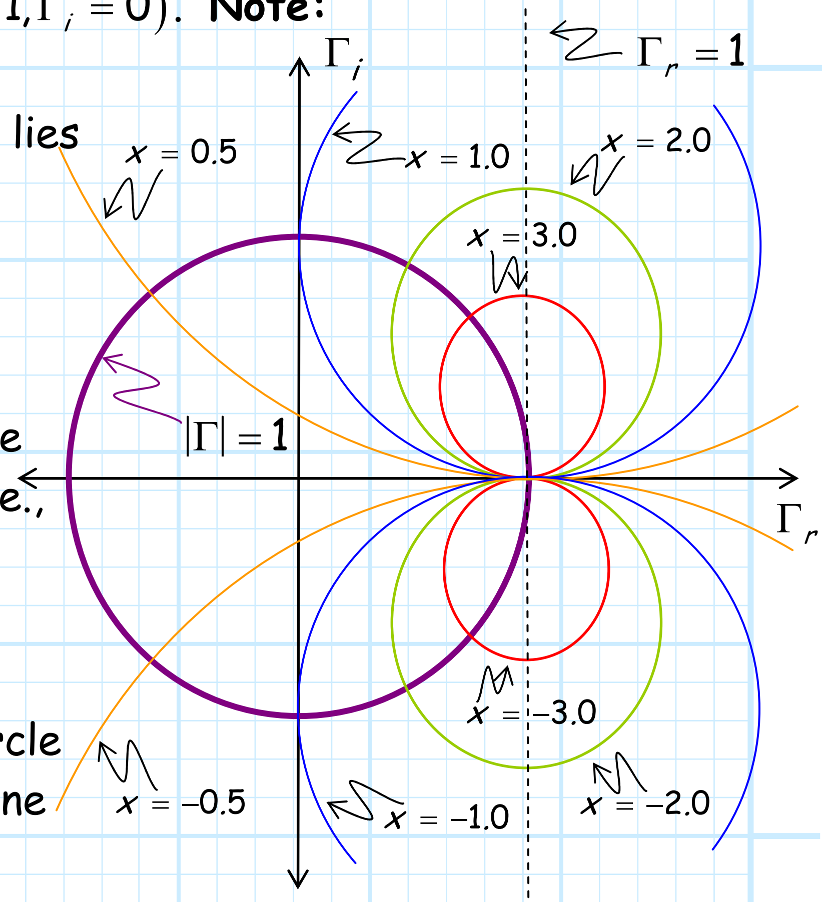
$$a = \frac{1}{|c_i|}$$

We perform a few of these **mappings** and see where these circles lie on the complex Γ plane →



Some more important stuff to notice

We see that as the **magnitude** of constant c_i **increases**, the radius of the circle **decreases**, and its **center** moves toward the point $(\Gamma_r = 1, \Gamma_i = 0)$. **Note:**



1. If $c_i > 0$ (i.e., reactance is **inductive**) then the circle lies entirely in the **upper half** of the complex Γ plane (i.e., where $\Gamma_i > 0$)—the upper half-plane is known as the **inductive** region.

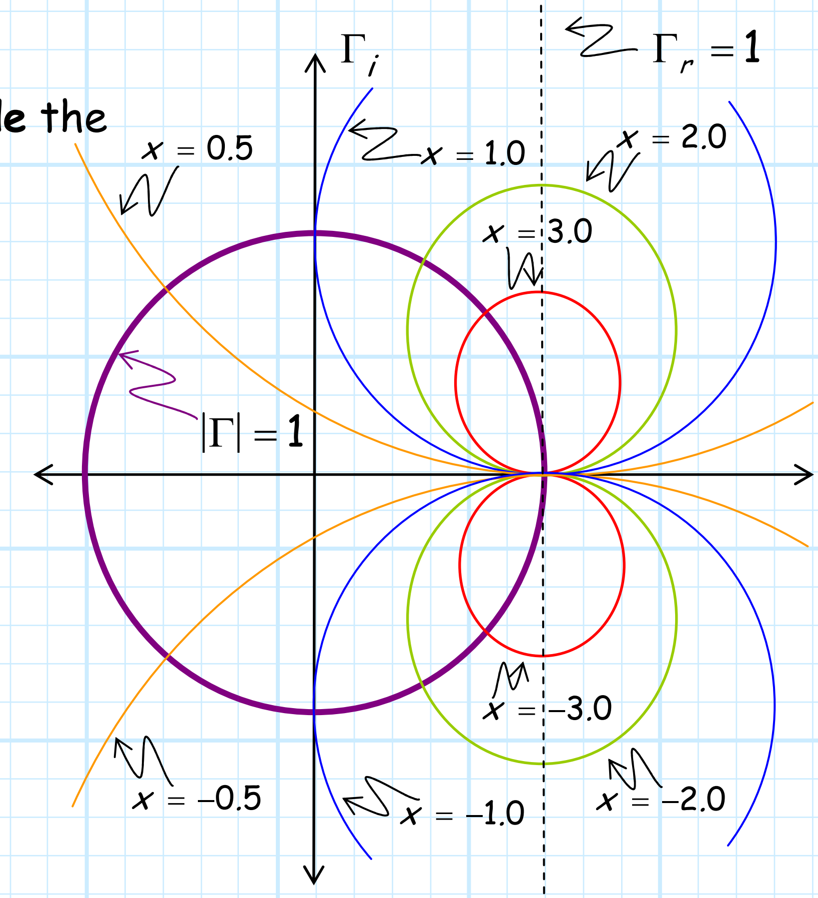
2. If $c_i < 0$ (i.e., reactance is **capacitive**) then the circle lies entirely in the **lower half** of the complex Γ plane (i.e., where $\Gamma_i < 0$)—the lower half-plane is known as the **capacitive** region.

3. If $c_i = 0$ (i.e., a **purely resistive** impedance), the circle has an infinite radius, such that it lies **entirely** on the line $\Gamma_i = 0$.

4. If $c_i = \pm\infty$, then the **radius** of the circle is **zero**, and its **center** is at the point $\Gamma_r = 1, \Gamma_i = 0$ (i.e., $\Gamma = 1e^{j0}$). In other words, the **entire** vertical line $x = \infty$ or $x = -\infty$ on the normalized impedance plane is mapped onto just a **single point** on the complex Γ plane!

But of course, this **makes sense**! If $x = \infty$, the impedance is **infinite** (an **open** circuit), **regardless** of what the value of the resistive component r is.

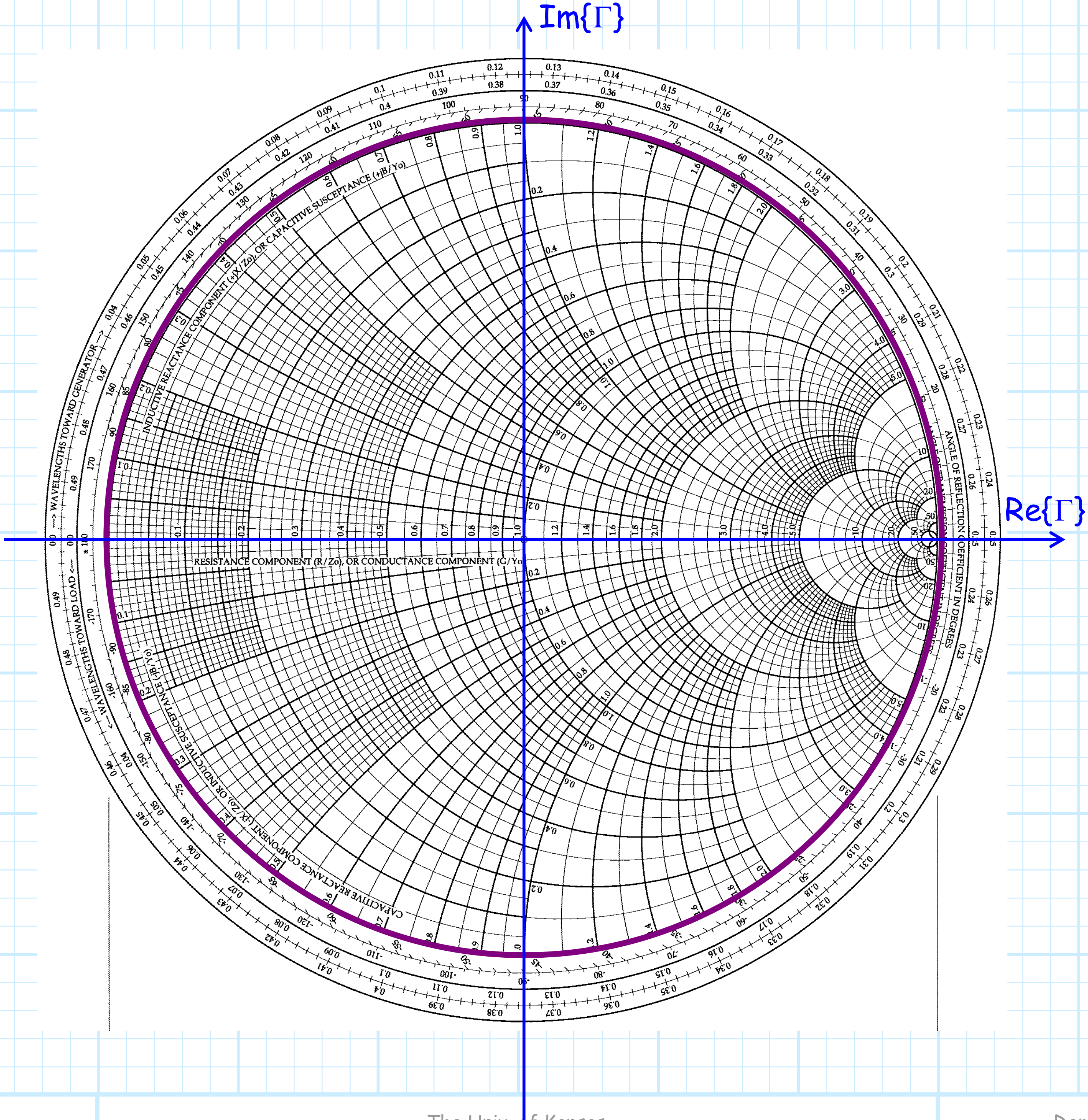
5. Note also that **much** of the circle formed by mapping $x = c_i$ onto the complex Γ plane lies **outside** the circle $|\Gamma| = 1$.



This **makes sense**! The portions of the circles laying **outside** $|\Gamma| = 1$ circle correspond to impedances where the **real** (resistive) part is **negative** (i.e., $r < 0$).

Thus, we typically can completely **ignore** the portions of the circles that lie **outside** the $|\Gamma| = 1$ circle !

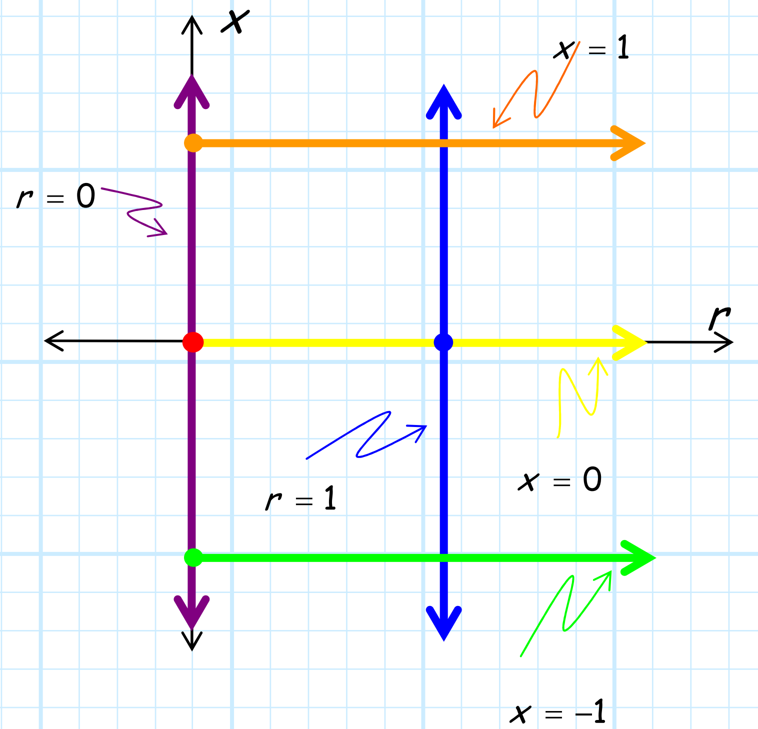
Mapping **many** lines of the form $r = c_r$ and $x = c_i$ onto circles on the complex Γ plane results in tool called the **Smith Chart**……


Im{Γ}
Re{Γ}
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD

Rectilinear and Curvilinear Grids

Note the Smith Chart is simply the vertical lines $r = c_r$ and horizontal lines $x = c_i$ of the normalized **impedance** plane, **mapped** onto the two types of **circles** on the complex Γ plane.

For the normalized **impedance** plane, a vertical line $r = c_r$ and a horizontal line $x = c_i$ are always **perpendicular** to each other when they intersect. We say these lines form a **rectilinear grid**.

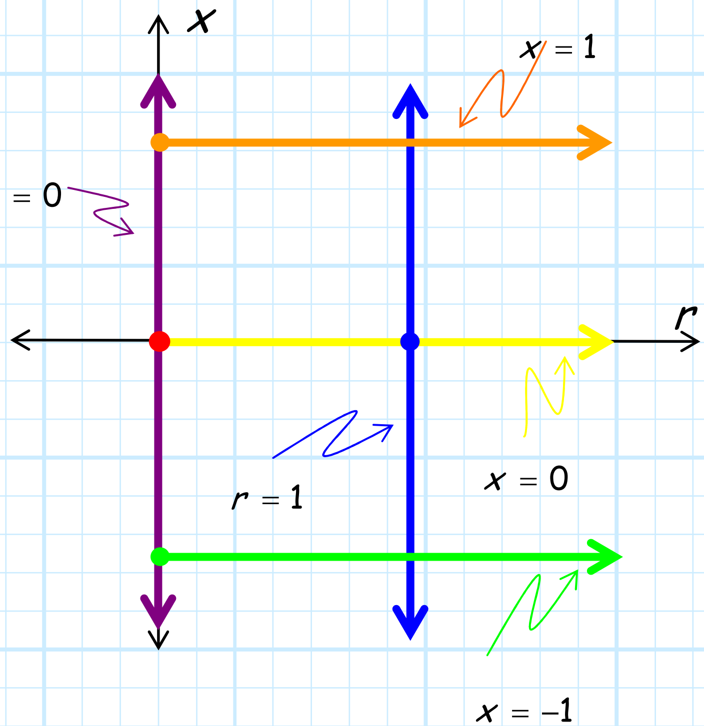


However, a similar thing is true for the **Smith Chart**! When a mapped circle $r = c_r$ **intersects** a mapped circle $x = c_i$, the two circles are **perpendicular** at that intersection point. We say these circles form a **curvilinear grid**.

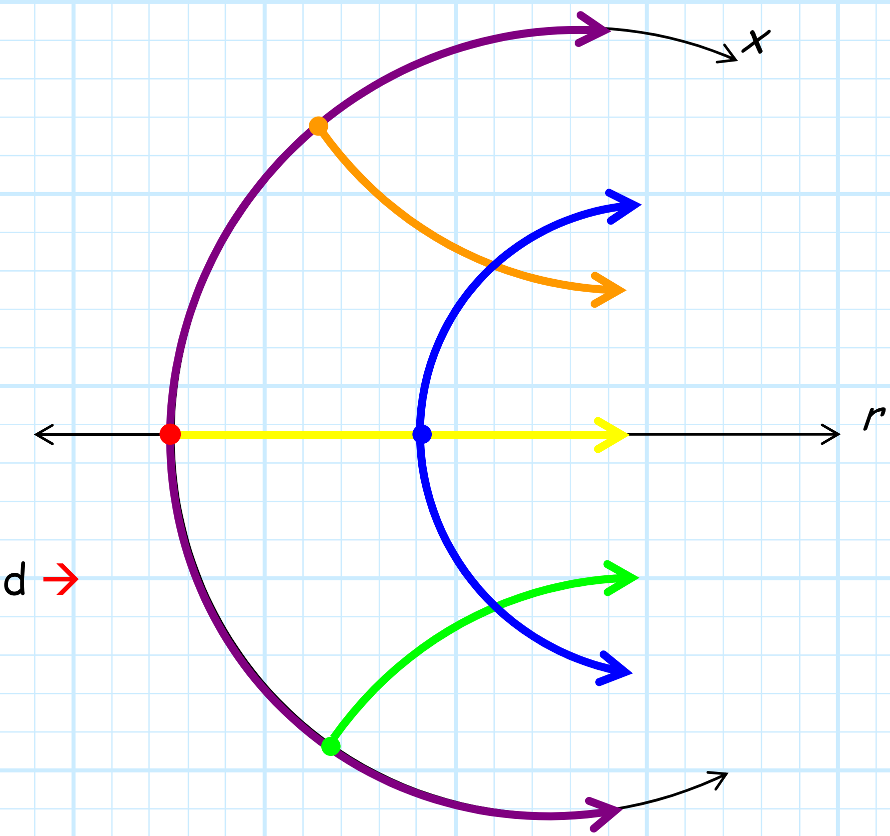
In fact, the Smith Chart is formed by **distorting** the **rectilinear** grid of the normalized impedance plane into the **curvilinear** grid of the Smith Chart!

The proverbial square peg..

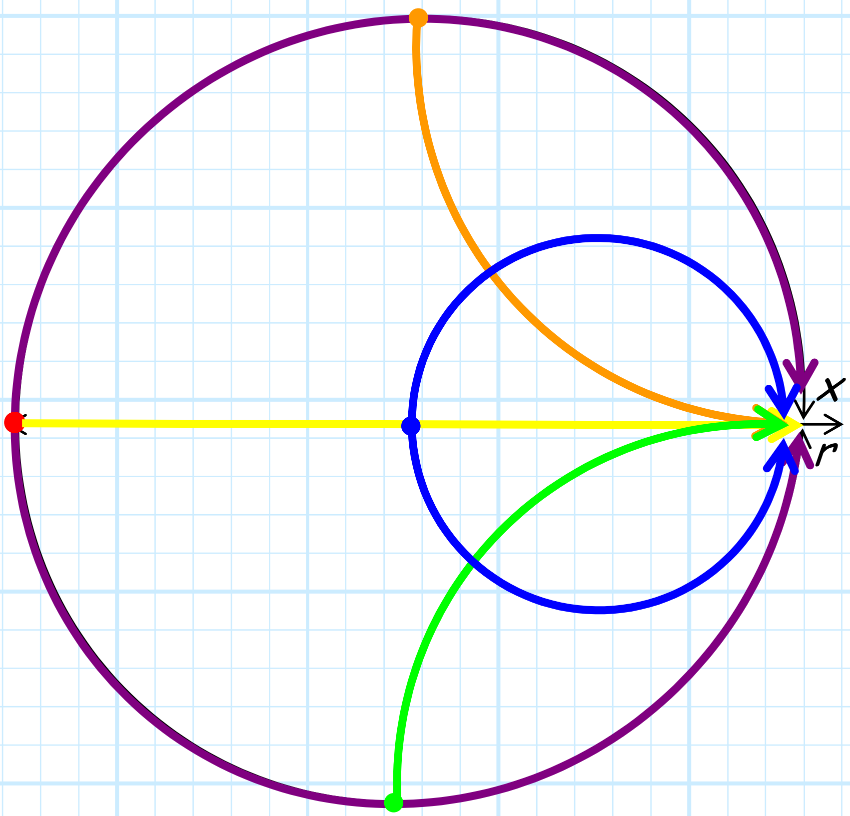
The rectilinear grid of the complex impedance plane:

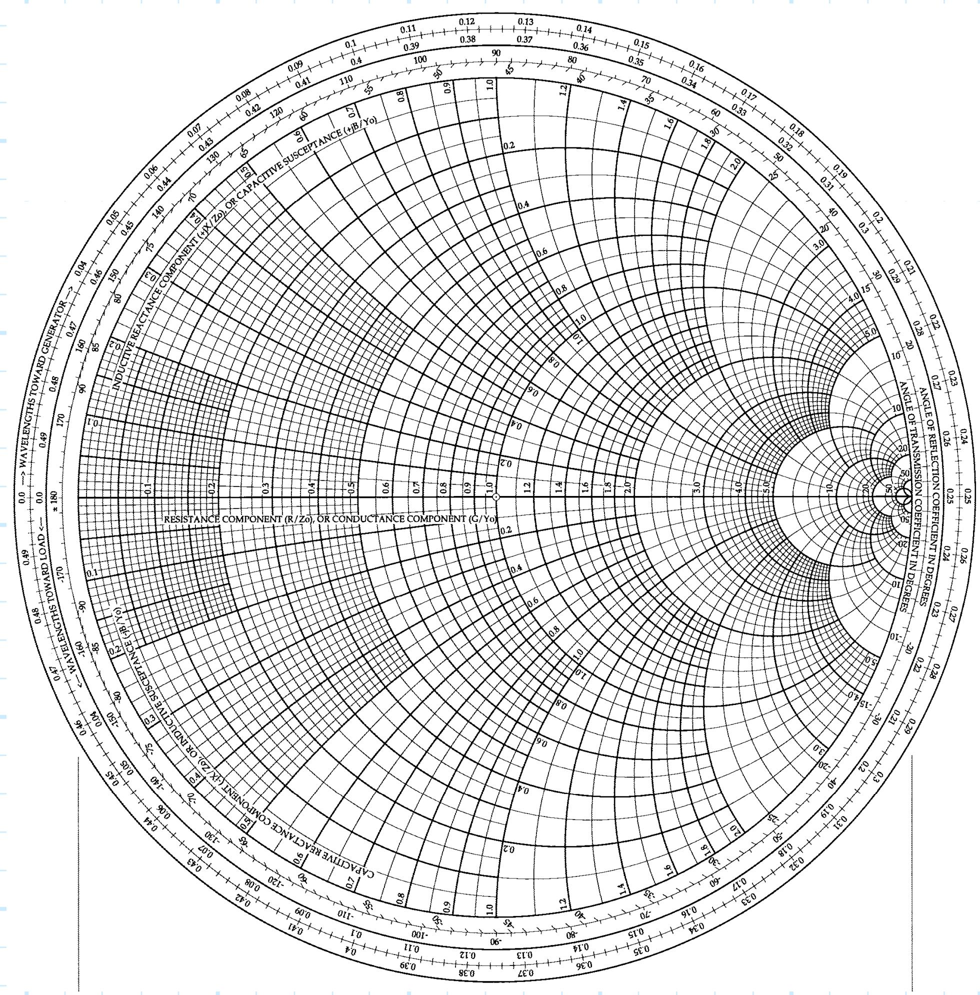


Distorting this rectilinear grid →



And then **distorting** some more—we have the **curvilinear** grid of the Smith Chart!





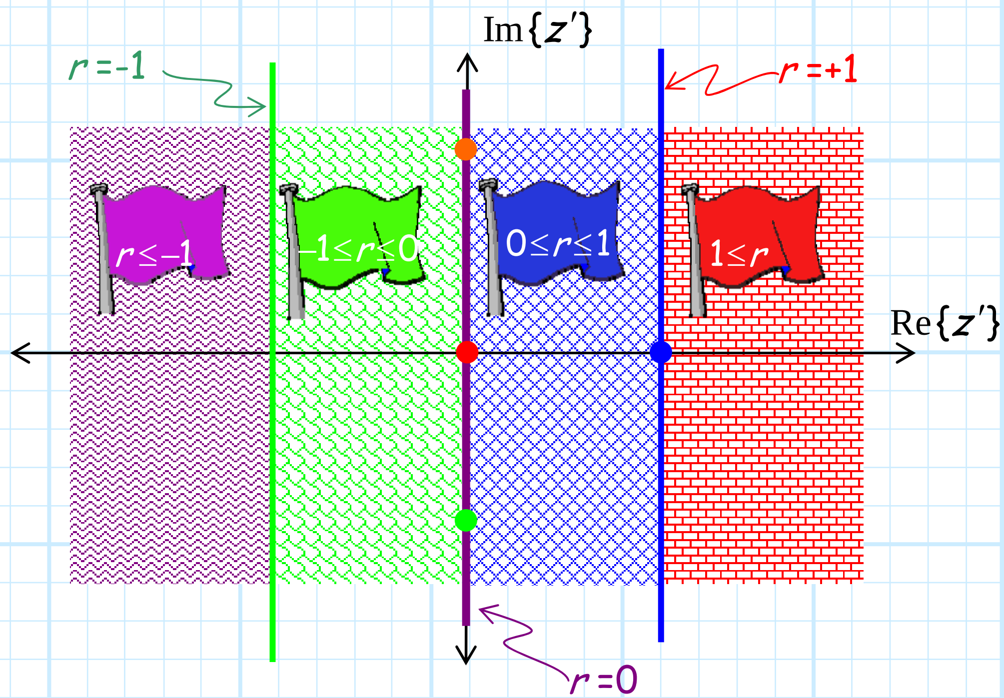
Smith Chart Geography

We have located specific **points** on the complex impedance plane, such as a **short circuit** or a **matched load**.

We've also identified **contours**, such as $r=1$ or $x=-2$.

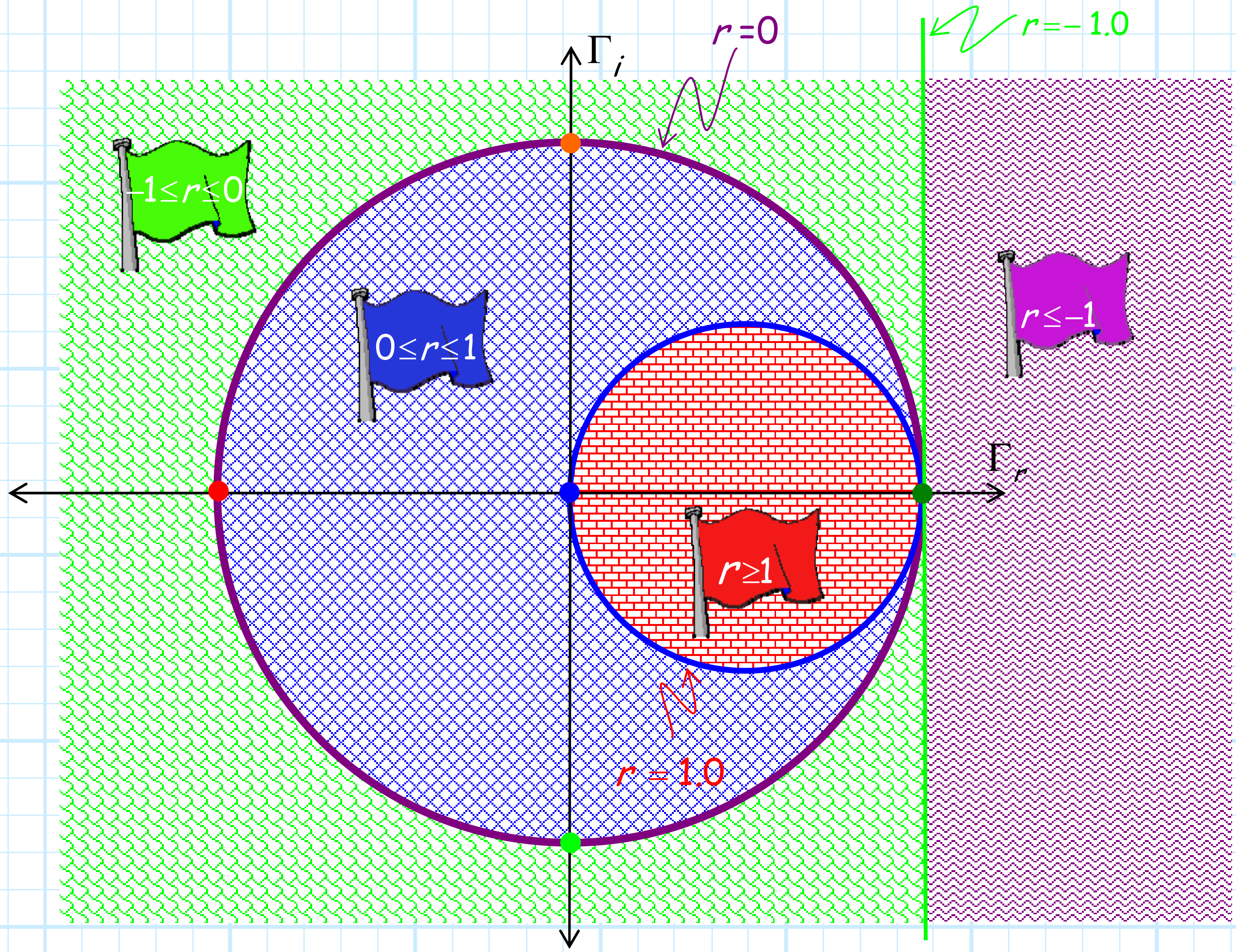
We can likewise identify **whole regions (!)** of the complex **impedance** plane, providing a bit of a **geography lesson** of the complex impedance plane.

For example, we can divide the complex impedance plane into **four** regions based on normalized **resistance** value *r*:



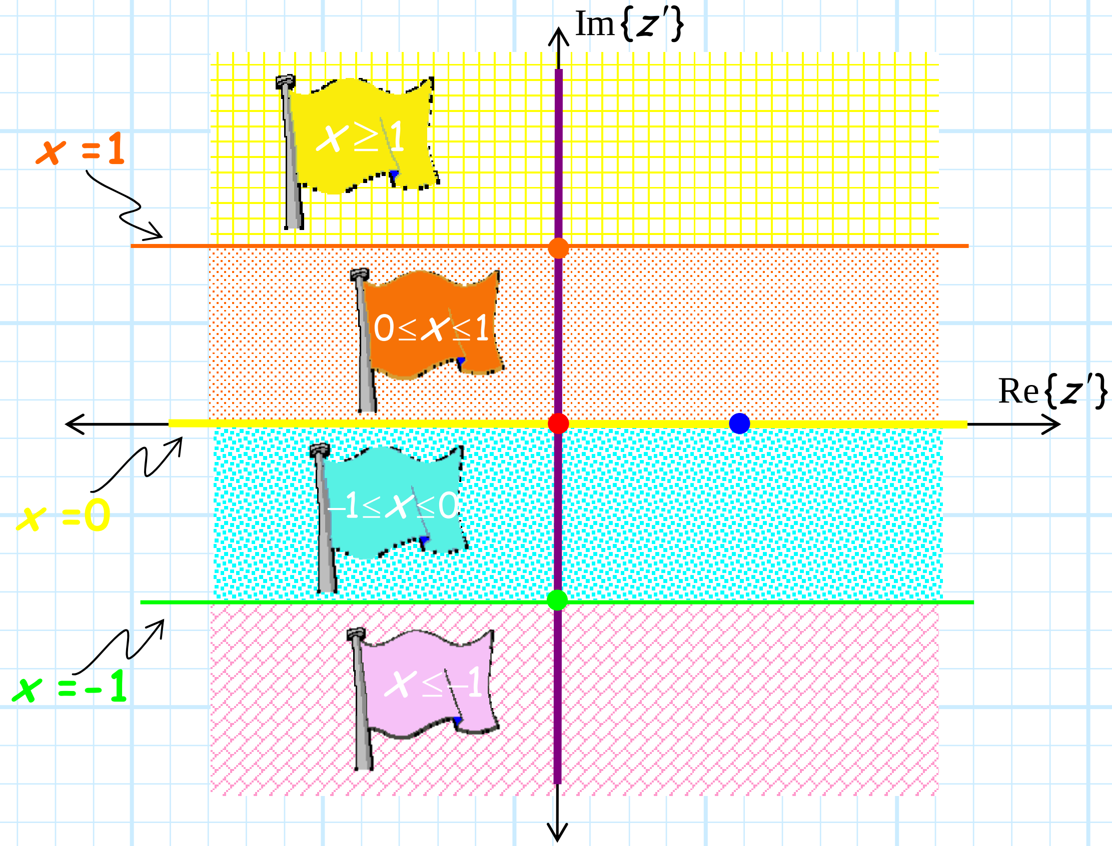
Mapping onto the Γ Plane

Just like points and contours, these regions of the complex impedance plane can be **mapped** onto the **complex gamma plane**!



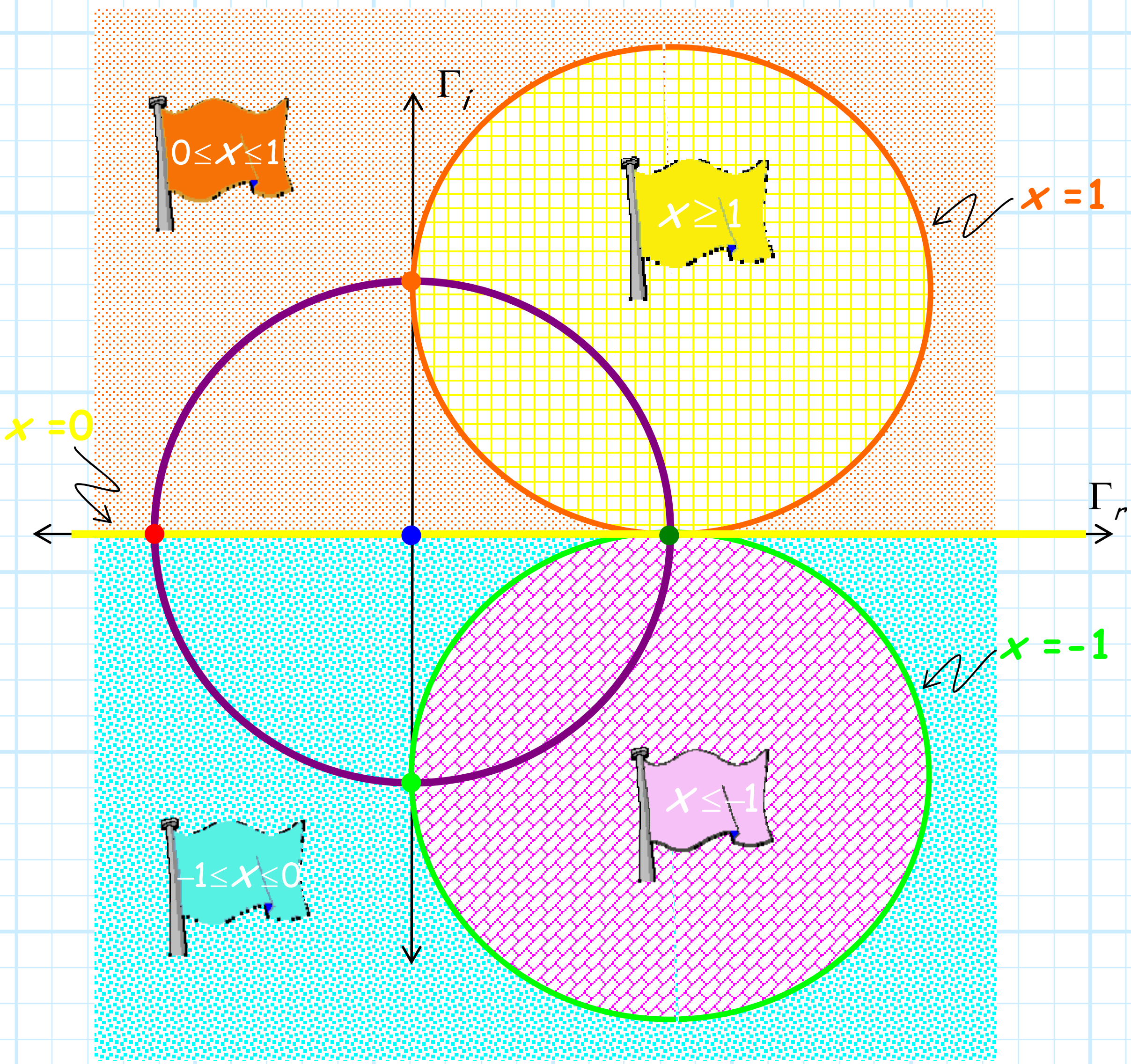
Reactive Boundaries and Borders

Instead of dividing the complex impedance plane into regions based on normalized resistance r, we could divide it based on **normalized reactance** x:



Mapping onto the Γ Plane

These four regions can likewise be mapped onto the **complex gamma plane**:

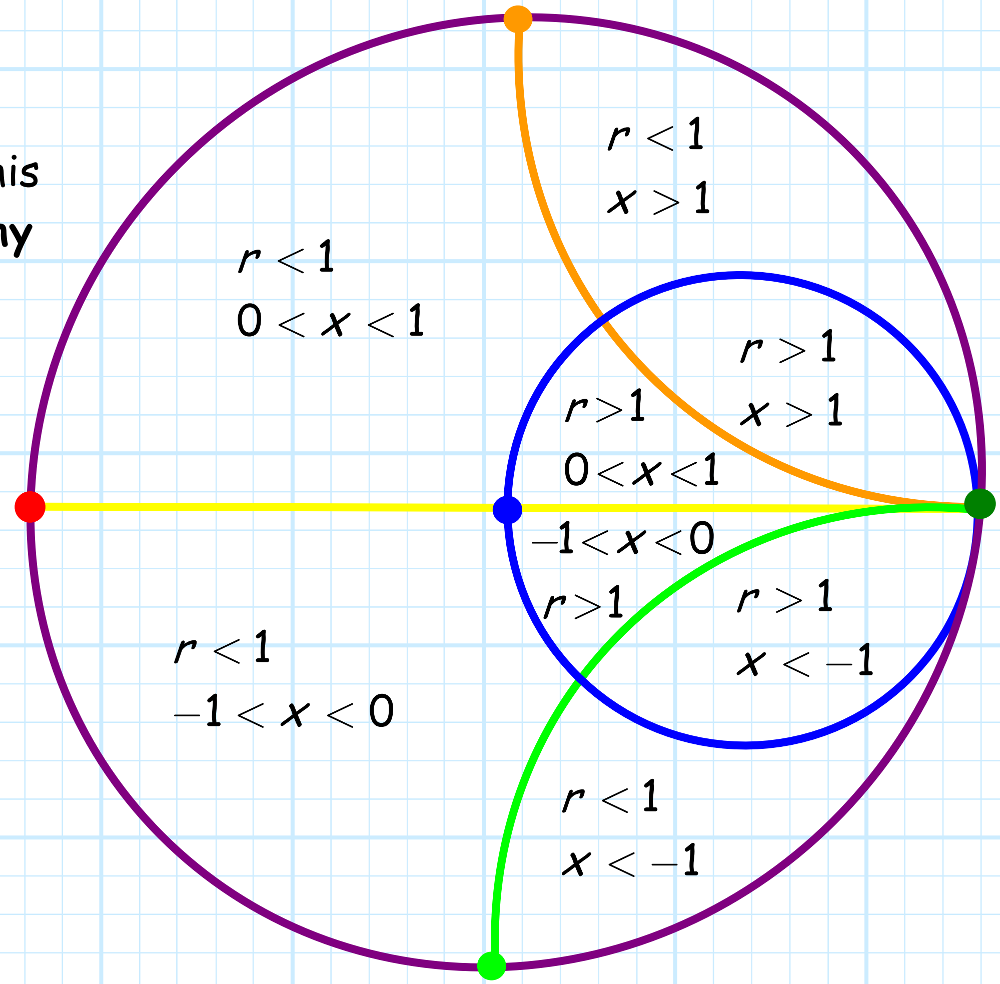


Smith Chart Geography

Note the four resistance regions and the four reactance regions combine to from **16 separate regions** on the complex impedance and complex gamma planes!

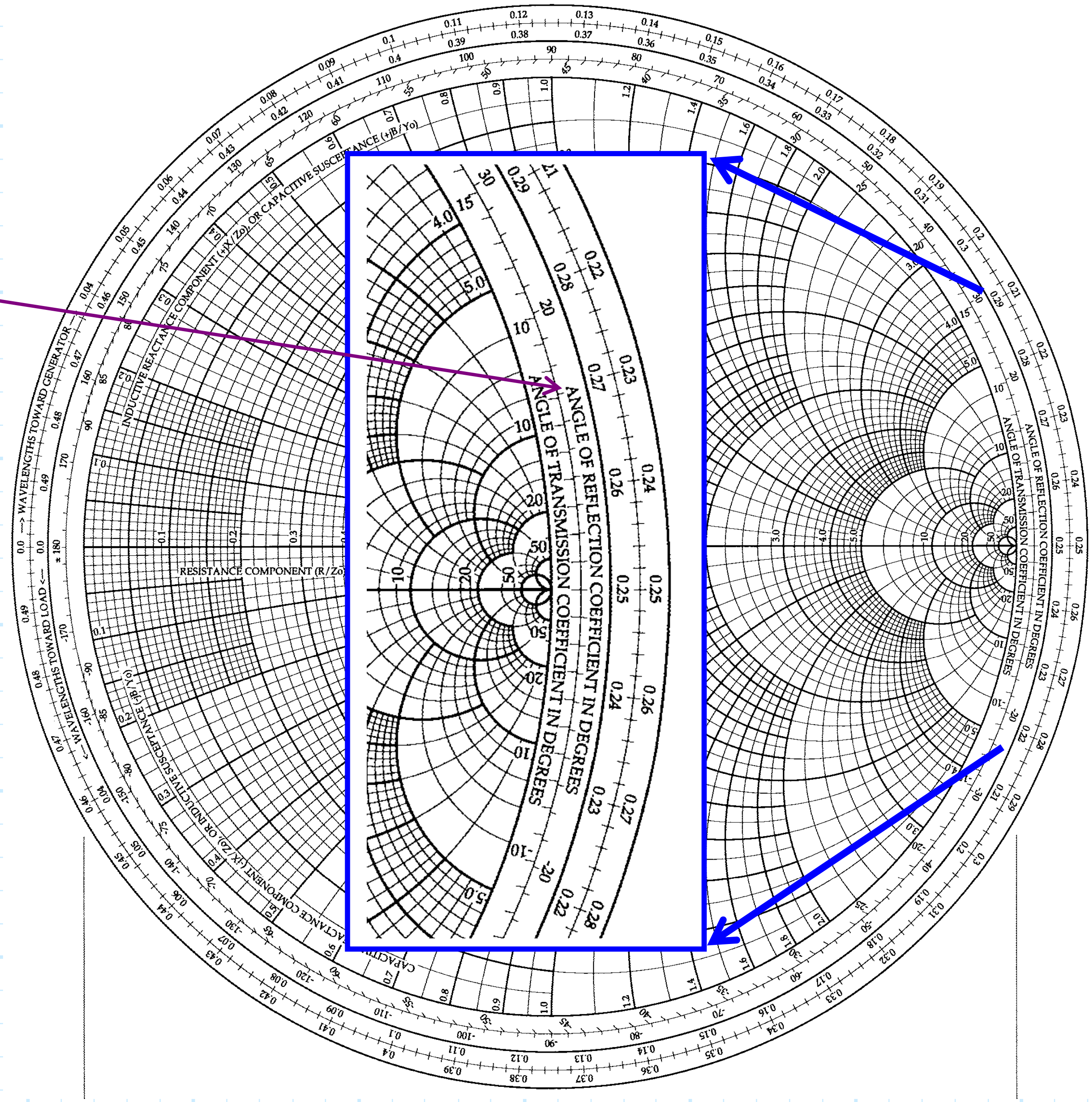
Eight of these sixteen regions lie in the **valid region** (i.e., $r > 0$), while the other eight lie entirely in the invalid region.

Make sure **you** can locate the eight impedance regions on a **Smith Chart**—this understanding of **Smith Chart geography** will help you understand your design and analysis results!



The Outer Scale

Note that around the **outside** of the Smith Chart there is a scale indicating the **phase angle** θ_Γ (i.e., $\Gamma = |\Gamma| e^{j\theta_\Gamma}$), from $-180^\circ < \theta_\Gamma < 180^\circ$.



Line position z and phase angle are related!

Recall however, for a **terminated** transmission line, the reflection coefficient function is:

$$\Gamma(z) = \Gamma_0 \, e^{j2\beta z} = |\Gamma_0| e^{j2\beta z + \theta_0}$$

Thus, the **phase** of the reflection coefficient function depends on transmission line **position *z*** as:

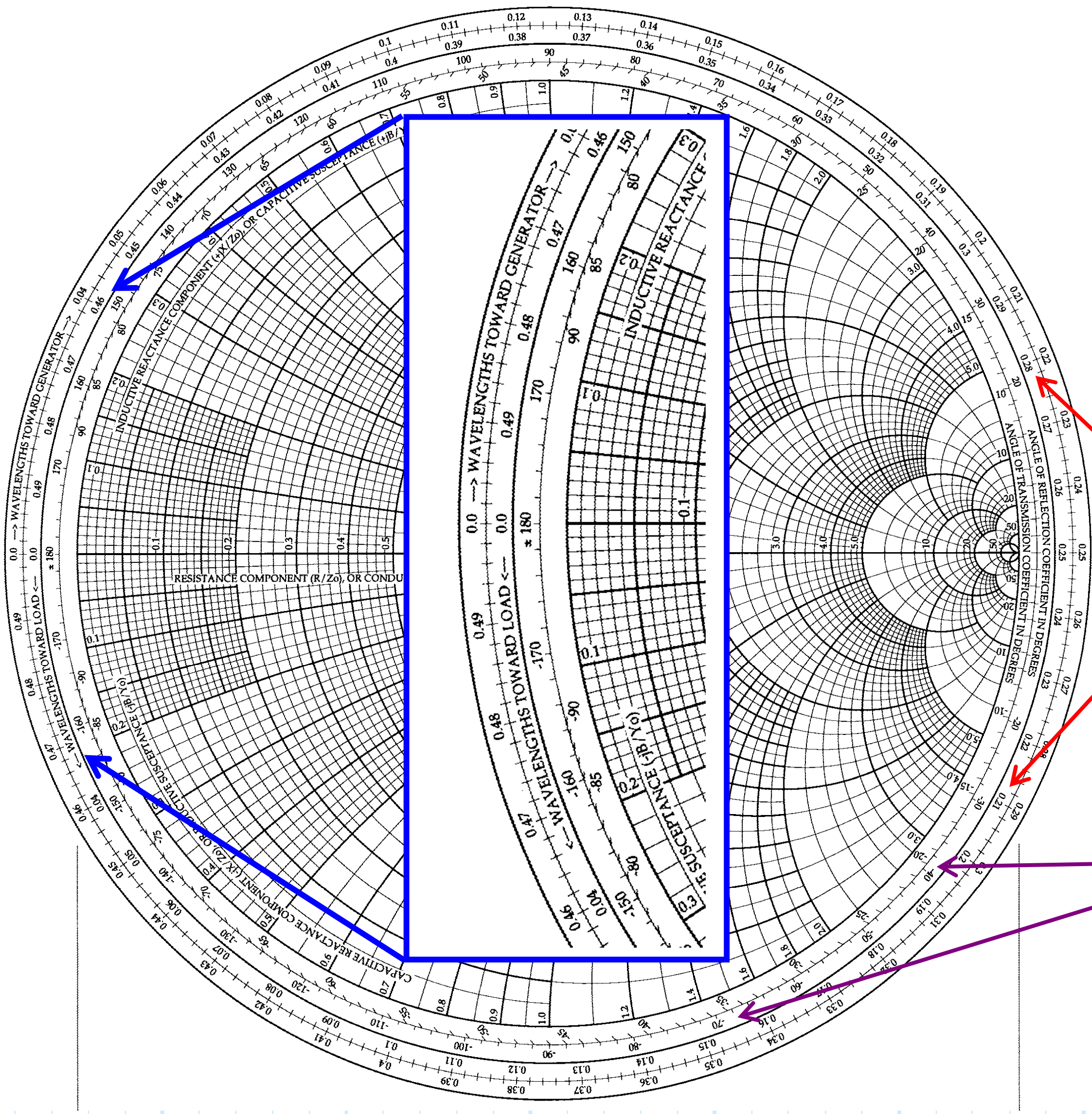
$$\theta_\Gamma(z) = 2\beta z + \theta_0 = 2\left(\frac{2\pi}{\lambda}\right) z + \theta_0 = 4\pi\left(\frac{z}{\lambda}\right) + \theta_0$$

As a result, a **change** in line position z (i.e., Δz) results in a **change** in reflection coefficient phase θ_Γ (i.e., $\Delta\theta_\Gamma$):

$$\Delta\theta_\Gamma = 4\pi\left(\frac{\Delta z}{\lambda}\right)$$

For example, a change of position equal to one-quarter wavelength $\Delta z = \lambda/4$ results in a phase change of π radians—we rotate **half-way** around the complex Γ plane (otherwise known as the Smith Chart).

A second outer scale



The Smith Chart thus has a **second scale** (besides θ_Γ) that surrounds it—one that relates transmission line position in **wavelengths** (i.e., $\Delta z/\lambda$) to the reflection coefficient phase:

$$\frac{z}{\lambda} = \frac{1}{4} + \frac{\theta_\Gamma}{4\pi}$$

$$\Updownarrow$$

$$\theta_\Gamma = 4\pi\left(\frac{z}{\lambda} - \frac{1}{4}\right)$$

This second scale is very useful!

Since the phase scale on the Smith Chart extends from $-180^\circ < \theta_\Gamma < 180^\circ$ (i.e., $-\pi < \theta_\Gamma < \pi$), this **electrical length scale** extends from:

$$0 < z/\lambda < 0.5$$

Note for this mapping the reflection coefficient phase at location $z = 0$ is $\theta_L = -\pi$. Therefore, $\theta_0 = -\pi$, and we find that:

$$\Gamma_0 = |\Gamma_0| e^{j\theta_0} = |\Gamma_0| e^{-j\pi} = -|\Gamma_0|$$

In other words, Γ_0 is a **negative real** value.

Q: *But, Γ_0 could be **anything!** What is the likelihood of Γ_0* being a real and negative value*? Most of the time this is **not** the case—this second Smith Chart scale seems to **be nearly useless**!?*

A: Quite the contrary! This electrical length scale is in fact **very useful**—you just need to understand how to utilize it!

The first of many analogies

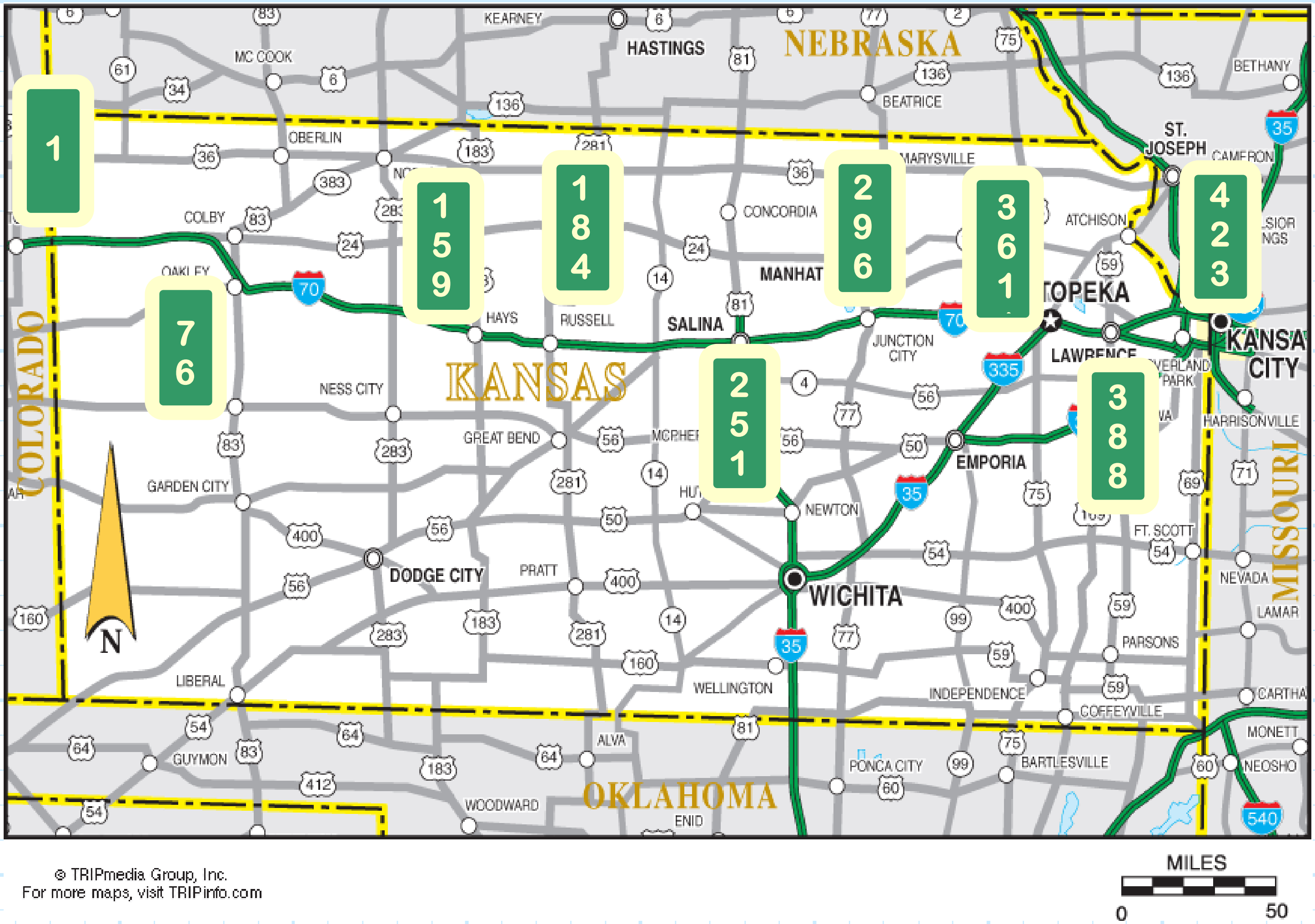
This electrical length scale is very much like the **mile markers** you see along an interstate highway; although the specific numbers are quite arbitrary, they are still very useful.

Take for example **Interstate 70**, which stretches across Kansas. The **western end** of I-70 (at the Colorado border) is denoted as **mile 1**.

At each mile along I-70 a new marker is placed, such that the **eastern end** of I-70 (at the **Missouri** border) is labeled **mile 423**—Interstate 70 runs for 423 miles across Kansas!

A Kansas geography lesson

The location of various **towns** and burgs along I-70 can thus be specified in terms of these **mile markers**. For example, along I-70 we find:

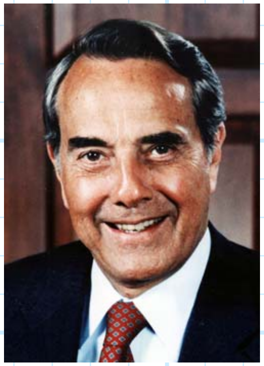


Oakley at 76
Hays at 159
Russell at 184
Salina at 251
Junction City 296
Topeka at 361
Lawrence at 388

Mile markers: the key to successful navigation

So say you are traveling **eastbound** (→) along I-70, and you want to know the distance to **Topeka**. Topeka is at mile marker **361**, but this does **not** of course mean you are **361 miles** from Topeka.

Instead, you **subtract from 361** the value of the mile marker denoting **your position** along I-70.



For **example**, if you find yourself in the lovely borough of **Russell** (mile marker **184**), you have precisely 361-184 = **177 miles** to go before reaching Topeka!

Q: *I'm confused! Say I'm in **Lawrence** (mile marker 388); using **your** logic I am a distance of 361-388 = **-27 miles** from Topeka! How can I be a **negative** distance from something??*

A: The mile markers across Kansas are arranged such that their value **increases** as we move from west to east across the state. Take the value of the mile marker denoting to where you are traveling, and **subtract** from it the value of the mile marker where you are.

If this value is **positive,** then your destination is **East** of you; if this value is **negative**, it is **West** of your current position (hopefully you're in the westbound lane!).

Its not rocket science!

For example, say you're traveling to Salina (mile marker 251). If you are in Oakley (mile marker 76) then:

251 - 76 = **175** → Salina is 175 miles **East** of Oakley

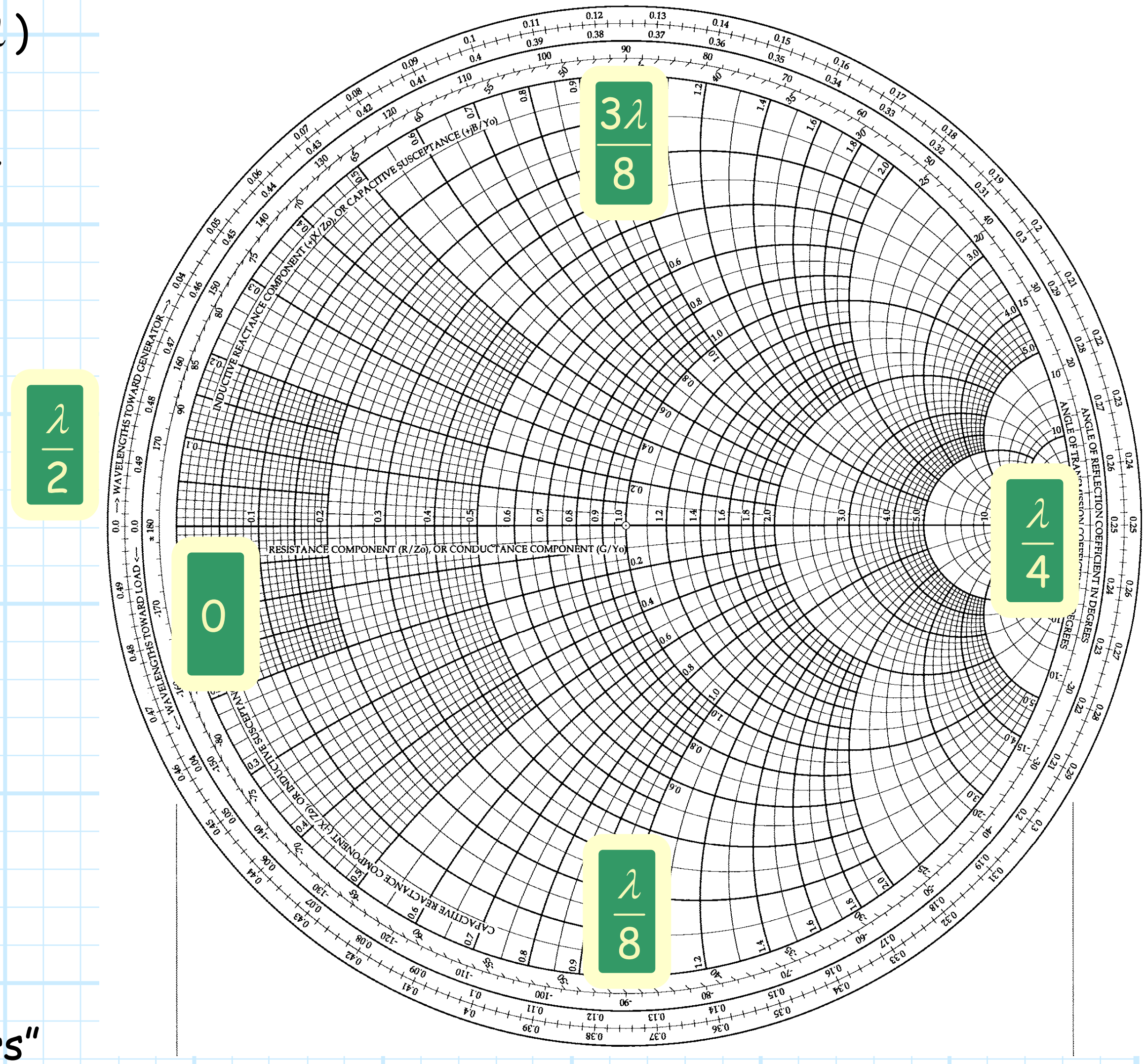
If, on the other hand, you begin your journey from Junction City (mile marker 296), we find:

251 - 296 = **-45** → Salina is 45 miles **West** of Junction City

Please tell me this is useful

Q: *But just what the &()#$% does this discussion have to do with* ***SMITH CHARTS*** *!!?!?*

A: The electrical length scale (z/λ) around the perimeter of the Smith Chart is precisely **analogous** to mile markers along an interstate!

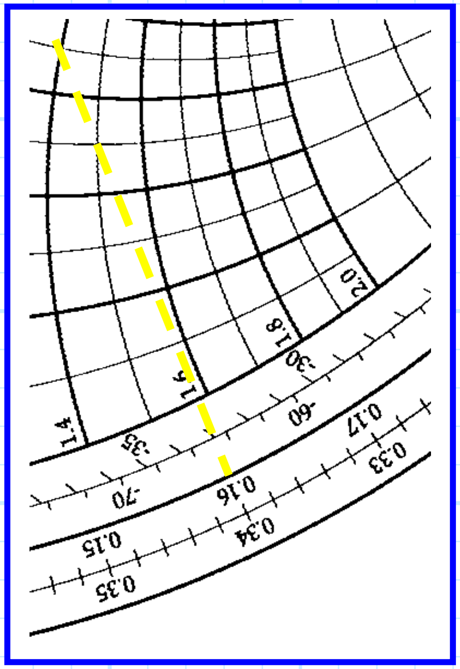


Recall that the change in **phase** ($\Delta\theta_\Gamma$) of the reflection coefficient function is related to the change in **distance** (Δz) along a transmission line as:

$$\Delta\theta_\Gamma = 4\pi\left(\frac{\Delta z}{\lambda}\right)$$

The value $\Delta z/\lambda$ can be determined from the **outer scale** of the Smith Chart, simply by taking the **difference** of the two "mile markers" values.

For example …



For **example**, say you're at some location $z = z_1$ along a transmission line. The value of the **reflection coefficient** function at that point happens to be:

$$\Gamma(z = z_1) = 0.685\, e^{-j65^\circ}$$

Finding the **phase angle** of $\theta_\Gamma = -65^\circ$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.160\lambda$$

Note this tells us **nothing** about the location $z = z_1$. This does **not** mean that $z_1 = 0.160\lambda$, for example!

Continued …

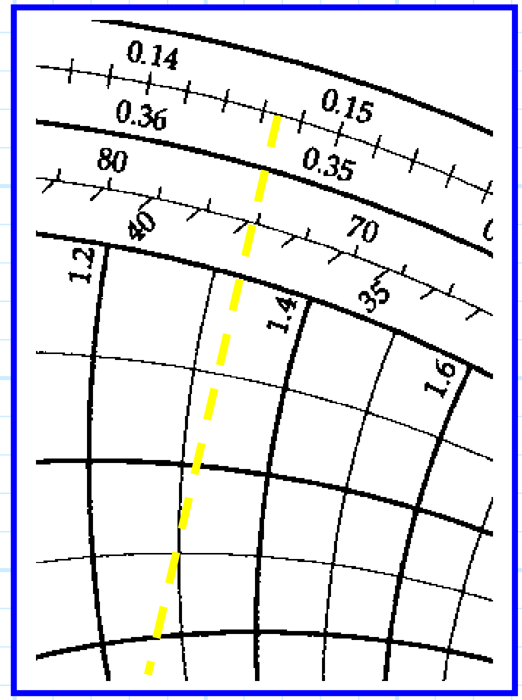
Now, say we **move a short distance** Δz (i.e., a distance less than $\lambda/2$) along the transmission line, to a **new location** denoted as $z = z_2$.

We find that this new location that the **reflection coefficient** function has a value of:

$$\Gamma(z = z_2) = 0.685\, e^{+j74^\circ}$$

Now finding the **phase angle** of $\theta_\Gamma = +74^\circ$ on the **outer scale** of the Smith Chart, we note that the corresponding **electrical length** value is:

$$0.353\lambda$$



Note this tells us **nothing** about the location $z = z_2$. This does **not** mean that $z_1 = 0.353\lambda$, for example!

See the analogy?

Q: *So what do the values* 0.160λ *and* 0.353λ *tell us?*

A: They allow us to determine the **distance between** points z_2 and z_1 on the transmission line:

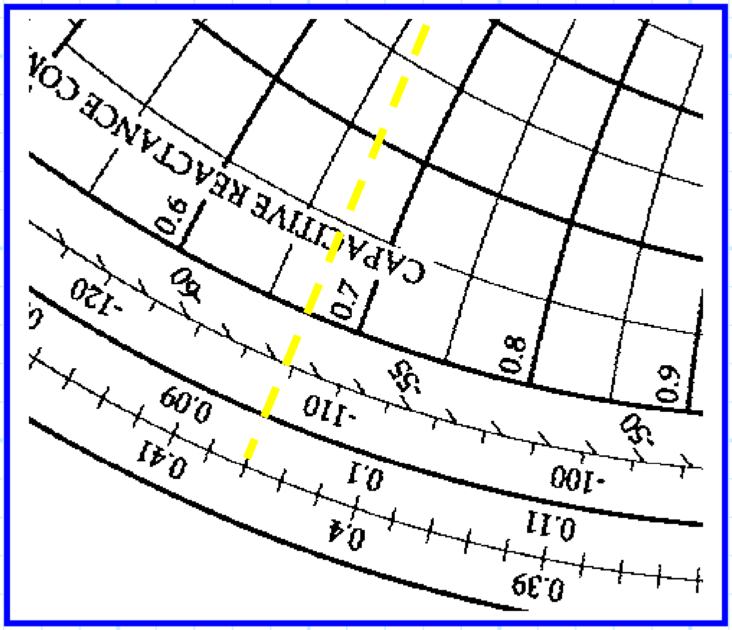
$$\frac{\Delta z}{\lambda} = \frac{z_2}{\lambda} - \frac{z_1}{\lambda} \quad !!!$$

Thus, for this example, the **distance between** locations z_2 and z_1 is:

$$\Delta z = 0.353\lambda - 0.160\lambda = 0.193\lambda$$

→ The transmission line location z_2 is a distance of 0.193λ from location z_1!

The power of negative thinking



Q: *But, say the reflection coefficient at some point z_3 has a phase value of $\theta_\Gamma = -112^\circ$. This maps to a value of:*

$$0.094\lambda$$

on the outer scale of the Smith Chart.

The ***distance*** *between z_3 and z_1 would then turn out to be:*

$$\frac{\Delta z}{\lambda} = 0.094 - 0.160 = -0.066$$

What does the ***negative*** *value mean??*

A: Just like our I-70 mile marker analogy, the **sign** (plus or minus) indicates the **direction** of movement from one point to another.

This isn't rocket science either

In the **first** example, we find that $\Delta z > 0$, meaning $z_2 > z_1$:

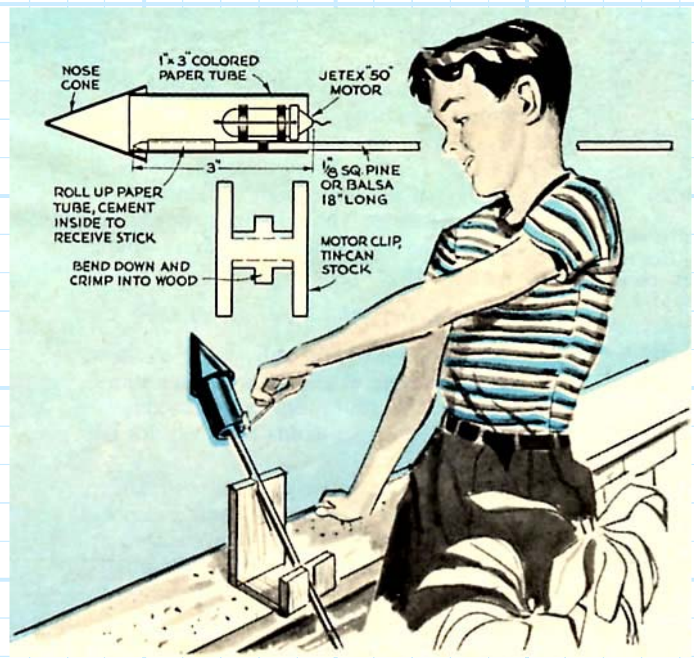
$$z_2 = z_1 + 0.094\lambda$$

Clearly, the location z_2 is **further** down the transmission line (i.e., **closer to the load**) than is location z_1.

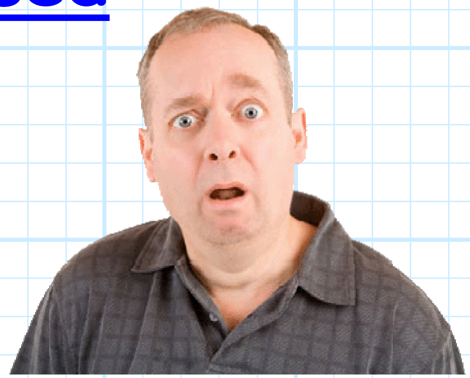
For the **second** example, we find that $\Delta z < 0$, meaning $z_3 < z_1$:

$$z_3 = z_1 - 0.066\lambda$$

Conversely, in this second example, the location z_3 is **closer to the beginning** of the transmission line (i.e., farther from the load) than is location z_1.



You shouldn't have be surprised



This is completely **consistent** with what we **already** know to be true!

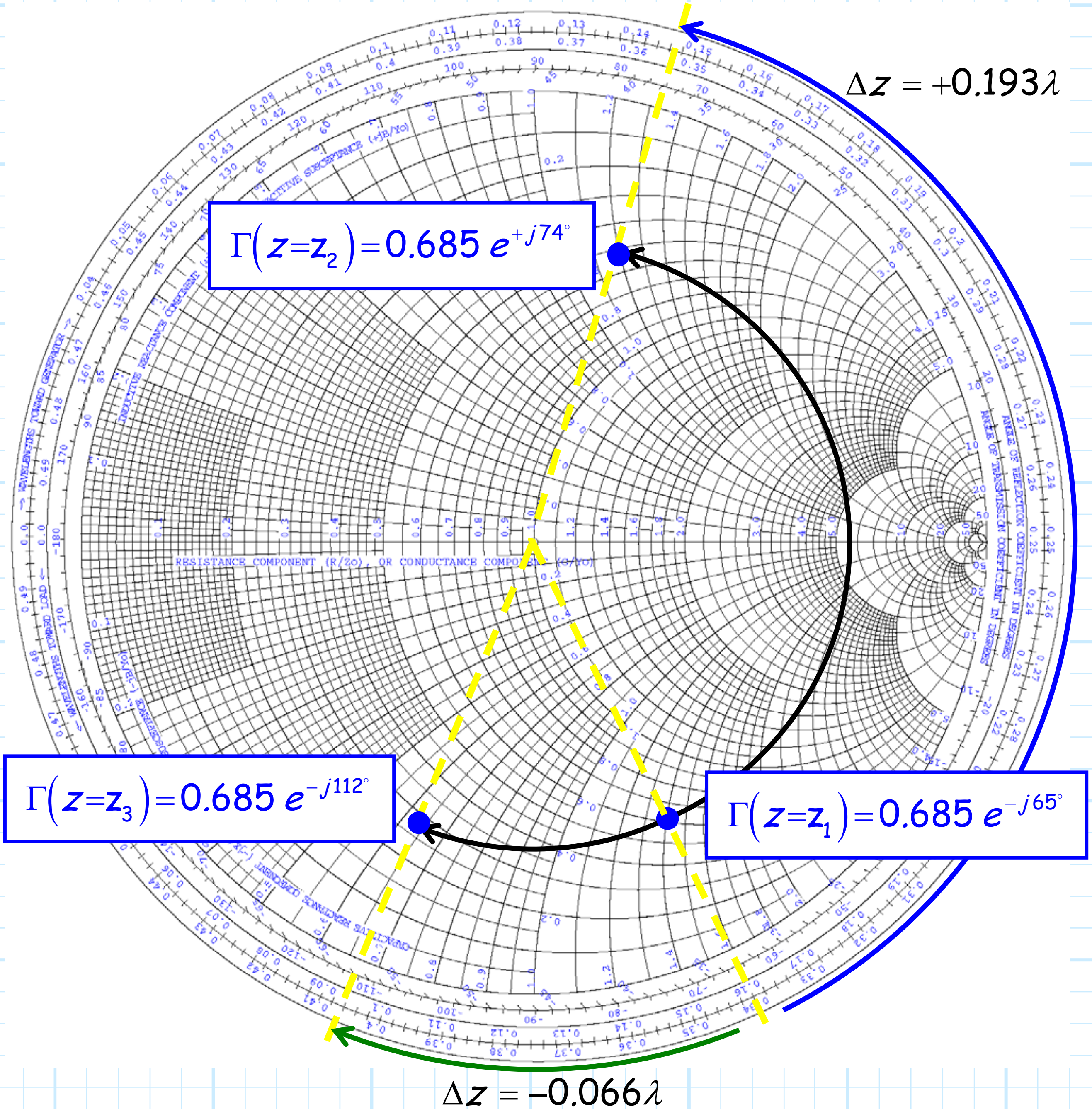
In the first case, the **positive** value $\Delta z = 0.193\lambda$ maps to a phase change of $\Delta\theta_\Gamma = 74^\circ - (-65^\circ) = 139^\circ$.

In other words, as we move **toward the load** from location z_1 to location z_2, we **rotate counter-clockwise** around the Smith Chart.

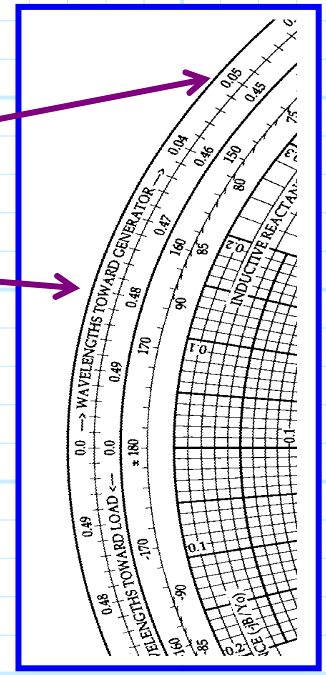
Likewise, the **negative** value $\Delta z = -0.066\lambda$ maps to a phase change of $\Delta\theta_\Gamma = -112^\circ - (-65^\circ) = -47^\circ$.

In other words, as we move **away from the load** (toward the source) from a location z_1 to location z_3, we **rotate clockwise** around the Smith Chart.

A graphical summary of what I just said



Yet another outer scale

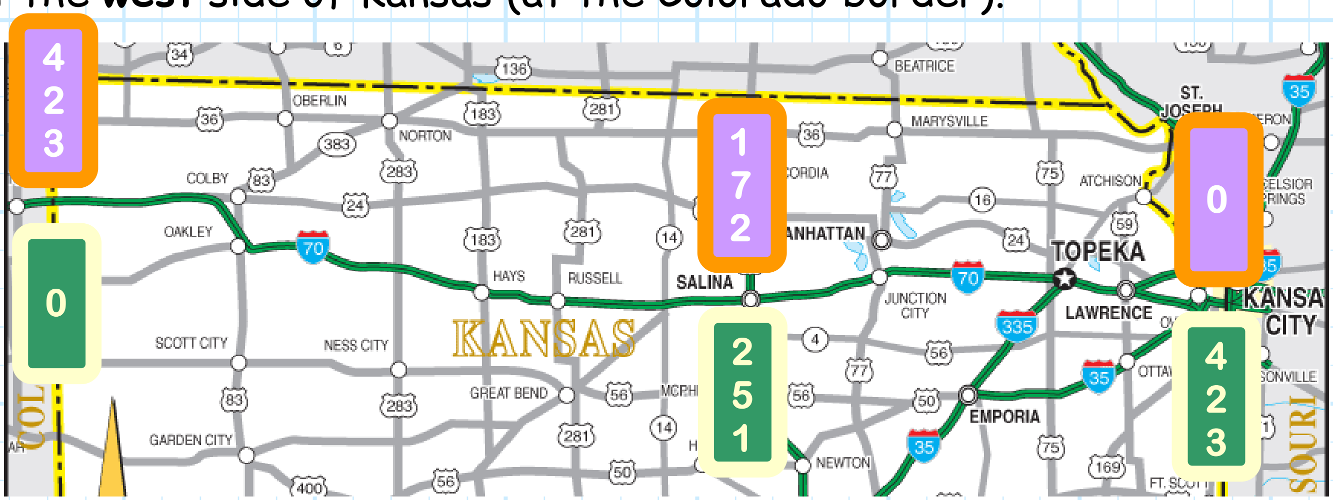


Q: *I notice that there is a **second** electrical length scale on the Smith Chart. Its values increase as we move **clockwise** from an initial value of zero to a maximum value of* 0.5λ*. What's up with that?*

A: This scale uses an **alternative mapping** between θ_Γ and z/λ:

$$\frac{z}{\lambda} = \frac{1}{4} - \frac{\theta_\Gamma}{4\pi} \qquad \Leftrightarrow \qquad \theta_\Gamma = 4\pi\left(\frac{1}{4} - \frac{z}{\lambda}\right)$$

This scale is **analogous** to a situation wherein a second set of mile markers were placed along I-70. These mile markers **begin** at the **east** side of Kansas (at the Missouri border), and **end** at the **west** side of Kansas (at the Colorado border).



What's the point?

Q: *What **good** would this second set of markers do? Does it serve any purpose?*

A: **Not much** really. After all, this second set is **redundant**—it does not provide any **new** information that the original set already provides.

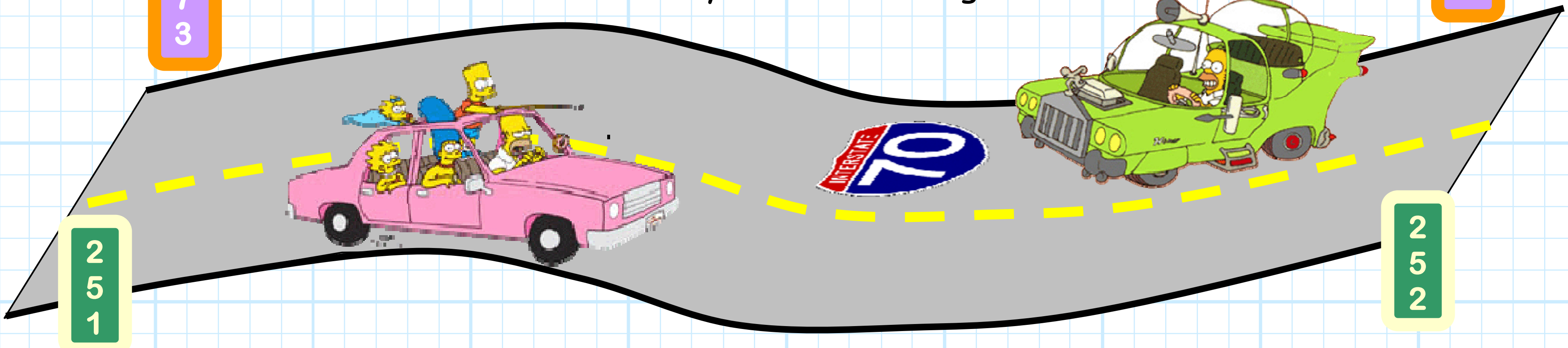
Yet, if we were to place this new set along I-70, we almost certainly would place the **original** mile markers along the **eastbound** lanes, and this new set along the **westbound** lanes.

In this manner, all I-70 motorists (eastbound or westbound) would see an **increase** in the mile markers as they traverse the **Sunflower State**.

As a result, a **positive** distance to their destination indicates to **all** drivers that their destination is in **front** of them (in the direction they are driving), while a **negative** distance indicates to **all** drivers that their destination is **behind** the (they better **turn around**!).

The power of positive thinking

Thus, it could be argued that each set of mile markers is **optimized for a specific direction** of travel—the original set if you are traveling **east**, and this second set if you are traveling **west**.

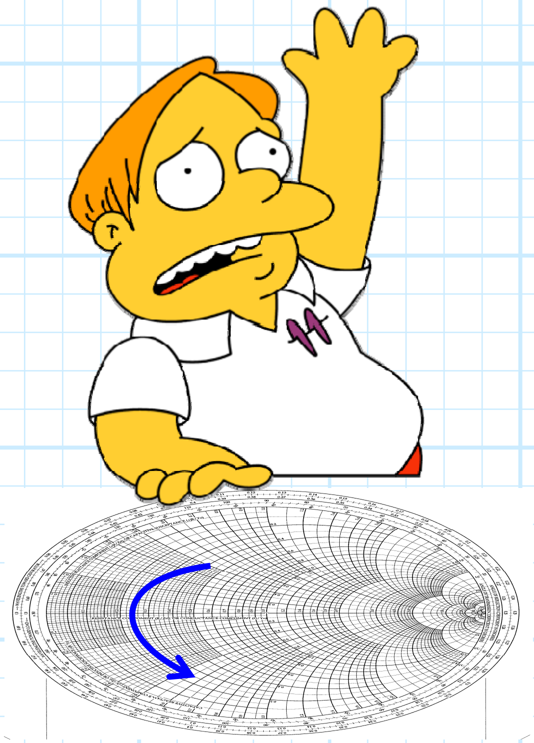


Similarly, the two electrical length scales on the Smith Chart are meant for two different "directions of travel". If we move down the transmission line **toward the load**, the value Δz will be **positive**.

Conversely, if we move up the transmission line and **away from the load** (i.e., "toward the generator"), this second electrical length scale will also provide a **positive** value of Δz.

Again, these two electrical length scales are **redundant**—you will get the correct answer **regardless** of the scale you use, but be careful to interpret negative signs properly.

Oh, so you noticed



Q: *Wait! I just used a Smith Chart to analyze a transmission line problem in the manner you have just explained. At one point on my transmission line the phase of the reflection coefficient is* $\theta_\Gamma = +170^\circ$*, which is denoted as* 0.486λ *on the* ***"wavelengths toward load"*** *scale.*

I then moved a short distance along the line ***toward the load****, and found that the reflection coefficient phase was* $\theta_\Gamma = -144^\circ$*, which is denoted as* 0.050λ *on the "wavelengths toward load" scale.*

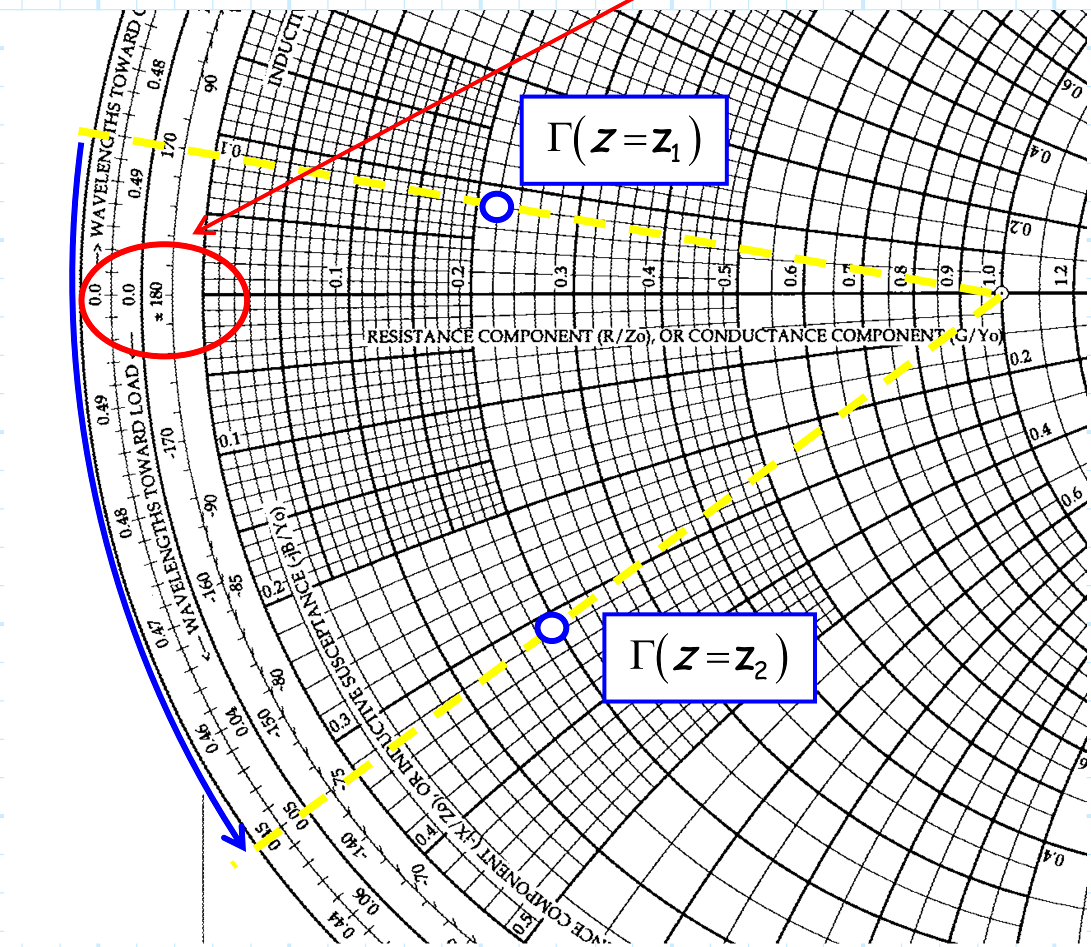
According to ***your*** *"instruction", the distance between these two points is:*

$$\Delta z = 0.050\lambda - 0.486\lambda = -0.436\lambda$$

A large ***negative*** *value! This says that I moved nearly a half wavelength* ***away*** *from the load, but I know that I moved just a short distance* ***toward*** *the load!* ***What happened?***

Here's the problem

A: Note the electrical length scales on the Smith Chart **begin and end** where $\theta_\Gamma = \pm\pi$ (by the short circuit!).



In your example, when rotating counter-clockwise around the chart (i.e., moving toward the load) you **passed by** this **transition**. This makes the calculation of Δz a bit more problematic.

Yet another enlightening analogy

To see why, let's again consider our **I-70 analogy**. Say we are Lawrence, and wish to drive eastbound on Interstate 70 until we reach **Columbia, Missouri**.

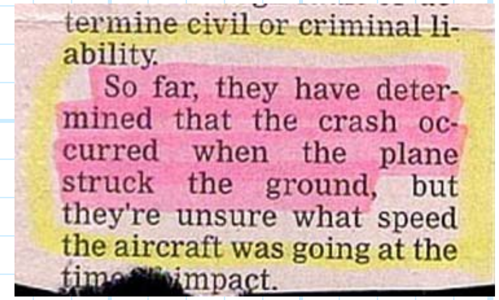
The mile marker for Lawrence is of course **388**, and Columbia Missouri is located at mile marker **126**. We **might** conclude that the distance from Lawrence to Columbia is:

$$126 - 388 = -262 \text{ miles}$$

Q: *Yikes! According to this, Columbia is 262 miles* ***west*** *of Lawrence—should we turn the* ***car around?***

A: Columbia, Missouri is most decidedly **east** of Lawrence, Kansas. The problem is that mile markers **"reset"** to zero once we reach a **state border**, and then again **increase** as we travel eastward.

The painfully obvious*


termine civil or criminal liability.
So far, they have determined that the crash occurred when the plane struck the ground, but they're unsure what speed the aircraft was going at the tim[illegible]impact.

Thus, to **accurately** determine the **distance** between Lawrence and Columbia, we need to break the problem into **two steps**:

Step 1: Determine the distance between **Lawrence** (mile marker 388) , and the **last mile marker** before the state line (mile marker 423):


Survey finds fewer deer after hunt

$$423 - 388 = 35 \text{ miles}$$

Step 2: Determine the distance between the **first mile marker** after the state line (mile marker 0) and **Columbia** (mile marker 126):

$$126 - 0 = 126 \text{ miles}$$


REGION
Homicide victims rarely talk to police

Thus, the distance between Lawrence and Columbia is the distance between Lawrence and the state line (35 miles), **plus** the distance from the state line to Columbia (126 miles):

$$35 + 126 = 161 \text{ miles}$$


DIANA WAS STILL ALIVE HOURS BEFORE SHE DIED
A SERIES of previously unseen CCTV images have revealed that Diana was alive

Columbia, Missouri is **161 miles east** of Lawrence, Kansas!

*** Don't complain; it's far superior to the obviously painful.**

Back to the real world

Now back to the **Smith Chart problem**; as we rotate counter-clockwise around the Smith Chart, the "wavelengths toward load" scale increases in value, until it reaches a **maximum** value of 0.5λ (at $\theta_\Gamma = \pm\pi$) .

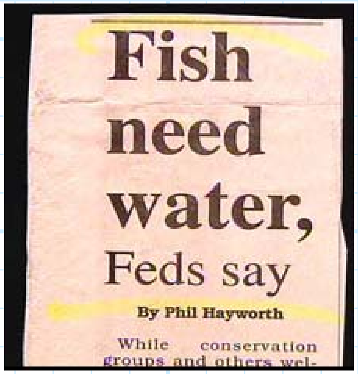
At that point, the scale "resets" to its **minimum** value of **zero**. We have **metaphorically** "crossed the state line" of this scale.

Thus, to accurately determine the electrical length moved along a transmission line, we must divide the problem into **two steps**:

Step 1: Determine the electrical length from the **initial** point to the **"end"** of the scale at 0.5λ .

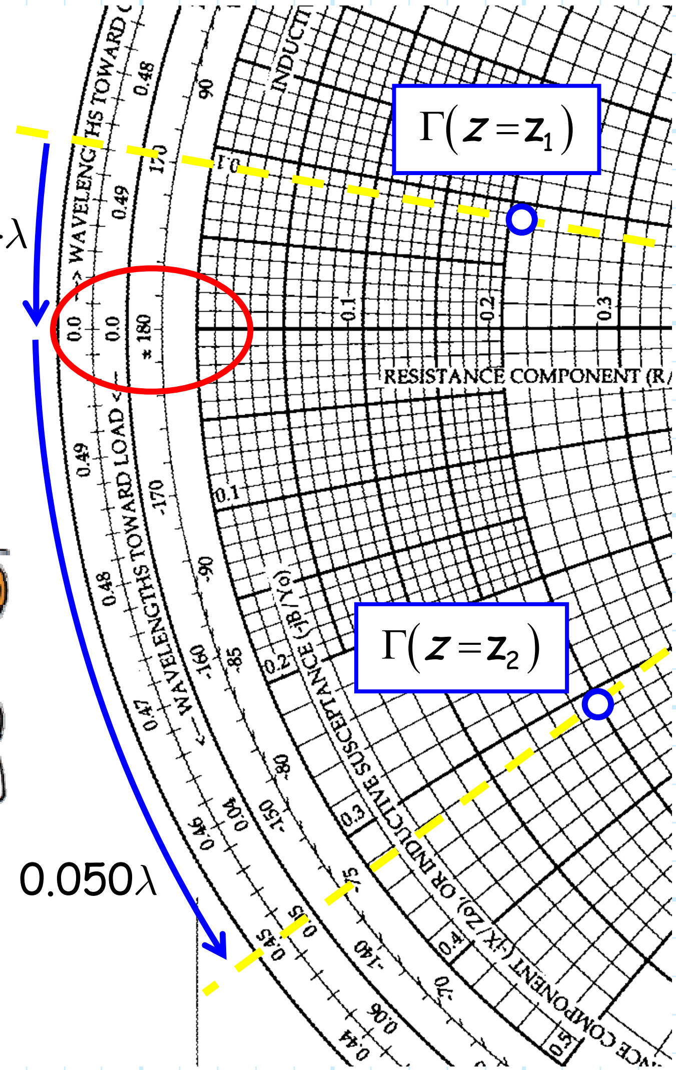
Step 2: Determine the electrical distance from the **"beginning"** of the scale (i.e., 0) and the **second location** on the transmission line.

Add the results of steps 1 and 2, and you have your answer!



Your problem is solved

For **example**, let's look at the case that originally gave us the erroneous result. The distance from the initial location to the **end of the scale** is:



And the distance from the **beginning of the scale** to the second point is:

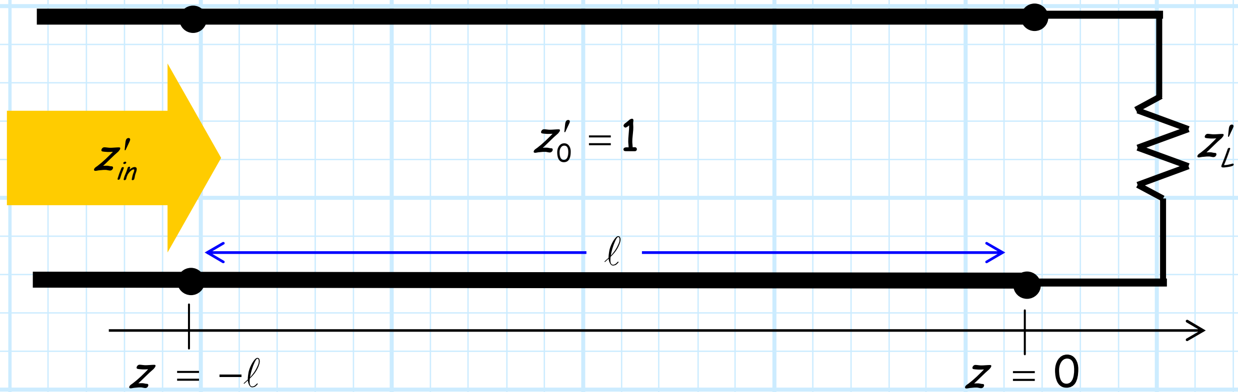
$$0.050\lambda - 0.000\lambda = +0.050\lambda$$

Thus the **distance between** the two points is:

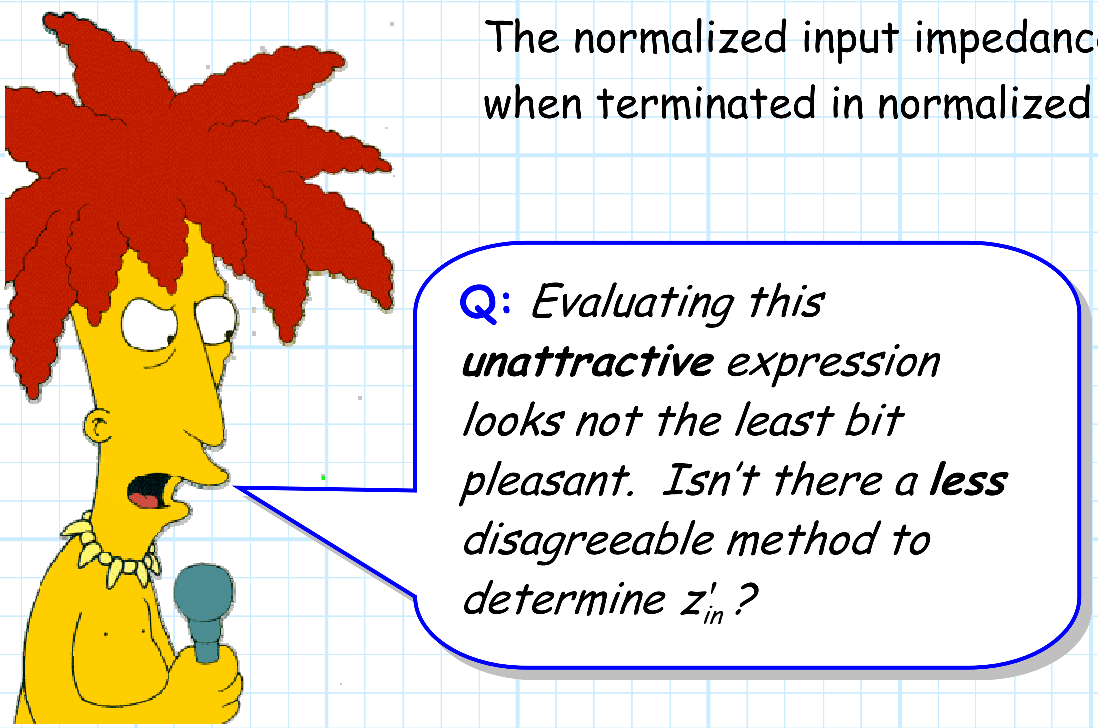
$$0.014\lambda + 0.050\lambda = +0.064\lambda$$

The second point is just a **little closer** to the load than the first !

Zin Calculations using the Smith Chart



The normalized input impedance z'_{in} of a transmission line **length** ℓ, when terminated in normalized **load** z'_L, **can** be determined as:



$$
\begin{aligned}
z'_{in} &= \frac{Z_{in}}{Z_0} \\
&= \frac{1}{Z_0} Z_0 \left(\frac{Z_L + j\, Z_0 \tan \beta\ell}{Z_0 + j\, Z_L \tan \beta\ell} \right) \\
&= \frac{Z_L / Z_0 + j \tan \beta\ell}{1 + j\, Z_L / Z_0 \tan \beta\ell} \\
&= \frac{z'_L + j \tan \beta\ell}{1 + j\, z'_L \tan \beta\ell}
\end{aligned}
$$

A: **Yes** there is! Instead, we could determine this normalized input impedance by following these **three** steps:

1. Convert z'_L to Γ_L, using the equation:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z'_L - 1}{z'_L + 1}$$

2. Convert Γ_L to Γ_{in}, using the equation:

$$\Gamma_{in} = \Gamma_L \, e^{-j2\beta\ell}$$

3. Convert Γ_{in} to z'_{in}, using the equation:

$$z'_{in} = \frac{Z_{in}}{Z_0} = \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$$

Q: *But performing these **three** calculations would be even **more** difficult than the **single** step you described earlier. What short of dimwit would ever use (or recommend) this approach?*

A: The benefit in this last approach is that **each** of the three steps can be executed using a **Smith Chart—no** complex calculations are required!

1. **Convert z_L' to Γ_L**

Find the point z_L' from the impedance mappings on your Smith Chart. **Place you pencil at that point—you have now located the correct Γ_L on your complex Γ plane!**

For **example**, say $z_L' = 0.6 - j1.4$. We find on the Smith Chart the circle for r=0.6 and the circle for x=-1.4. The **intersection** of these two circles is the point on the complex Γ plane corresponding to normalized impedance $z_L' = 0.6 - j1.4$.

This point is a **distance** of 0.685 units from the origin, and is located at **angle** of –65 degrees. Thus the value of Γ_L is:

$$\Gamma_L = 0.685\, e^{-j65^\circ}$$

2. Convert Γ_L to Γ_{in}

Since we have correctly located the point Γ_L on the complex Γ plane, we merely need to **rotate** that point **clockwise** around a circle ($|\Gamma| = 0.685$) by an angle $2\beta\ell$.

When we **stop**, we are located at the point on the complex Γ plane where $\Gamma = \Gamma_{in}$!

For **example**, if the length of the transmission line terminated in $z'_L = 0.6 - j1.4$ is $\ell = 0.307\lambda$, we should rotate around the Smith Chart a total of $2\beta\ell = 1.228\pi$ radians, or 221°. We are now at the point on the complex Γ plane:

$$\Gamma = 0.685\, e^{+j74^\circ}$$

This is the value of Γ_{in}!

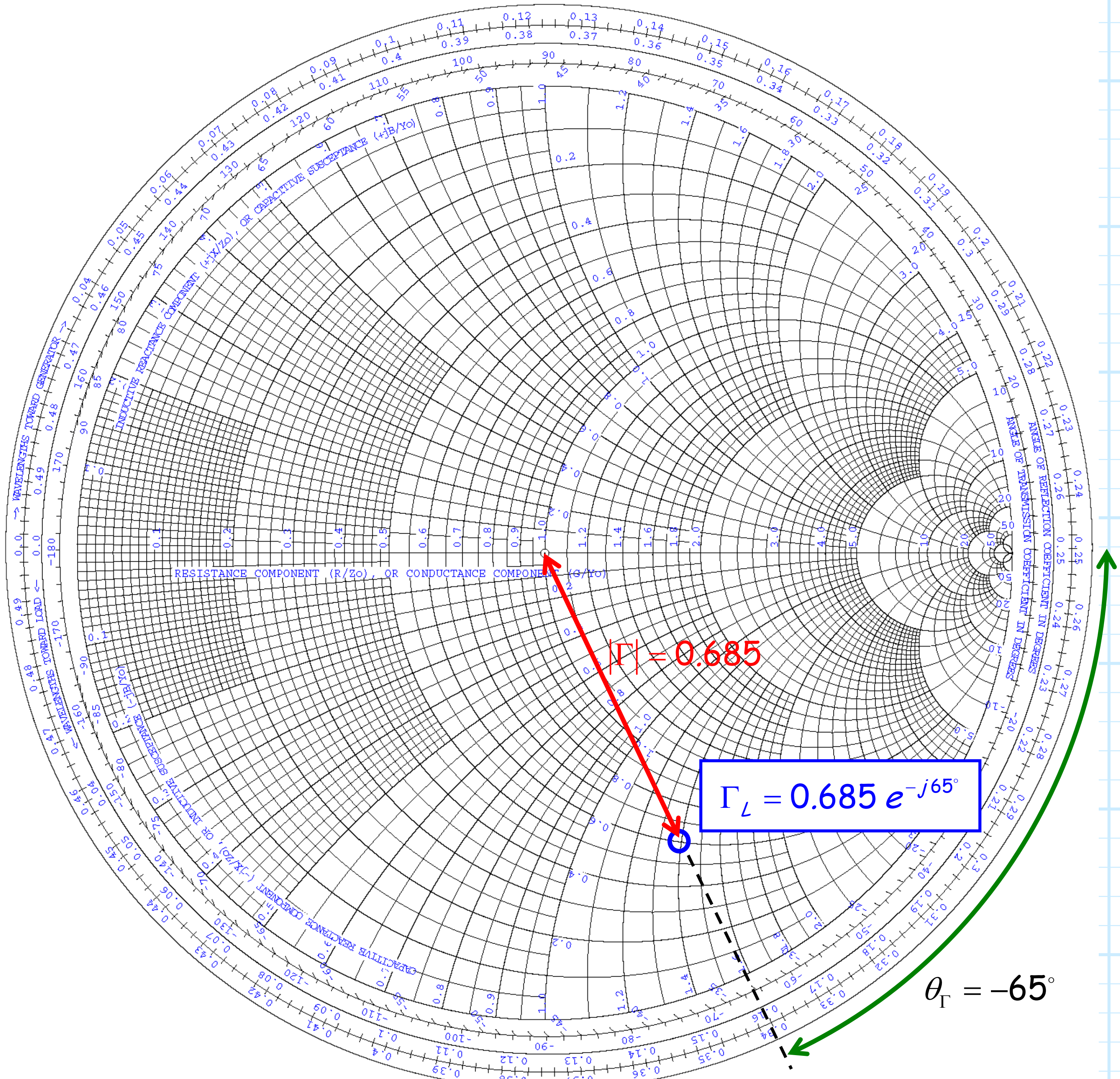
3. Convert Γ_{in} to z'_{in}

When you get finished rotating, and your pencil is located at the point $\Gamma = \Gamma_{in}$, **simply lift your pencil and determine the values *r* and *x* to which the point corresponds!**

For **example**, we can determine directly from the Smith Chart that the point $\Gamma_{in} = 0.685\, e^{+j74^\circ}$ is located at the **intersection** of circles $r = 0.5$ and $x = 1.2$. In other words:

$$z'_{in} = 0.5 + j1.2$$

Step 1

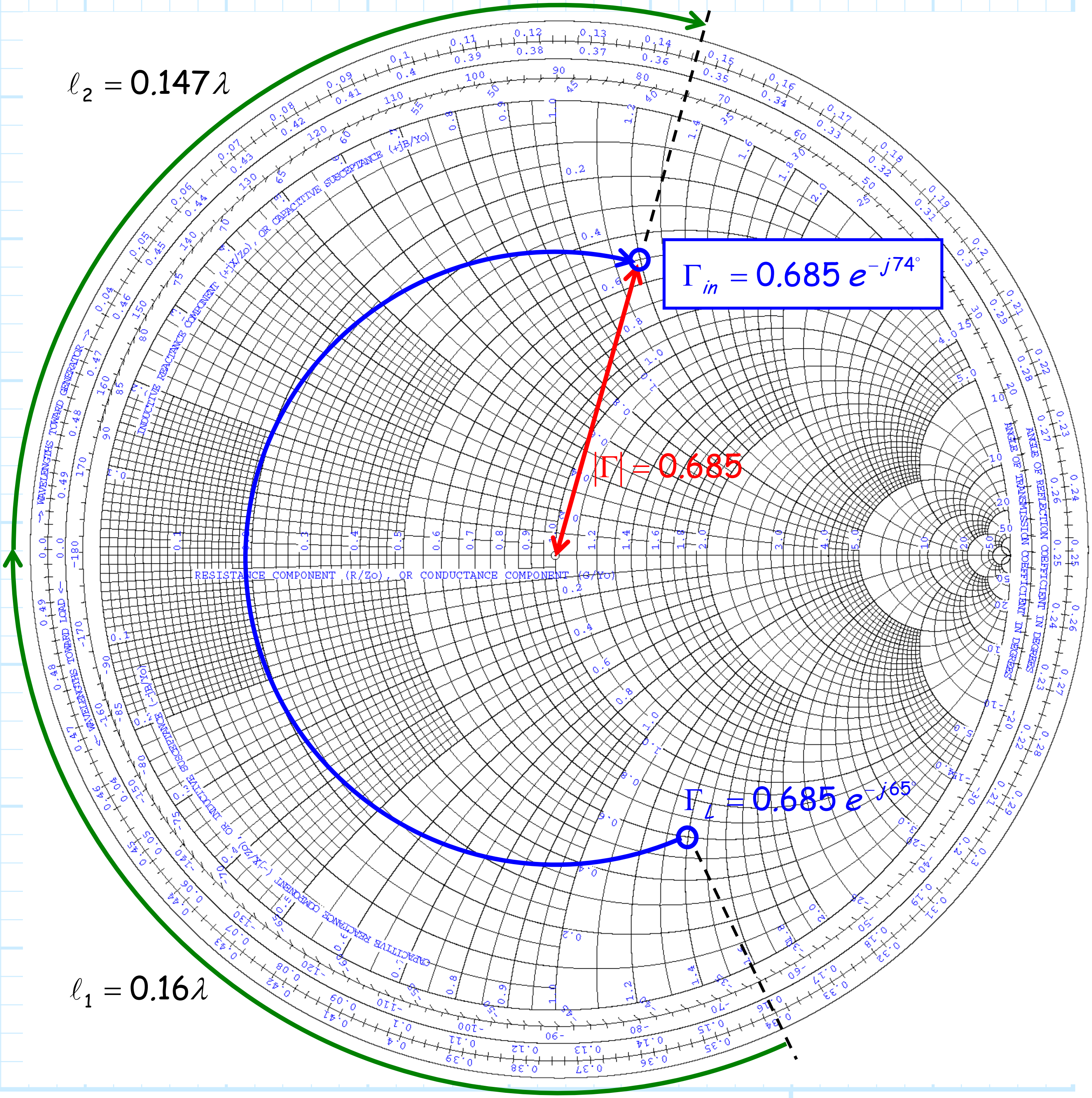


$|\Gamma| = 0.685$
$\Gamma_L = 0.685\ e^{-j65^\circ}$
$\theta_\Gamma = -65^\circ$
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)

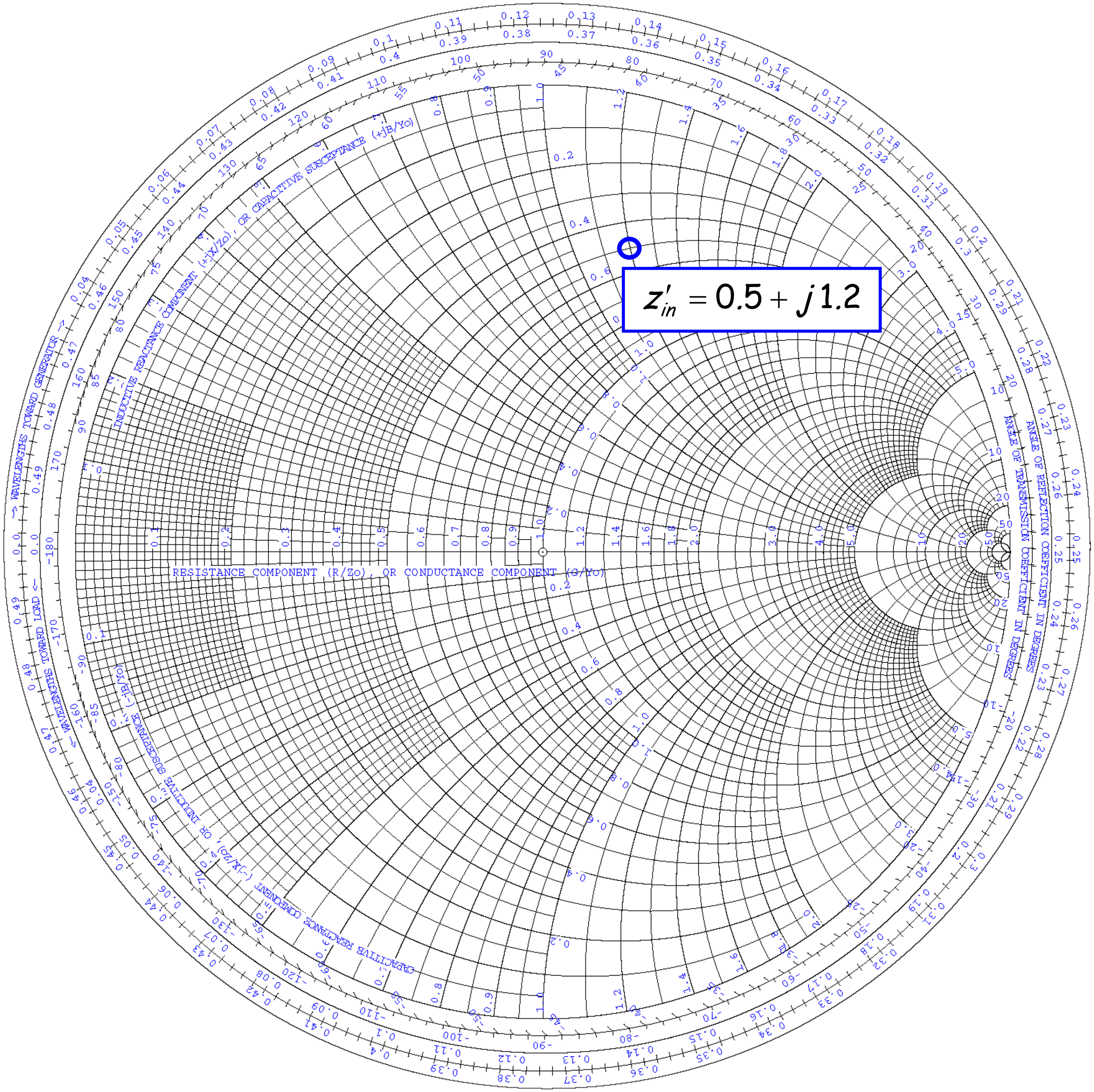
Step 2

$$\ell = \ell_1 + \ell_2$$
$$= 0.160\lambda + 0.147\lambda$$
$$= 0.307\lambda$$

$$2\beta\ell = 221^\circ$$



Step 3

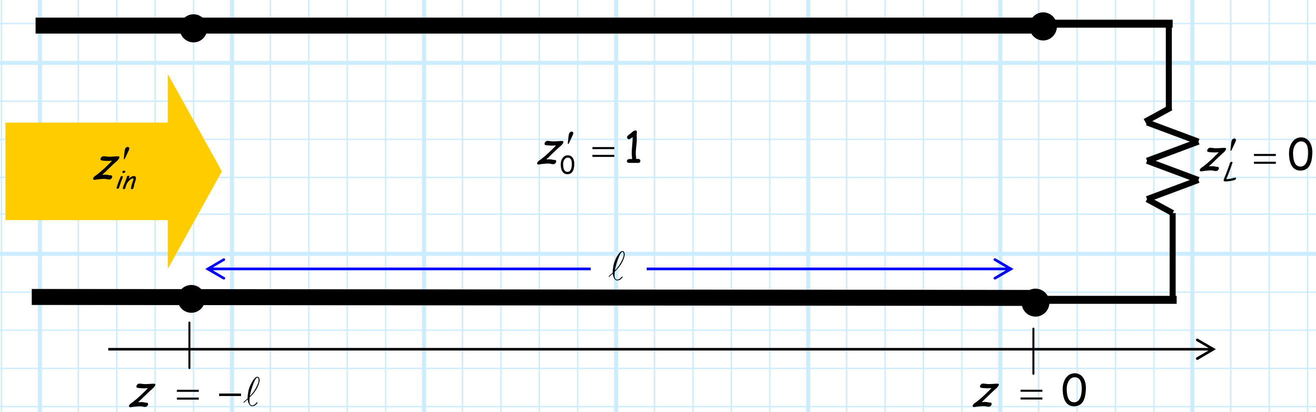

$z'_{in} = 0.5 + j1.2$
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)

Example: The Input Impedance of a Shorted Transmission Line

Let's determine the input impedance of a transmission line that is terminated in a **short circuit**, and whose length is:

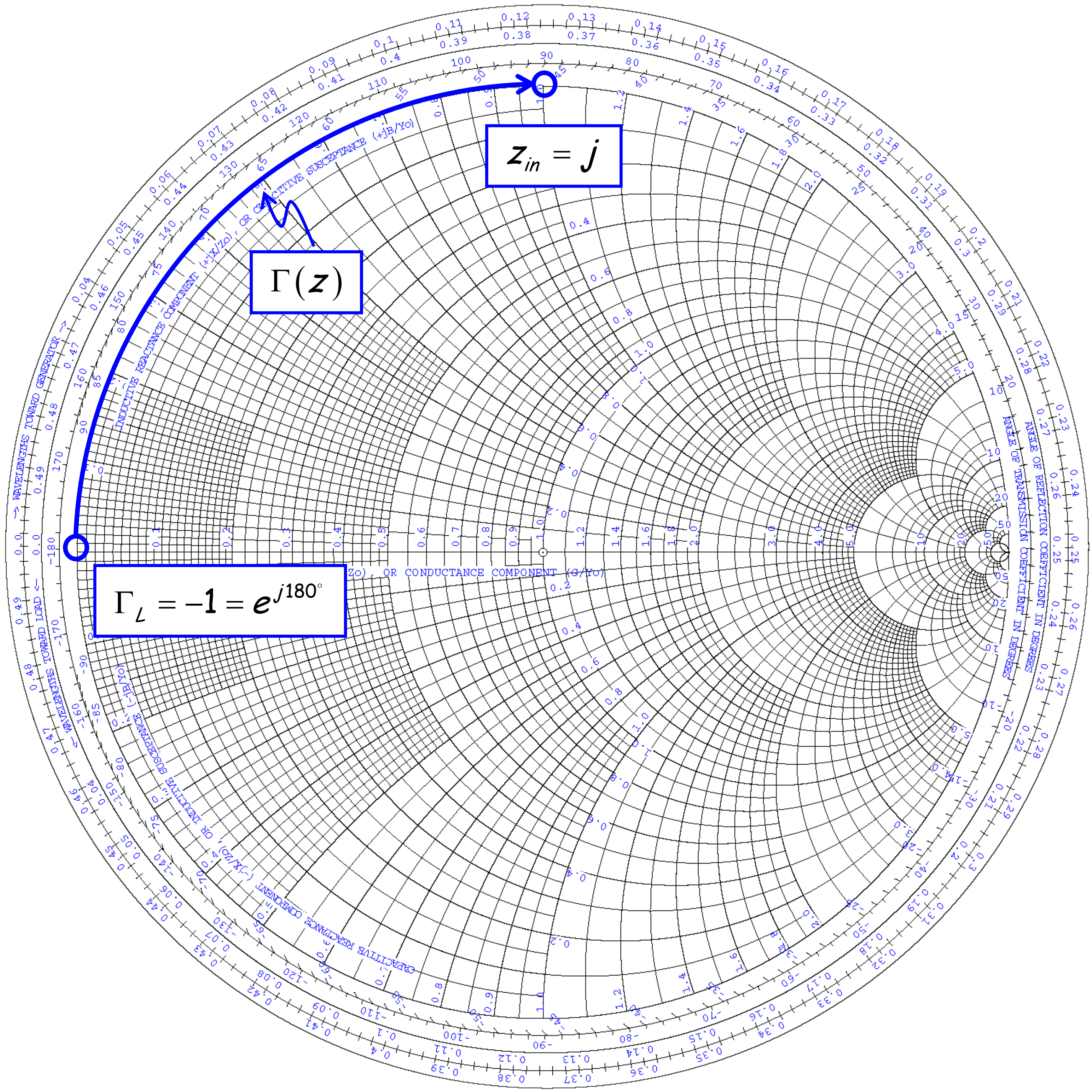
a) $\ell = \lambda/8 = 0.125\lambda \quad \Rightarrow \quad 2\beta\ell = 90^\circ$

b) $\ell = 3\lambda/8 = 0.375\lambda \quad \Rightarrow \quad 2\beta\ell = 270^\circ$



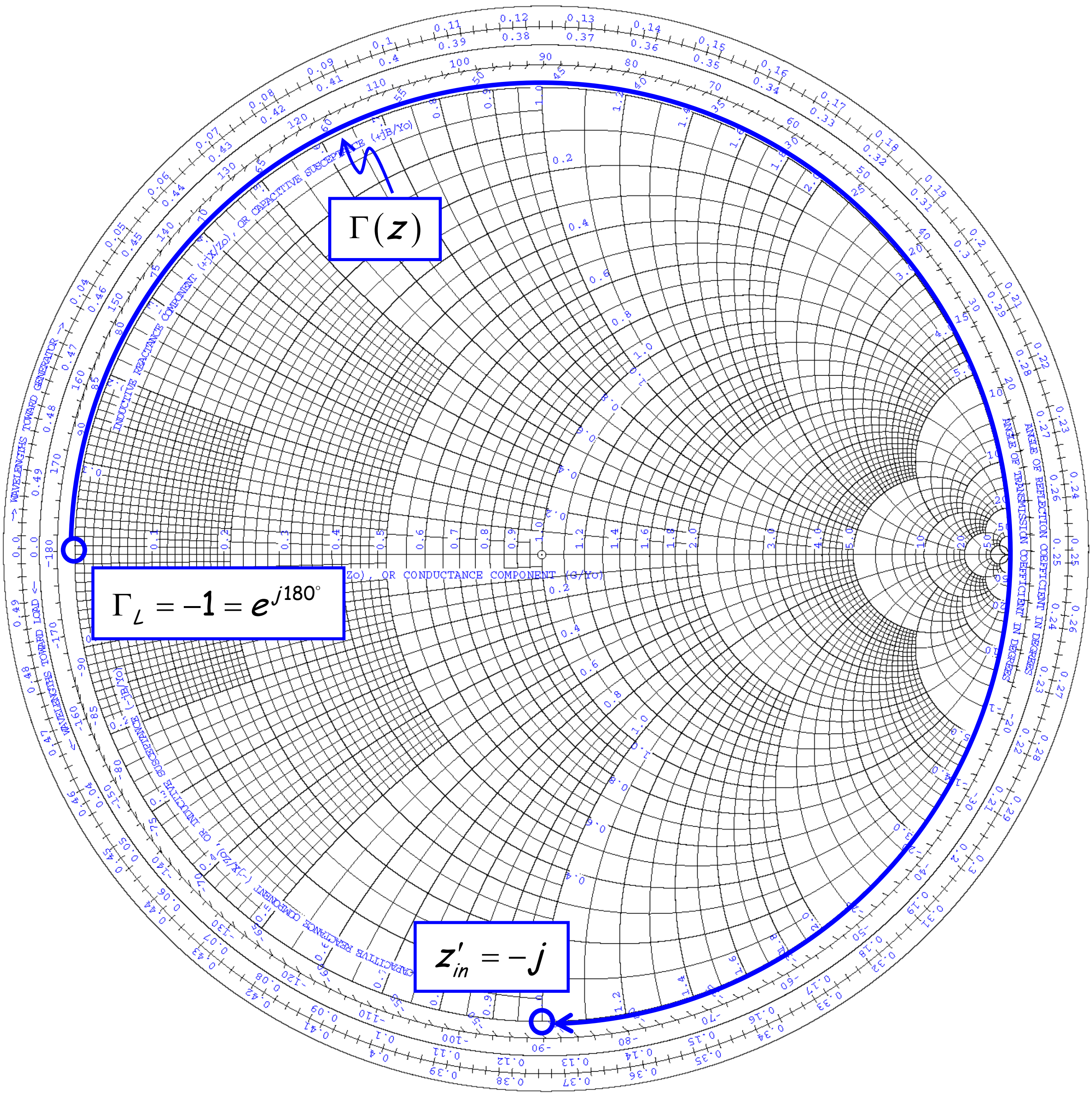
a) $\ell = \lambda/8 = 0.125\lambda \quad \Rightarrow \quad 2\beta\ell = 90^\circ$

Rotate **clockwise** 90° from $\Gamma = -1.0 = e^{j180^\circ}$ and find $z'_{in} = j$.



b) $\ell = 3\lambda/8 = 0.375\lambda \quad \Rightarrow \quad 2\beta\ell = 270^\circ$

Rotate **clockwise** 270° from $\Gamma = -1.0 = e^{j180^\circ}$ and find $z'_{in} = -j$.

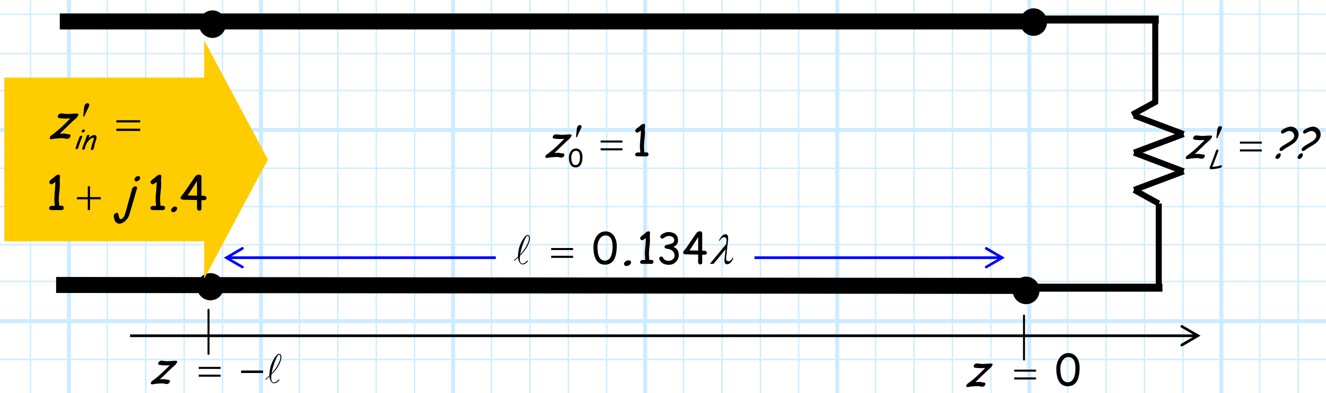


Example: Determining the Load Impedance of a Transmission Line

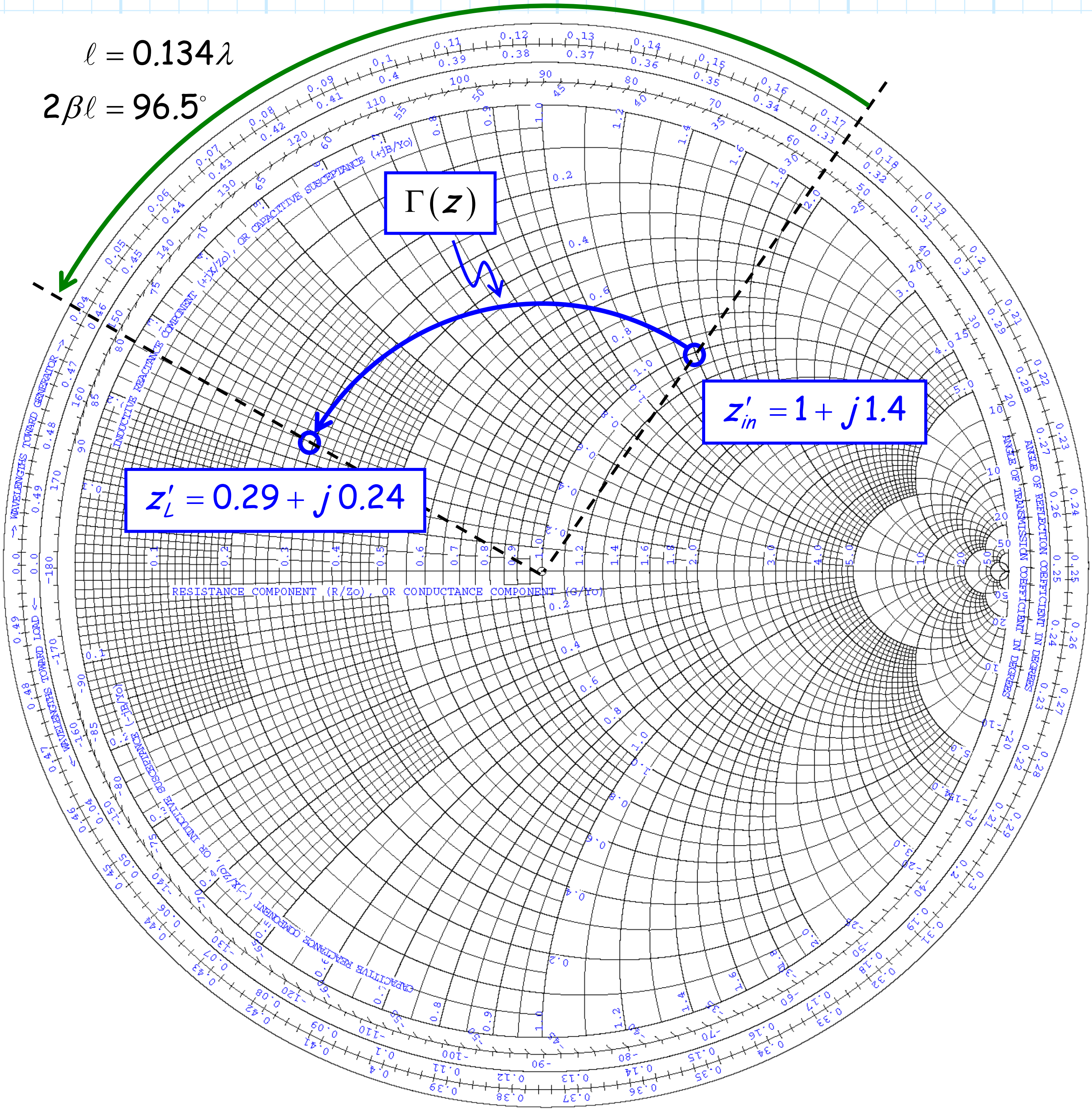
Say that we know that the **input** impedance of a transmission line length $\ell = 0.134\lambda$ is:

$$z'_{in} = 1.0 + j1.4$$

Let's determine the impedance of the **load** that is terminating this line.



Locate z'_{in} on the Smith Chart, and then rotate **counter-clockwise** (yes, I said **counter**-clockwise) $2\beta\ell = 96.5^{\circ}$. Essentially, you are removing the phase shift associated with the transmission line. When you stop, lift your pencil and find z'_L !



$\ell = 0.134\lambda$
$2\beta\ell = 96.5^\circ$
$\Gamma(z)$
$z'_{in} = 1 + j1.4$
$z'_L = 0.29 + j0.24$

Example: Determining Transmission Line Length

A load **terminating** at transmission line has a normalized impedance $z'_L = 2.0 + j2.0$. What should the **length** ℓ of transmission line be in order for its input impedance to be:

a) purely **real** (i.e., $x_{in} = 0$)?

b) have a real (resistive) part equal to **one** (i.e., $r_{in} = 1.0$)?

Solution:

a) Find $z'_L = 2.0 + j2.0$ on your Smith Chart, and then rotate **clockwise** until you "bump into" the contour $x = 0$ (recall this is contour lies on the Γ_r **axis!**).

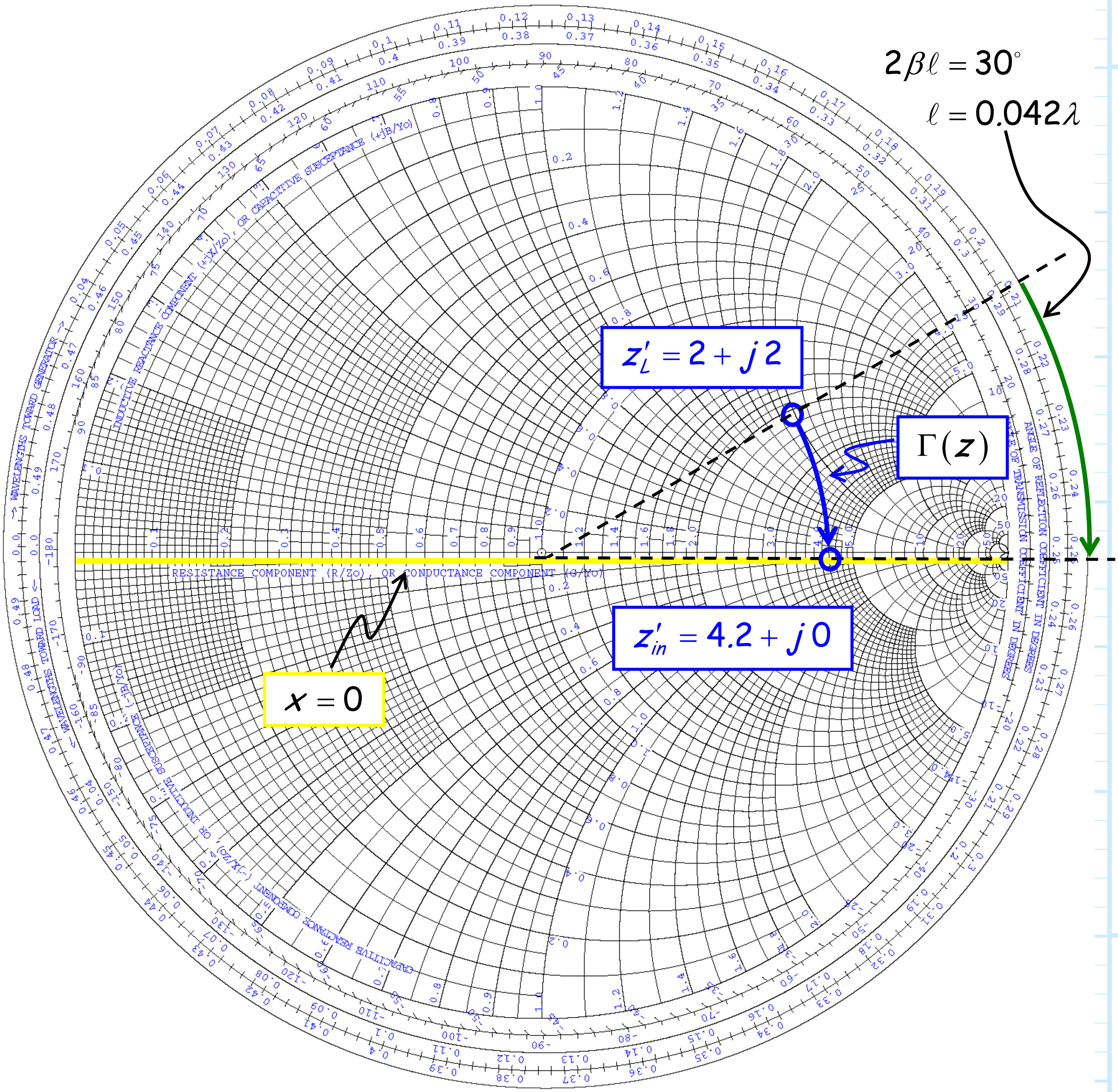
When you reach the $x = 0$ contour—**stop!** Lift your pencil and note that the impedance value of this location is **purely real** (after all, $x = 0$!).

Now, measure the **rotation angle** that was required to move clockwise from $z'_L = 2.0 + j2.0$ to an impedance on the $x = 0$ contour—this **angle** is equal to $2\beta\ell$!

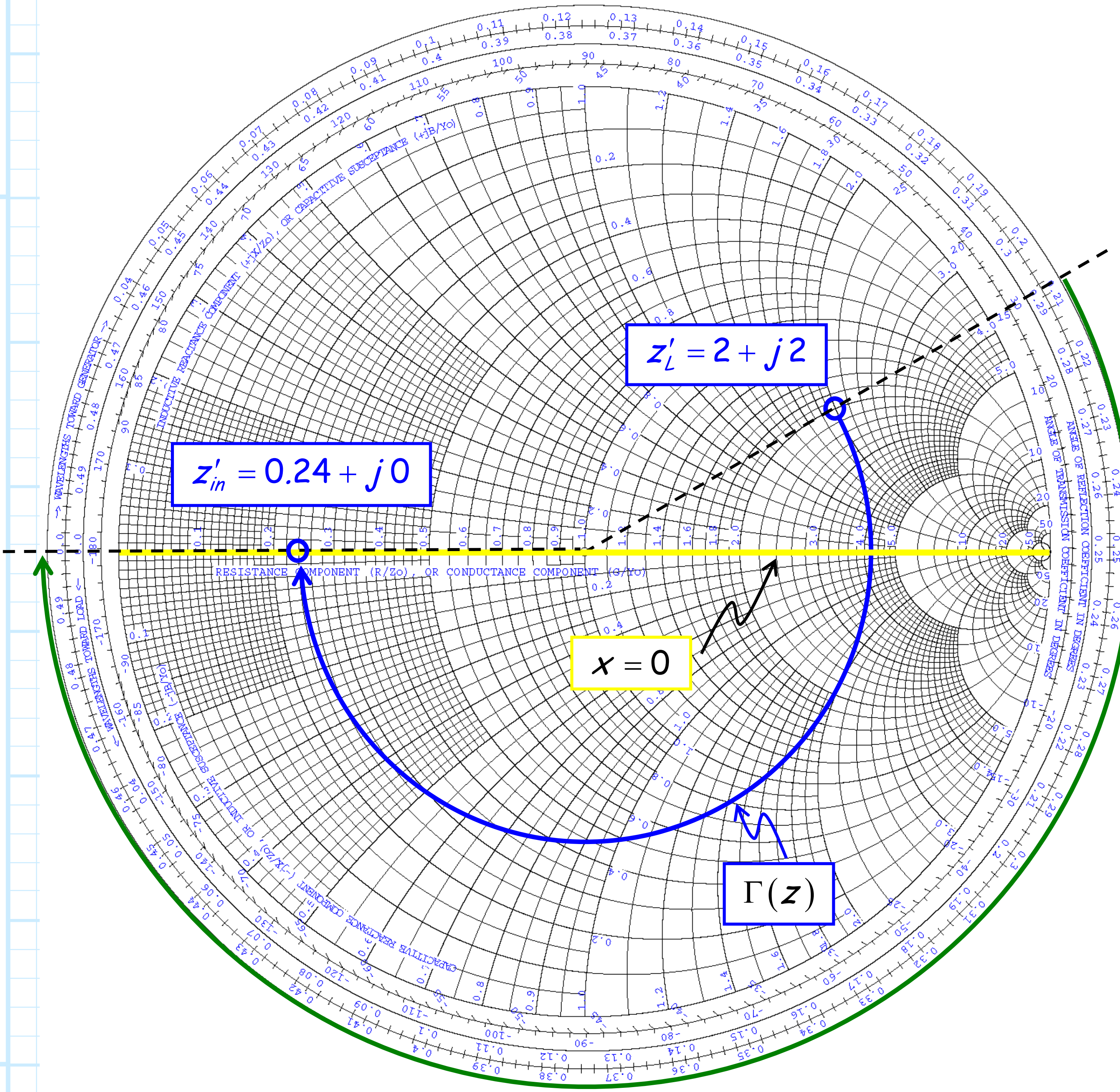
You can now **solve** for ℓ, or alternatively use the **electrical length scale** surrounding the Smith Chart.

One more important point—there are **two** possible solutions!

Solution 1:



Solution 2:



$$2\beta\ell = 210^\circ$$

$$\ell = 0.292\lambda$$

b) Find $z'_L = 2.0 + j2.0$ on your Smith Chart, and then rotate **clockwise** until you "bump into" the **circle** $r = 1$ (recall this circle intersects the **center** point or the Smith Chart!).

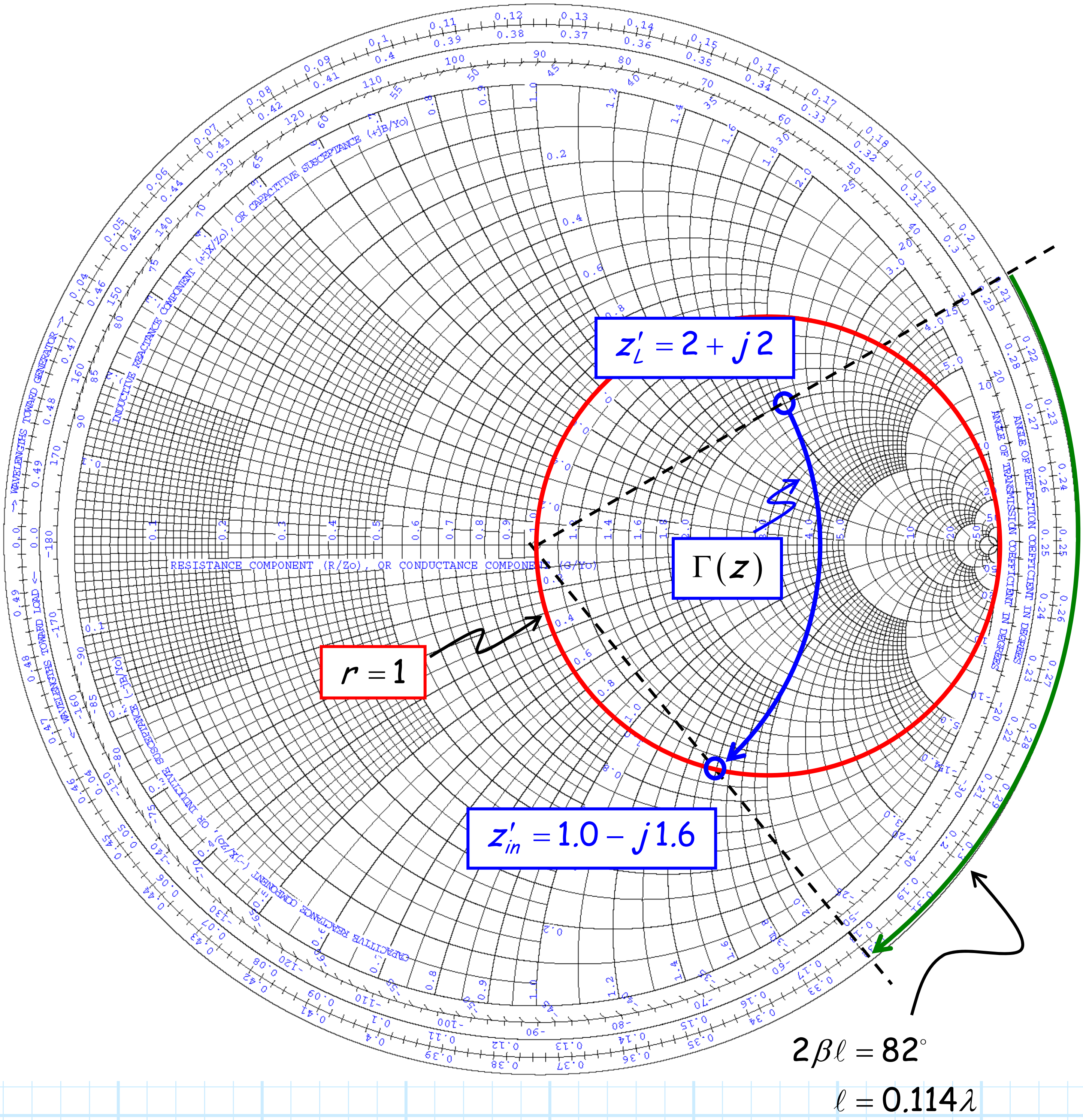
When you reach the $r = 1$ circle—**stop**! Lift your pencil and note that the impedance value of this location has a real value equal to **one** (after all, $r = 1$!).

Now, measure the **rotation angle** that was required to move clockwise from $z'_L = 2.0 + j2.0$ to an impedance on the $r = 1$ circle—this **angle** is equal to $2\beta\ell$!

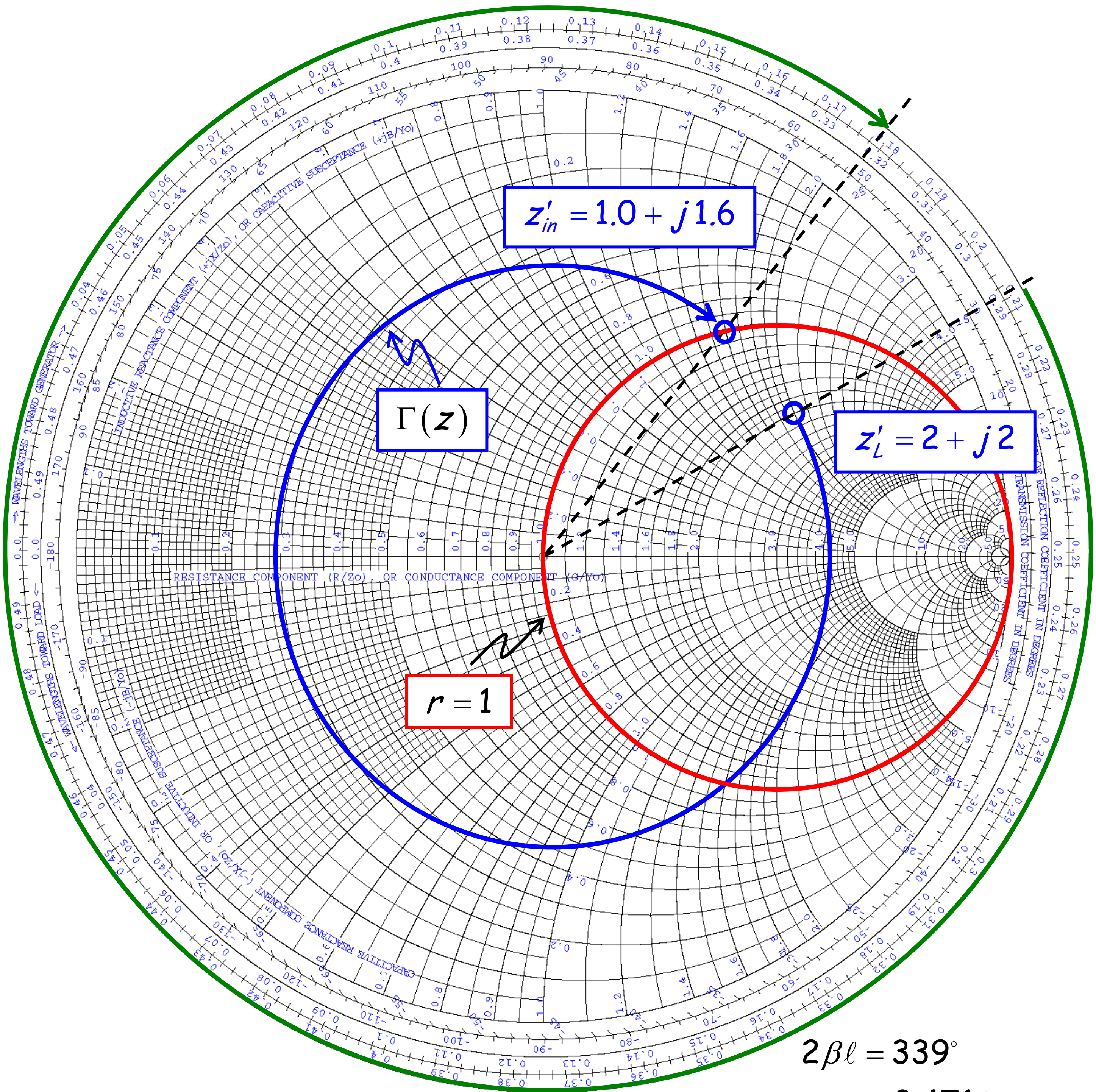
You can now **solve** for ℓ, or alternatively use the **electrical length scale** surrounding the Smith Chart.

Again, we find that there are **two** solutions!

Solution 1:



Solution 2:



$$2\beta\ell = 339^\circ$$

$$\ell = 0.471\lambda$$

Q: *Hey! For part b), the solutions resulted in $z'_{in} = 1 - j\,1.6$ and $z'_{in} = 1 + j\,1.6$ --the **imaginary** parts are equal but **opposite!** Is this just a coincidence?*

A: Hardly! Remember, the two impedance solutions must result in the **same magnitude** for Γ--for this example we find $\left|\Gamma(z)\right| = 0.625$.

Thus, for impedances where r =1 (i.e., $z' = 1 + j\,x$):

$$\Gamma = \frac{z'-1}{z'+1} = \frac{(1+jx)-1}{(1+jx)+1} = \frac{j\,x}{2+j\,x}$$

and therefore:

$$|\Gamma|^2 = \frac{|j\,x|^2}{|2+j\,x|^2} = \frac{x^2}{4+x^2}$$

Meaning:

$$x^2 = \frac{4\,|\Gamma|^2}{1-|\Gamma|^2}$$

of which there are **two** equal by opposite solutions!

$$x = \pm\frac{2\,|\Gamma|}{\sqrt{1-|\Gamma|^2}}$$

Which for **this** example gives us our solutions $x = \pm 1.6$.

Impedance & Admittance

As an alternative to impedance Z, we can define a complex parameter called **admittance** Y:

$$Y = \frac{I}{V}$$

where V and I are complex voltage and current, respectively.

Clearly, admittance and impedance are not independent parameters, and are in fact simply geometric **inverses** of each other:

$$Y = \frac{1}{Z} \qquad\qquad Z = \frac{1}{Y}$$

Thus, all the impedance parameters that we have studied can be **likewise** expressed in terms of admittance, e.g.:

$$Y(z) = \frac{1}{Z(z)} \qquad\qquad Y_L = \frac{1}{Z_L} \qquad\qquad Y_{in} = \frac{1}{Z_{in}}$$

Normalized Admittance

Moreover, we can define the **characteristic admittance** Y_0 of a transmission line as:

$$Y_0 = \frac{I^+(z)}{V^+(z)}$$

And thus it is similarly evident that characteristic impedance and characteristic admittance are geometric **inverses**:

$$Y_0 = \frac{1}{Z_0} \qquad\qquad Z_0 = \frac{1}{Y_0}$$

As a result, we can define a **normalized admittance** value y':

$$y' = \frac{Y}{Y_0}$$

An therefore (not surprisingly) we find:

$$y' = \frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{1}{z'}$$

Susceptance and Conductance

Now since admittance is a **complex** value, it has both a real and imaginary component:

$$Y = G + j\,B$$

where:

$$\text{Re}\{Y\} \doteq G = \text{ Conductance} \qquad\qquad \text{Im}\{Z\} \doteq B = \text{ Susceptance}$$

Now, since $Z = R + jX$, we can state that:

$$G + jB = \frac{1}{R + jX}$$

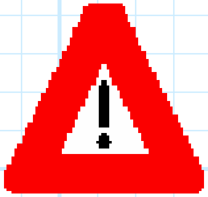
Steve Marcus / Reuters

Q: *Yes yes, I see, and from this we can conclude:*

$$G = \frac{1}{R} \quad \textit{and} \quad B = \frac{-1}{X}$$

and so forth. Please speed this up and quit wasting my valuable time making such **obvious** statements!

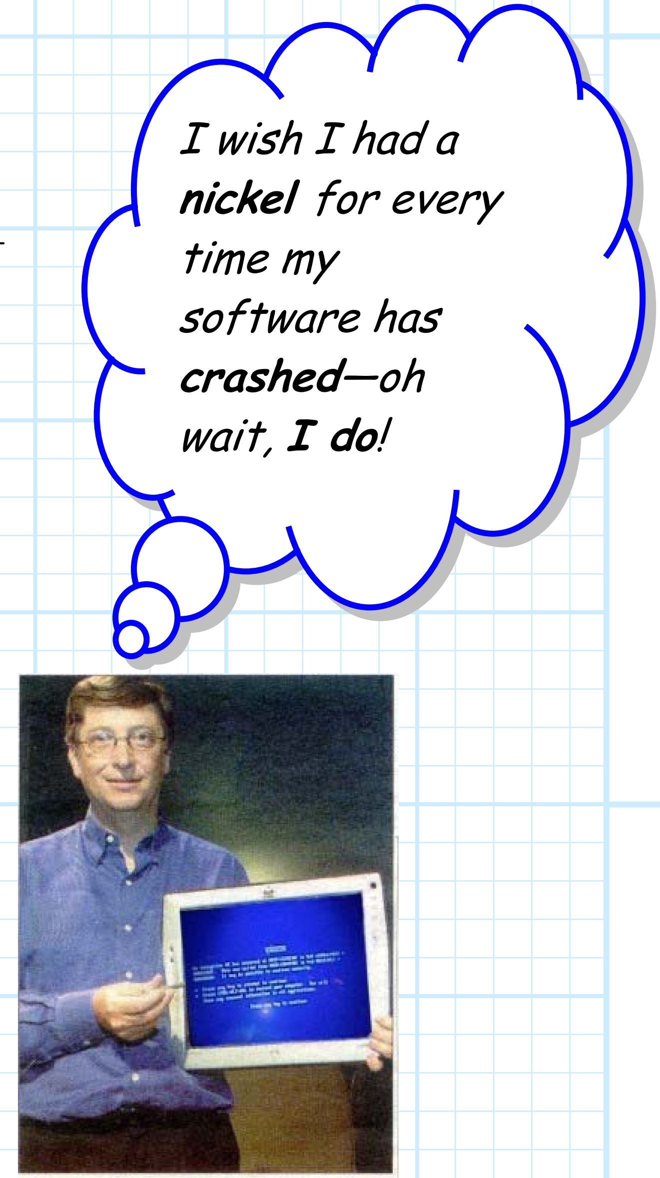
Be Careful!



A: **NOOOO!** We find that $G \neq 1/R$ and $B \neq 1/X$ (generally). Do **not** make this mistake!

In fact, we find that:

$$G = \frac{R}{R^2 + X^2} \qquad \text{and} \qquad B = \frac{-X}{R^2 + X^2}$$



Note then that **IF** $X = 0$ (i.e., $Z = R$), we get, as expected:

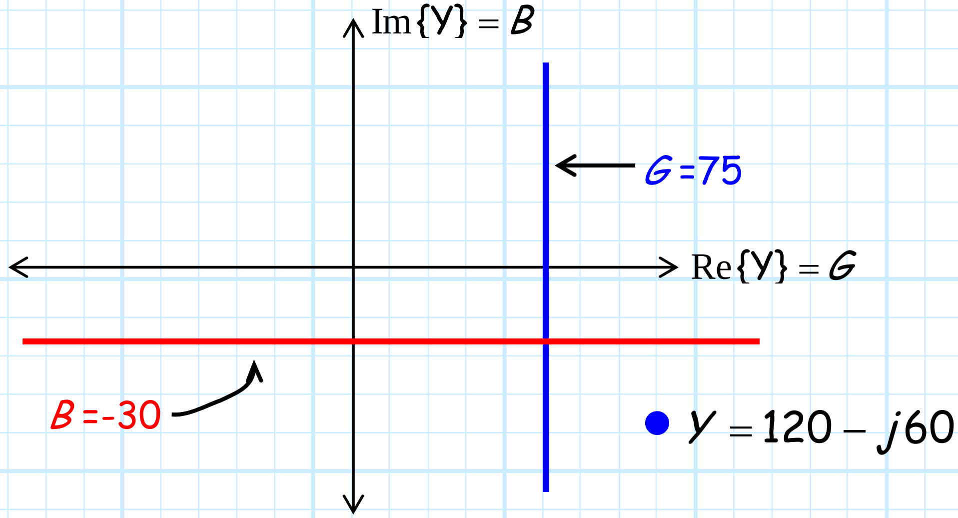
$$G = \frac{1}{R} \qquad \text{and} \qquad B = 0$$

And that **IF** $R = 0$ (i.e., $Z = R$), we get, as expected:

$$G = 0 \qquad \text{and} \qquad B = \frac{-1}{X}$$

Admittance and the Smith Chart

Just like the complex impedance plane, we can plot points and contours on the complex **admittance** plane:



Q: *Can we also map **these** points and contours onto the complex Γ plane?*

A: You bet! Let's first rewrite the refection coefficient function in terms of **line admittance** $Y(z)$:

$$\Gamma(z) = \frac{Y_0 - Y(z)}{Y_0 + Y(z)}$$

Rotation around the Smith Chart

Thus,

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} \qquad \text{and} \qquad \Gamma_{in} = \frac{Y_0 - Y_{in}}{Y_0 + Y_{in}}$$

We can therefore likewise express Γ in terms of **normalized** admittance:

$$\Gamma = \frac{Y_0 - Y}{Y_0 + Y} = \frac{1 - Y/Y_0}{1 + Y/Y_0} = \frac{1 - y'}{1 + y'}$$

Note this can likewise be expressed as:

$$\Gamma = \frac{1 - y'}{1 + y'} = -\frac{y' - 1}{y' + 1} = e^{j\pi}\,\frac{y' - 1}{y' + 1}$$

Contrast this to the mapping between normalized impedance and Γ:

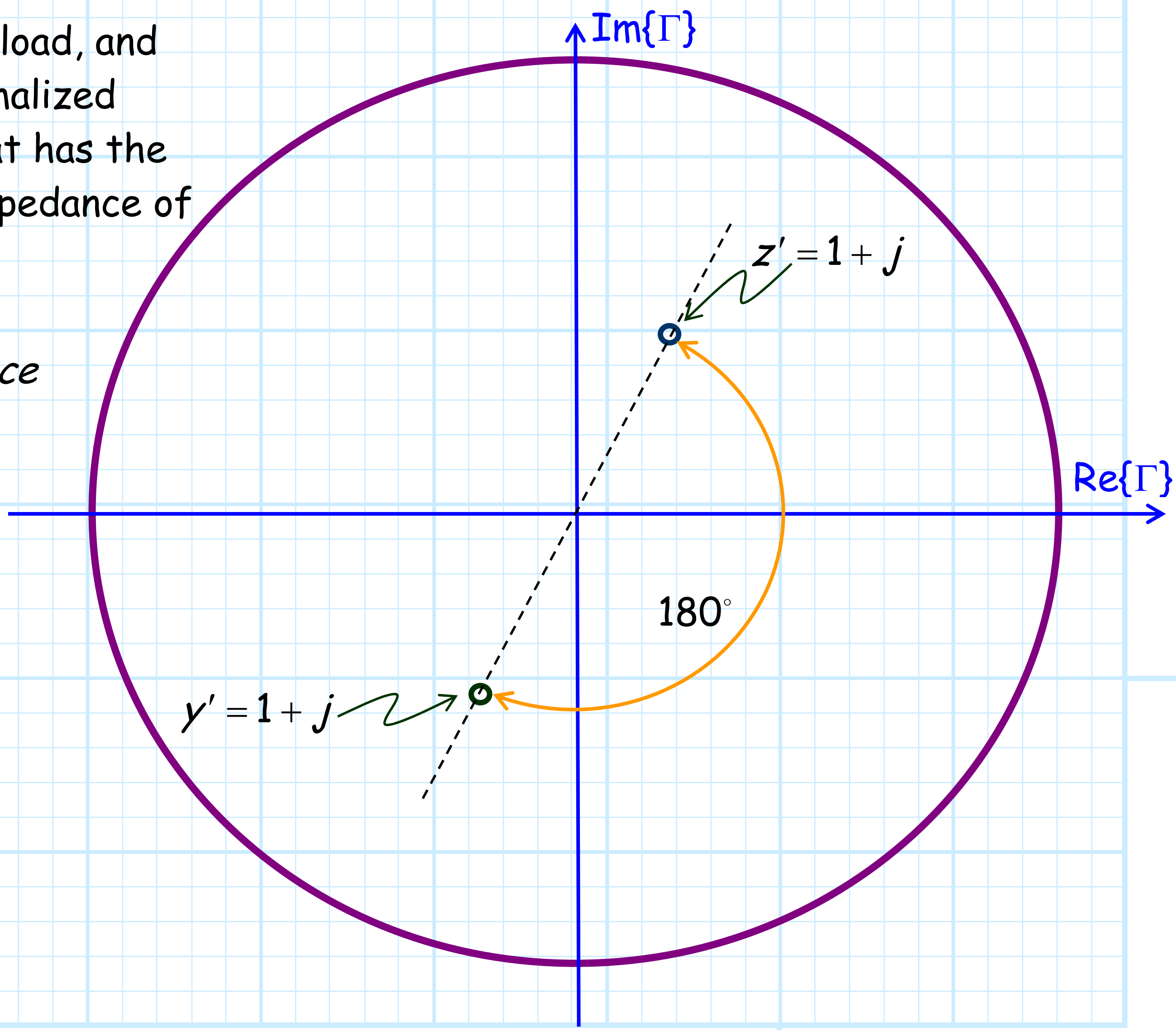
$$\Gamma = \frac{z' - 1}{z' + 1}$$

The difference between the two is simply the factor $e^{j\pi}$—a rotation of 180° around the Smith Chart!.

An example

For example, let's pick some load at random; $z' = 1 + j$, for instance. We know where this point is mapped onto the complex Γ plane; we can locate it on our **Smith Chart**.

Now let's consider a different load, and express it in terms of its normalized admittance—an admittance that has the same **numerical** value as the impedance of the first load (i.e., $y' = 1 + j$).



Q: ***Where** would this admittance value map onto the complex Γ plane?*

A: Start at the location $z' = 1 + j$ on the Smith Chart, and then rotate around the center 180°. You are now at the proper location on the complex Γ plane for the admittance $y' = 1 + j$!

We of course could just directly calculate Γ from the equation above, and then plot that point on the Γ plane.

Note the reflection coefficient for $z'=1+j$ is:

$$\Gamma = \frac{z'-1}{z'+1} = \frac{1+j-1}{1+j+1} = \frac{j}{2+j}$$

while the reflection coefficient for $y'=1+j$ is:

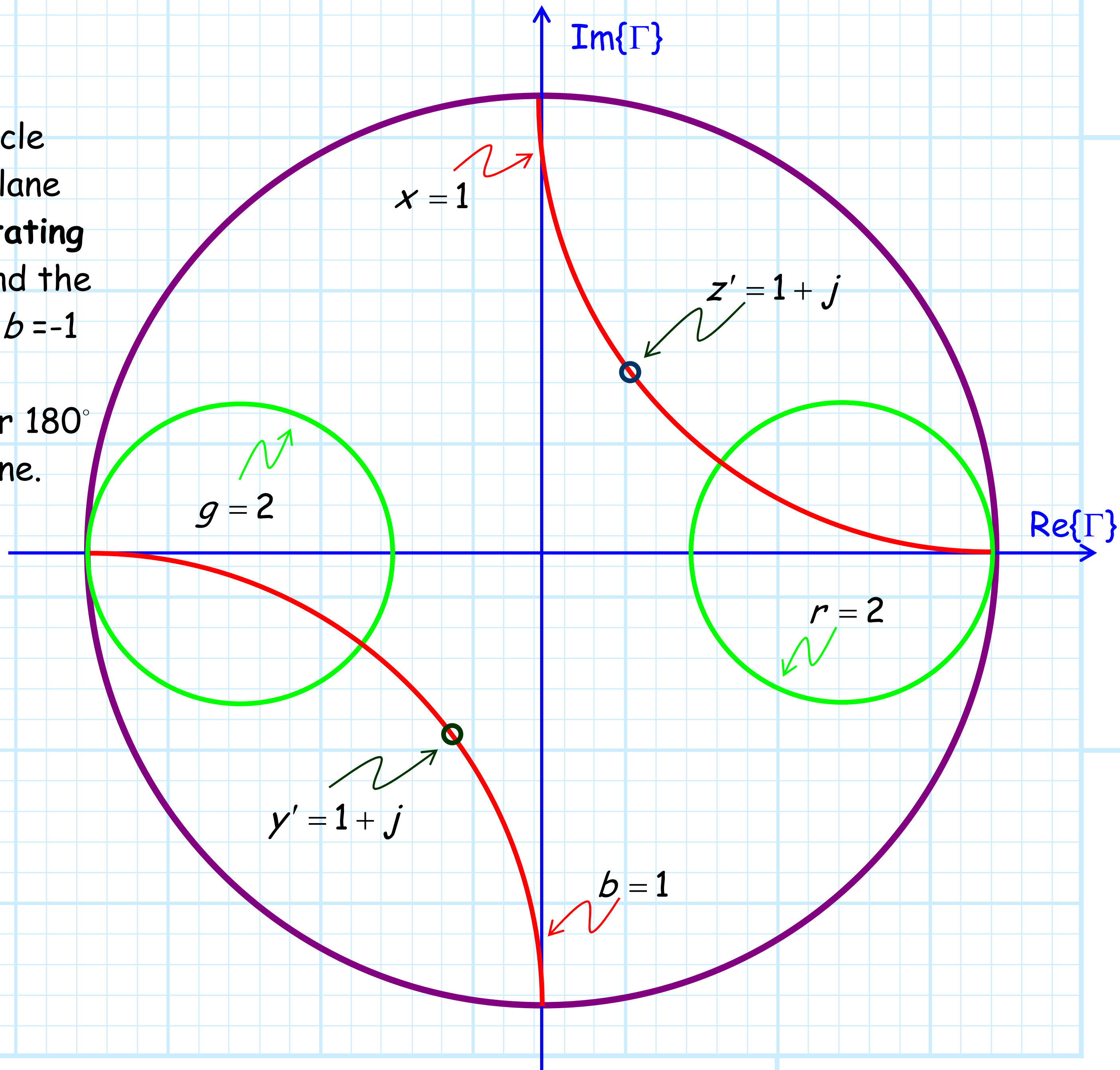
$$\Gamma = \frac{1-y'}{1+y'} = \frac{1-(1+j)}{1+(1+j)} = \frac{-j}{2+j}$$

Note the two results have **equal** magnitude, but are separated in **phase** by 180° $(-1=e^{j\pi})$. This means that the two loads occupy points on the complex Γ plane that are a 180° **rotation** from each other!

Moreover, this is a true statement not **just** for the point we randomly picked, but is true for **any** and **all** values of z' and y' mapped onto the complex Γ plane, provided that $z'=y'$.

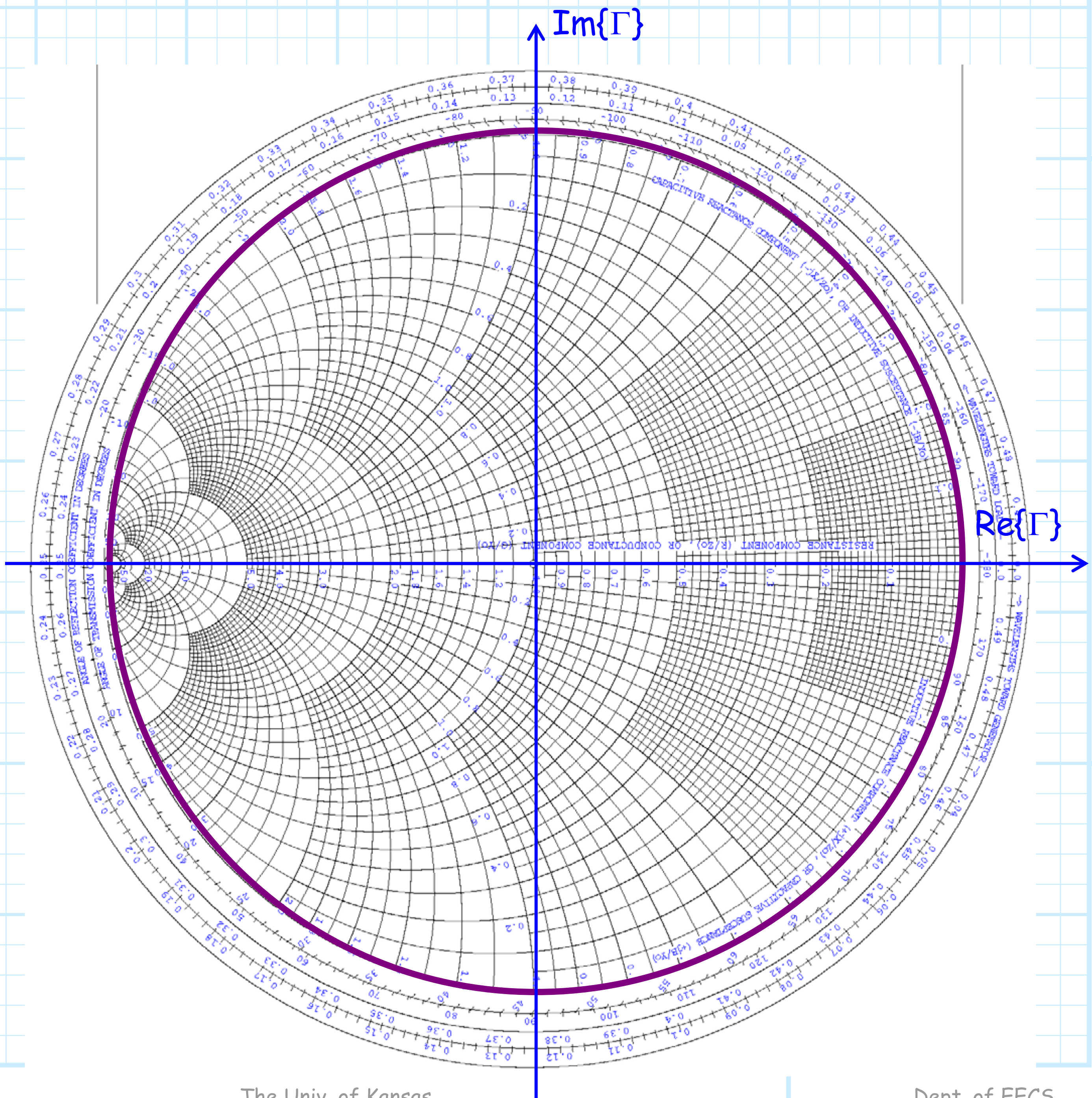
Another example

For example, the g=2 circle mapped on the complex plane can be determined by **rotating** the r=2 circle 180° around the complex Γ plane, and the b=-1 contour can be found by rotating the x=-1 contour 180° around the complex Γ plane.



The Admittance Smith Chart

Thus, rotating **all** the resistance circles and reactance contours of the Smith Chart 180° around the complex Γ plane provides us a mapping of complex **admittance** onto the complex Γ plane:



Note that circles and contours have been rotated with **respect** to the complex Γ plane—the complex Γ plane remains **unchanged**!

We're not surprised!

This result should **not** surprise us. Recall the case where a transmission line of length $\ell = \lambda/4$ is terminated with a load of impedance z'_L (or equivalently, an admittance y'_L). The input impedance (admittance) for this case is:

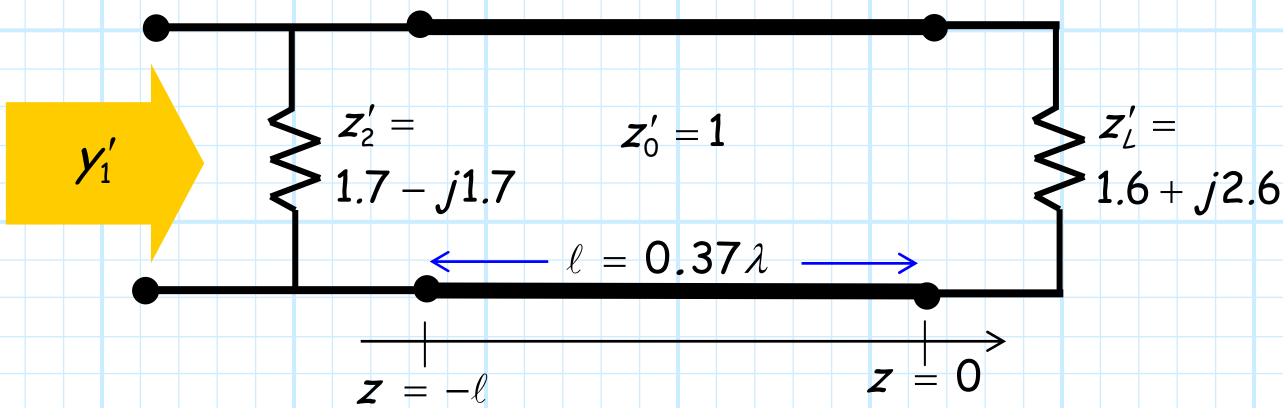
$$Z_{in} = \frac{Z_0^2}{Z_L} \quad \Rightarrow \quad \frac{Z_{in}}{Z_0} = \frac{Z_0}{Z_L} \quad \Rightarrow \quad z'_{in} = \frac{1}{z'_L} = y'_L$$

In other words, when $\ell = \lambda/4$, the input impedance is **numerically** equal to the load admittance—and **vice versa**!

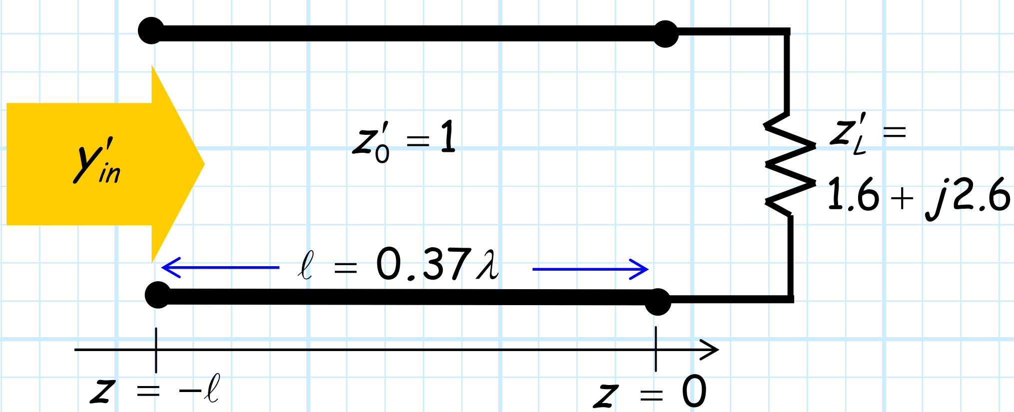
But note that if $\ell = \lambda/4$, then $2\beta\ell = \pi$ --a rotation around the Smith Chart of 180°!

Example: Admittance Calculations with the Smith Chart

Say we wish to determine the **normalized admittance** y_1' of the network below:



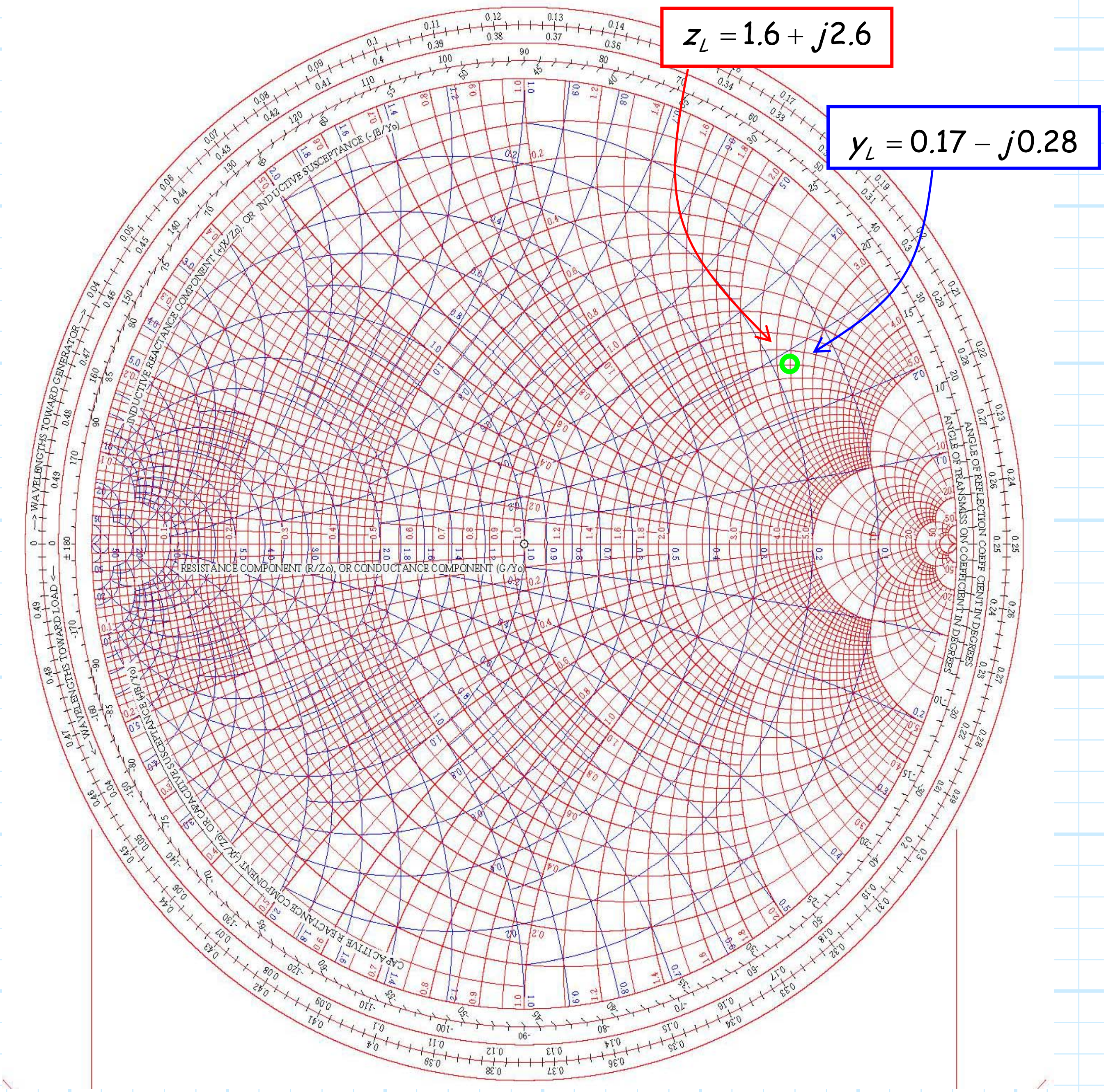
First, we need to determine the normalized **input** admittance of the transmission line:



There are **two ways** to determine this value!

Method 1

First, we express the load $z_L = 1.6 + j2.6$ in terms of its **admittance** $y'_L = 1/z_L$. We can calculate this complex value—or we can use a **Smith Chart**!

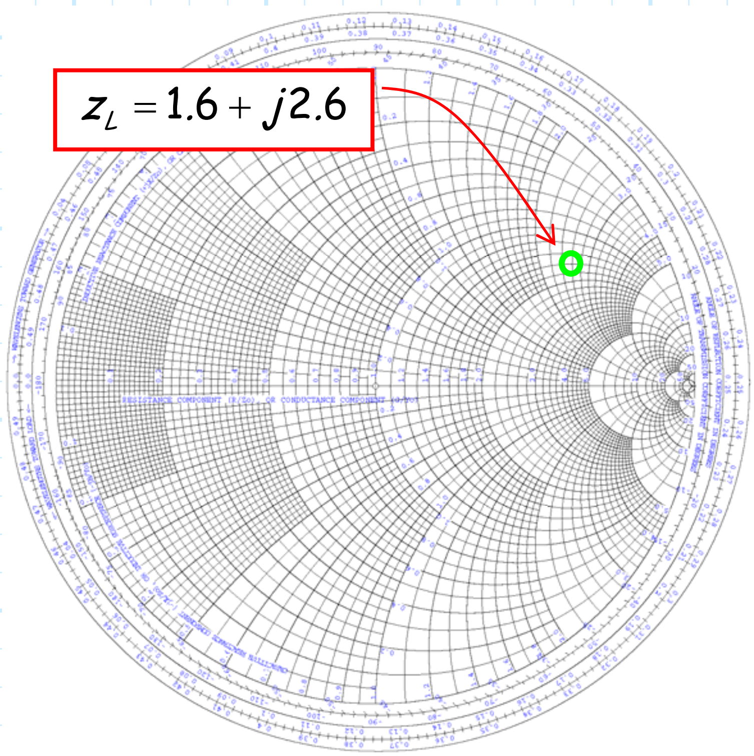


The Smith Chart above shows **both** the **impedance** mapping (red) and **admittance** mapping (blue). Thus, we can locate the impedance $z_L = 1.6 + j2.6$ on the impedance (red) mapping, and then determine the value of that **same** Γ_L point using the admittance (blue) mapping.

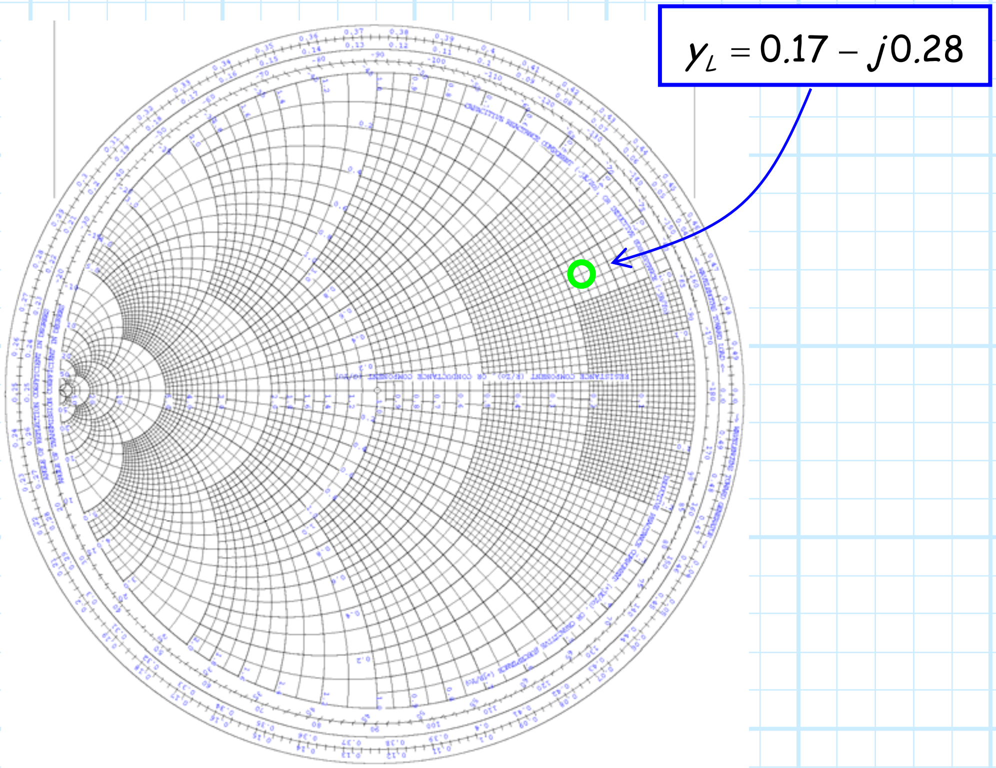
From the chart above, we find this admittance value is **approximately** $y_L = 0.17 - j0.28$.

Now, you may have noticed that the Smith Chart above, with both impedance and admittance mappings, is very **busy** and **complicated.** Unless the two mappings are printed in different colors, this Smith Chart can be very **confusing** to use!

But remember, the two mappings are precisely identical—they're just **rotated** 180° with respect to each other. Thus, we can **alternatively** determine y_L by again first locating $z_L = 1.6 + j2.6$ on the impedance mapping :



Then, we can rotate the **entire** Smith Chart 180°--while keeping the point Γ_L location on the complex Γ plane **fixed**.

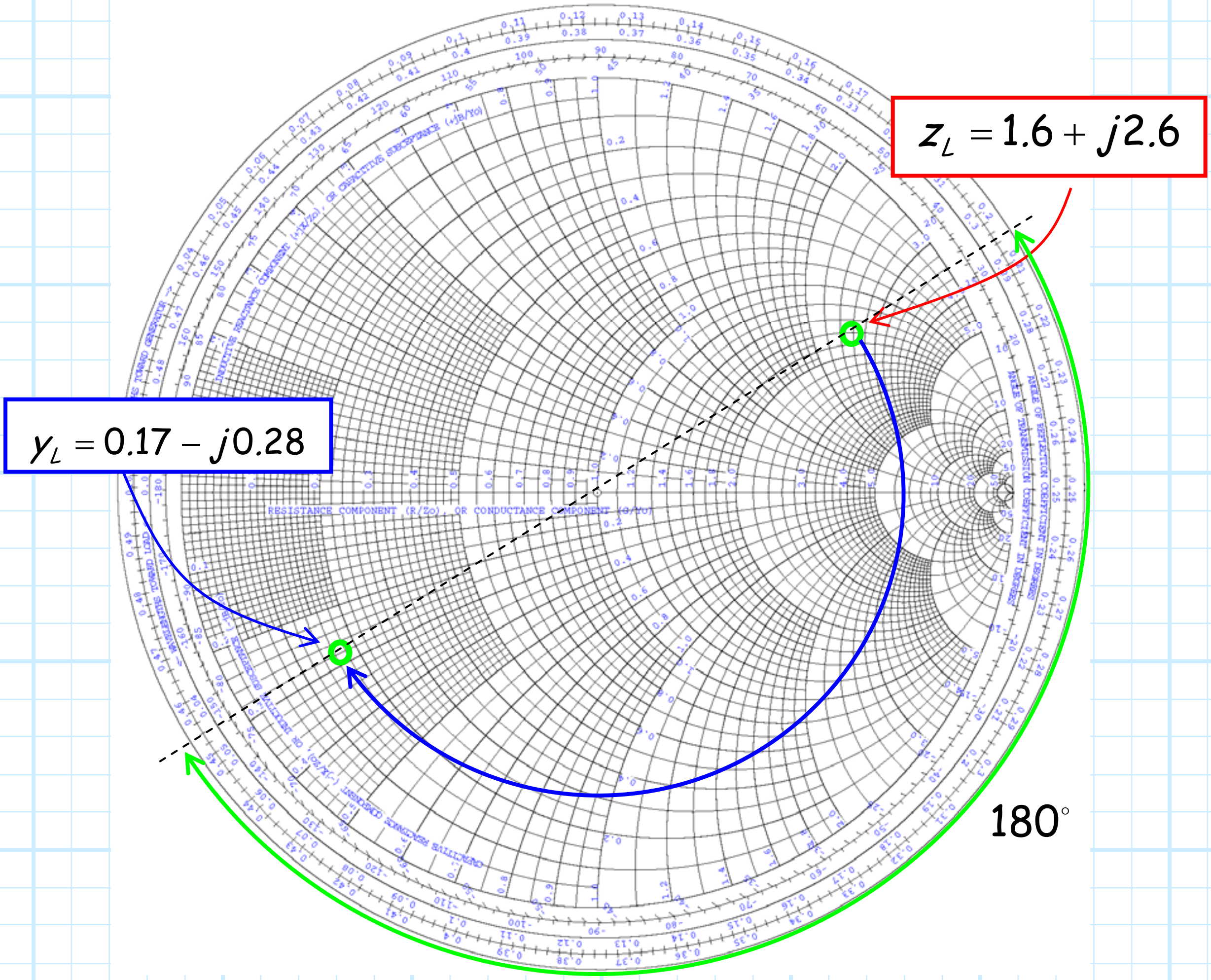


Thus, use the **admittance** mapping at that point to determine the admittance value of Γ_L.

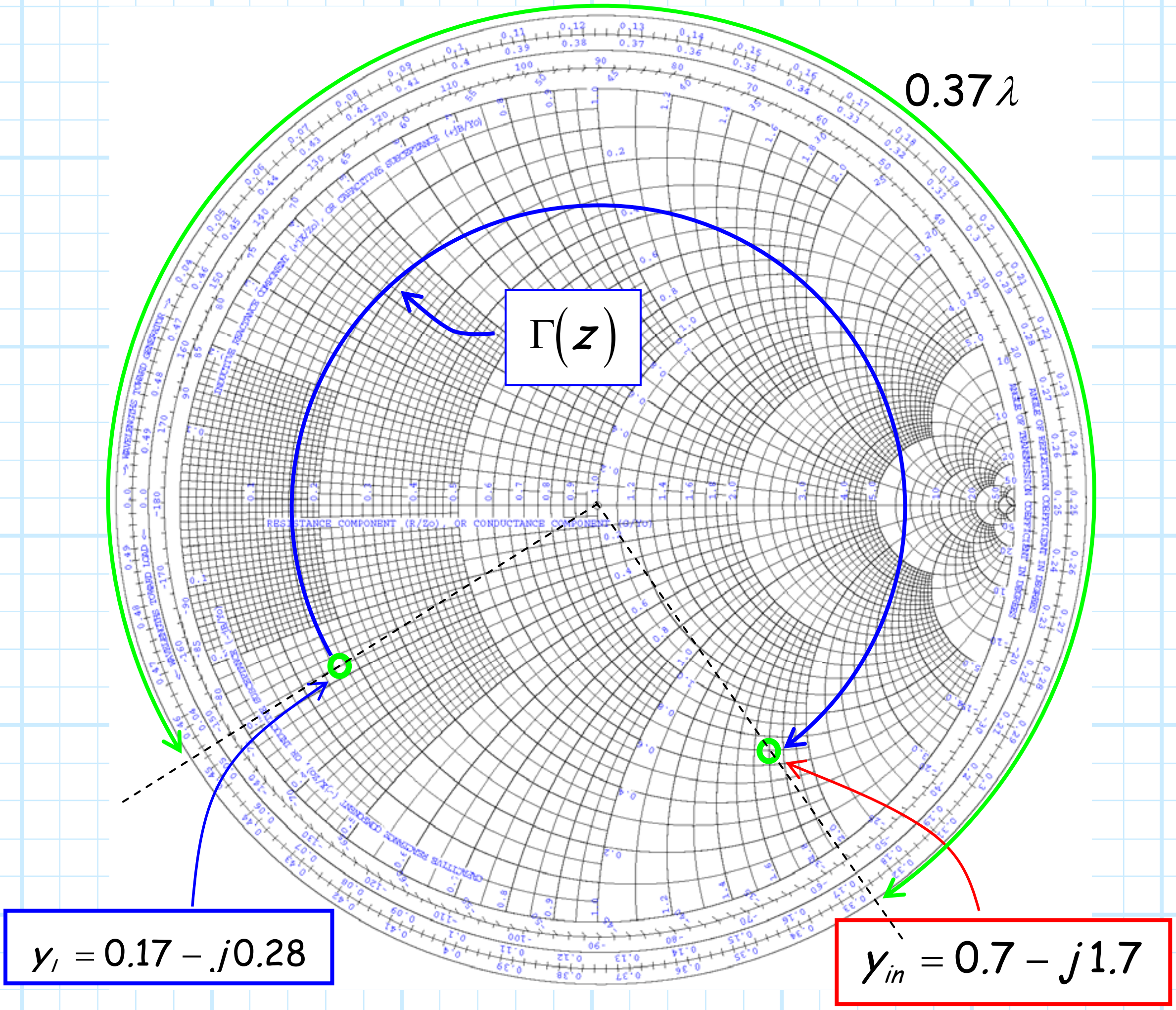
Note that rotating the **entire** Smith Chart, while keeping the point Γ_L fixed on the complex Γ plane, is a **difficult** maneuver to successfully—as well as accurately—execute.

But, realize that rotating the entire Smith Chart 180° with respect to point Γ_L is **equivalent** to rotating 180° the **point** Γ_L with respect to the entire Smith Chart!

This maneuver (rotating the **point** Γ_L) is **much** simpler, and the **typical** method for determining admittance.



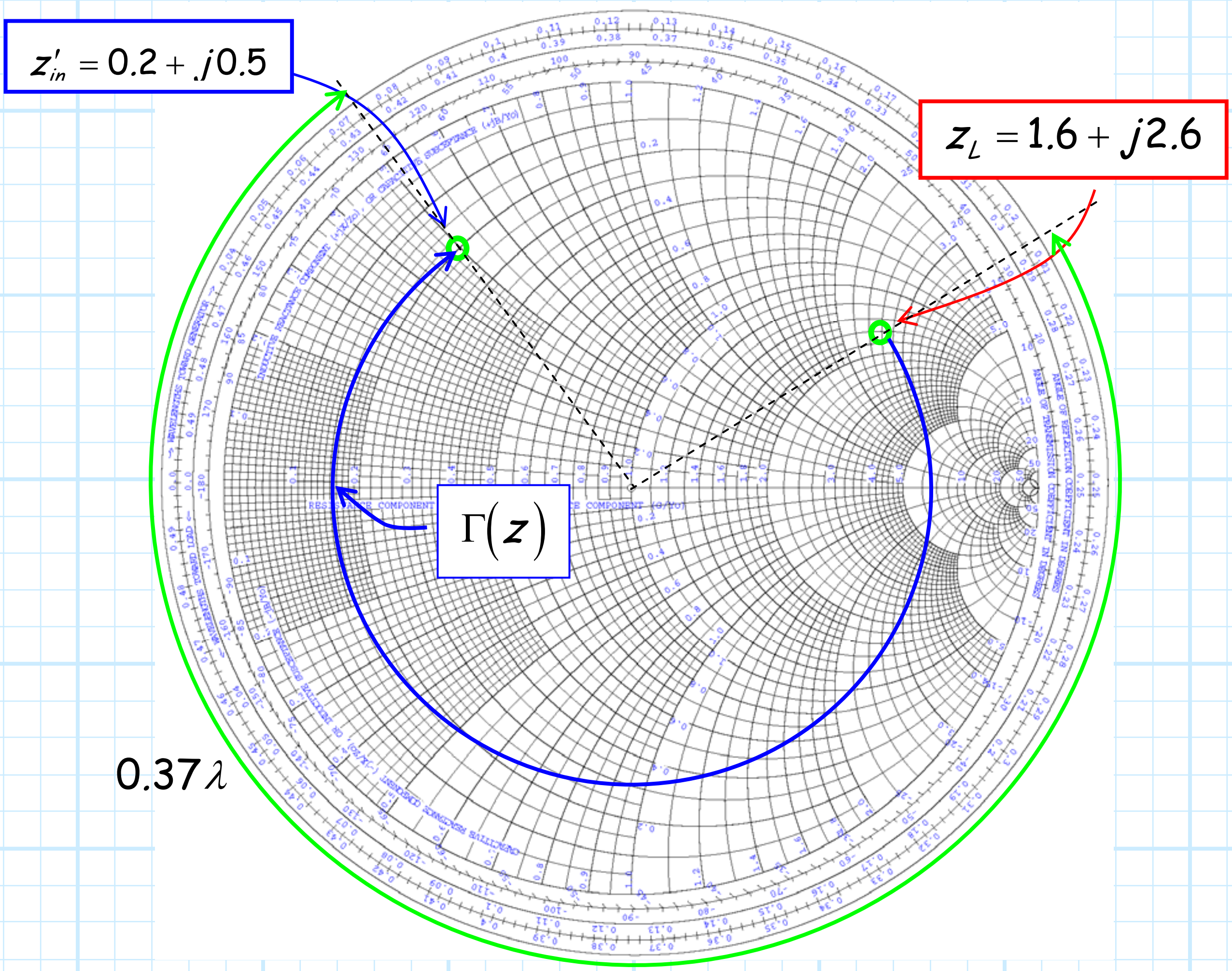
Now, we can determine the value of y'_{in} by simply **rotating** clockwise $2\beta\ell$ from y'_L, where $\ell = 0.37\lambda$:



Transforming the load admittance to the beginning of the transmission line, we have determined that $y'_{in} = 0.7 - j1.7$.

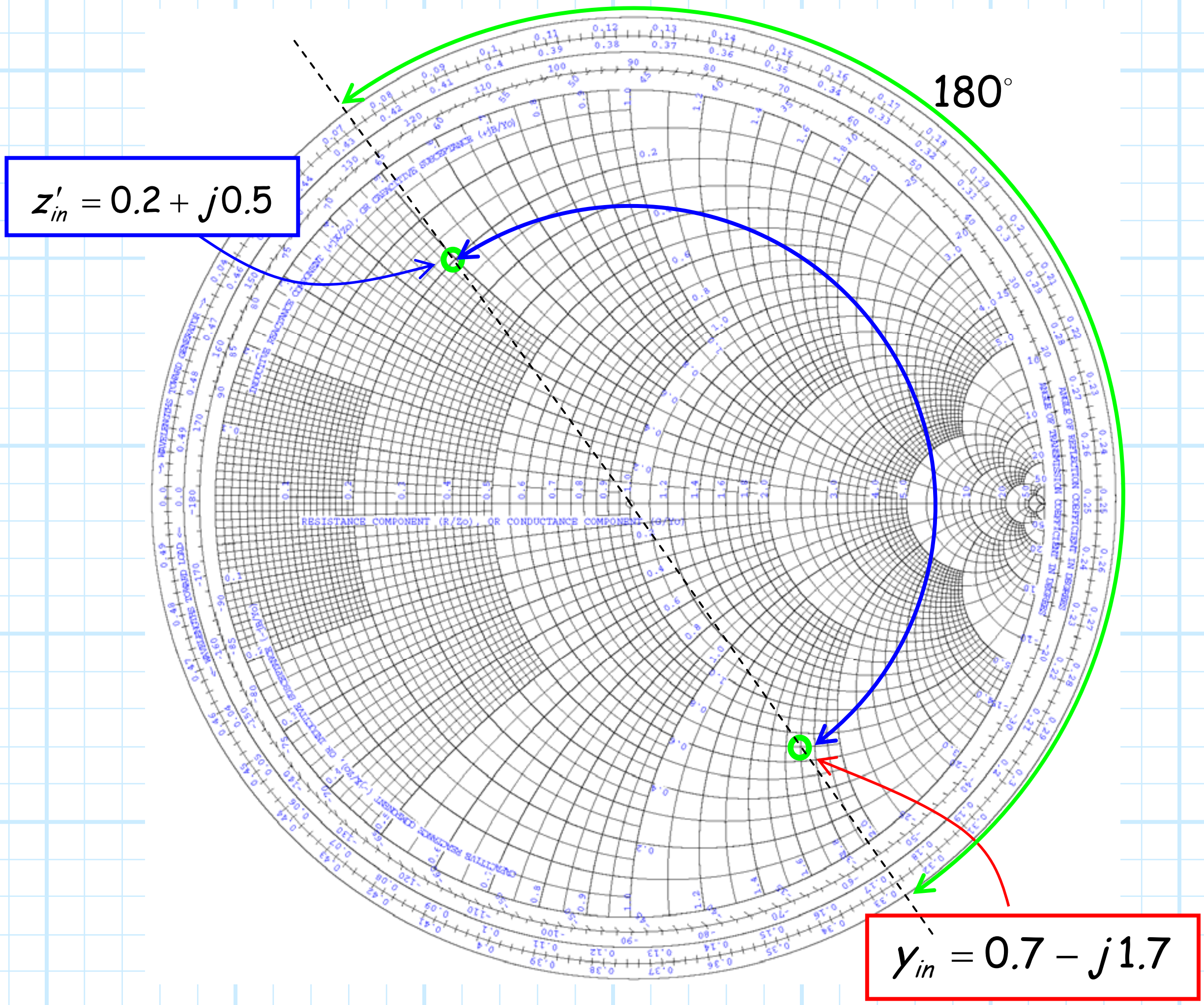
Method 2

Alternatively, we could have **first** transformed impedance z'_L to the **end** of the line (finding z'_{in}), and then determined the value of y'_{in} from the **admittance** mapping (i.e., rotate 180° around the Smith Chart).



The **input impedance** is determined after rotating clockwise $2\beta\ell$, and is $z'_{in} = 0.2 + j0.5$.

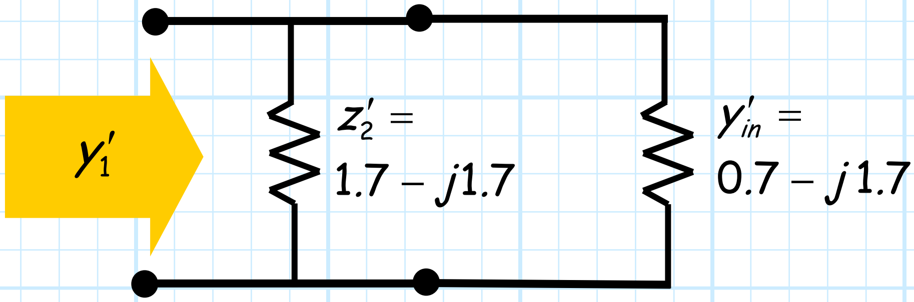
Now, we can rotate this point 180° to determine the **input admittance** value y'_{in}:



The result is the **same** as with the earlier method--
$y'_{in} = 0.7 - j1.7$.

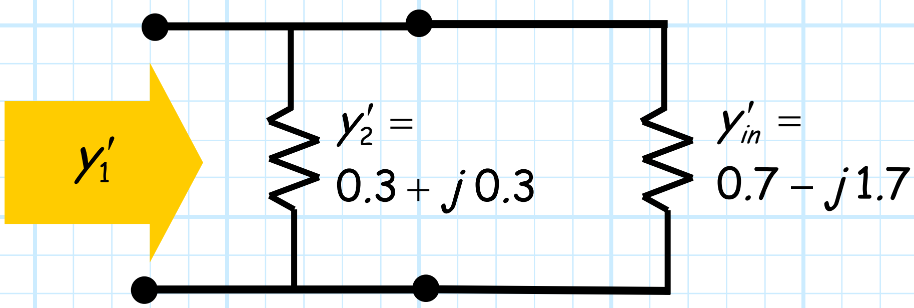
Hopefully it is **evident** that the two methods are equivalent. In method **1** we **first** rotate 180°, and **then** rotate $2\beta\ell$. In the second method we **first** rotate $2\beta\ell$, and **then** rotate 180°--the result is thus the **same**!

Now, the remaining **equivalent** circuit is:



Determining y_1' is just **basic circuit theory**. We first express z_2' in terms of its admittance $y_2' = 1/z_2'$.

Note that we could do this using a **calculator**, but could likewise use a **Smith Chart** (locate z_2' and then rotate 180°) to accomplish this calculation! Either way, we find that $y_2' = 0.3 + j\,0.3$.



Thus, y_1' is simply:

$$
\begin{aligned}
y_1' &= y_2' + y_{in}' \\
&= (0.3 + j\,0.3) + (0.7 - j\,1.7) \\
&= 1.0 - j\,1.4
\end{aligned}
$$