2.6 – Generator and Load Mismatches

Reading Assignment: *pp. 76-78*

Finally, let's **complete** our transmission line circuit, by placing at the beginning of the line a **source**—a device that does **not absorb** electromagnetic energy, but instead **delivers** energy to the circuit.

HO: CONNECTING SOURCE AND LOAD

There are a few special cases of source and load impedance that every electrical engineer **"knows"** to be true—or **do** they?

HO: SPECIAL CASES OF SOURCE AND LOAD

Q: *So, can we now **explicitly** determine the **plus**-wave $V^{+}(z)$ generated on a transmission line?*

A: Absolutely! We simply need to evaluate a **second boundary condition**.

HO: A TRANSMISSION LINE CONNECTING SOURCE AND LOAD

EXAMPLE: BOUNDARY CONDITIONS AND SOURCES

Q: *So, how can we determine the **power** delivered by a source?*

A: HO: DELIVERED POWER

Q: *So how do we insure that the delivered power is as* ***large as*** *possible?*

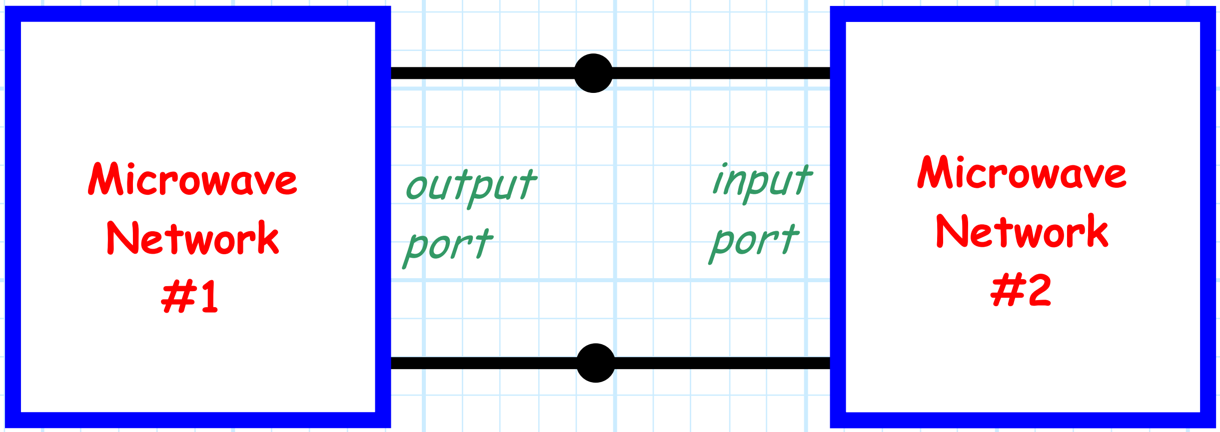
A: HO: SPECIAL CASES OF SOURCE IMPEDANCE

Make sure **you** understand how conservation of energy is applied, with respect to **delivered**, **incident**, **reflected**, and **absorbed** power!

EXAMPLE: CONSERVATION OF ENERGY AND YOU

Connecting a Source and Load

Say we wish to connect the **output** of one microwave network/component to the **input** of another microwave network/component.

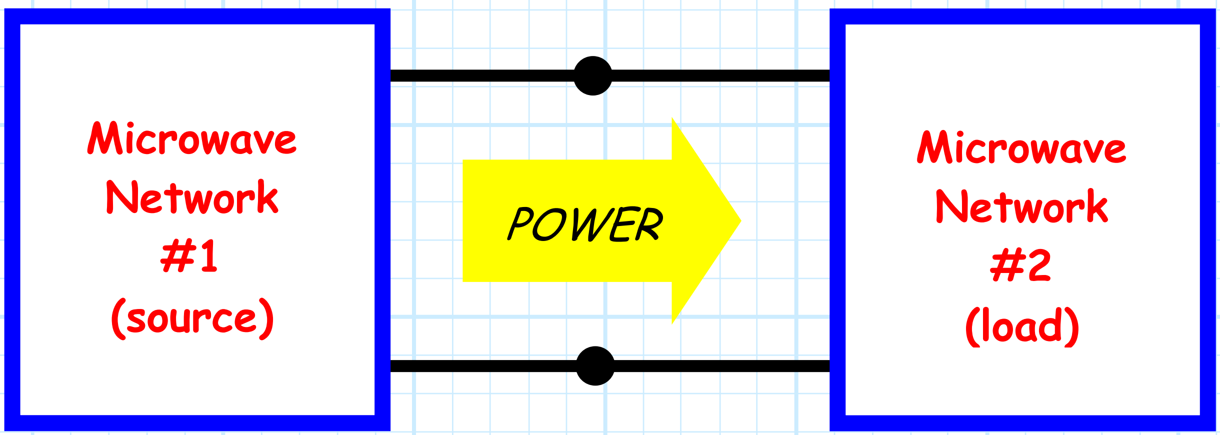


The terms "input" and "output" tells us that we wish for signal energy to flow **from** the output network **to** the input network.

Source Delivers; Load Absorbs

We can say that the **output delivers** signal power to the input, or equivalently, that the **input absorbs** power from the output.

In this case, the first network is the **source**, and the second network is the **load**—the **source delivers** power to the load, or equivalently, the **load absorbs** power from the source.

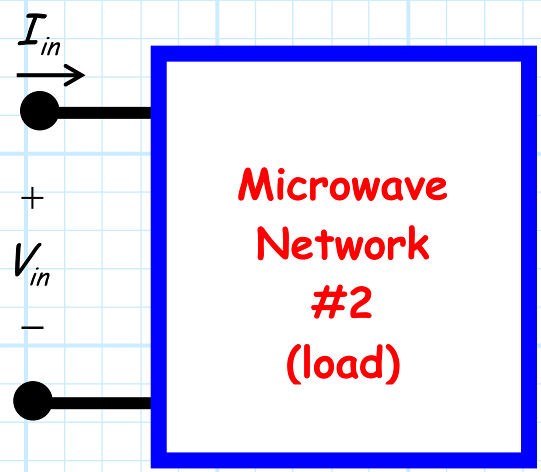


Each of these two networks may be quite complex, but we can always simply this problem by using **equivalent circuits**.

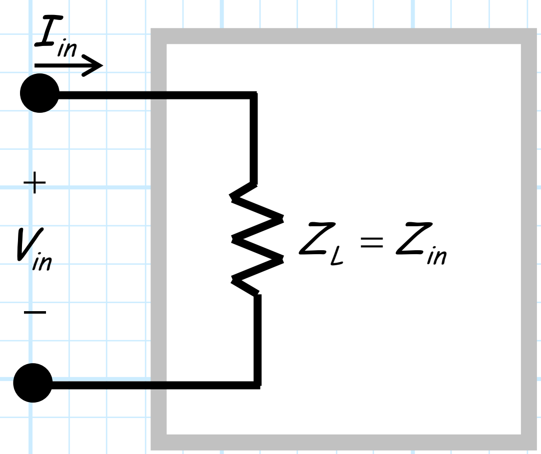
Input Impedance: The Equivalent Load

For example, if we assume time-harmonic signals (i.e., eigen functions!), the load can be modeled as a simple lumped **impedance** , with a **complex** value equal to the **input impedance** of the network.

$$Z_{in} = \frac{V_{in}}{I_{in}}$$



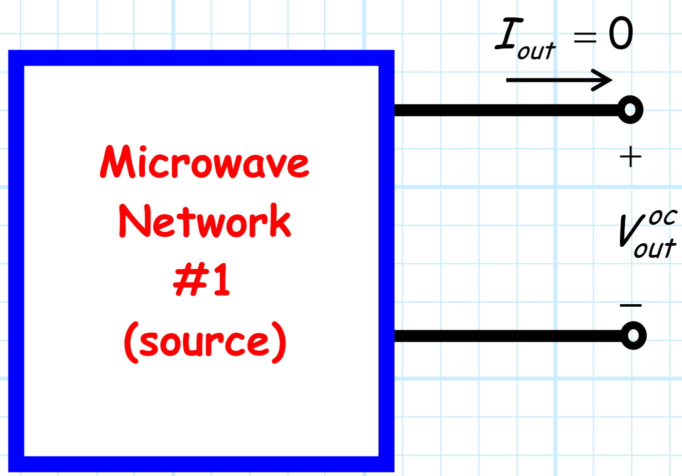
$$\boxed{V_{in} = Z_{in} I_{in}}$$



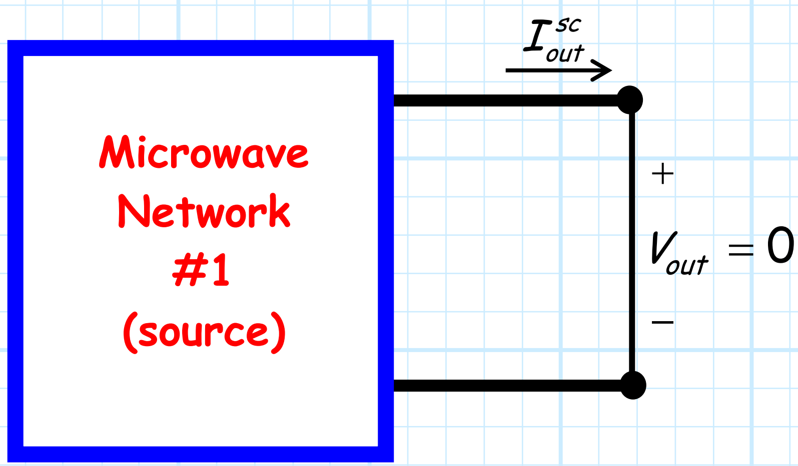
The Equivalent Source

The source network can likewise be modeled using either a Thevenin's or Norton's equivalent.

This equivalent circuit can be determined by first evaluating (or measuring) the **open-circuit** output **voltage** V_{out}^{oc}:



And likewise evaluating (or measuring) the **short-circuit** output **current** I_{out}^{sc}:

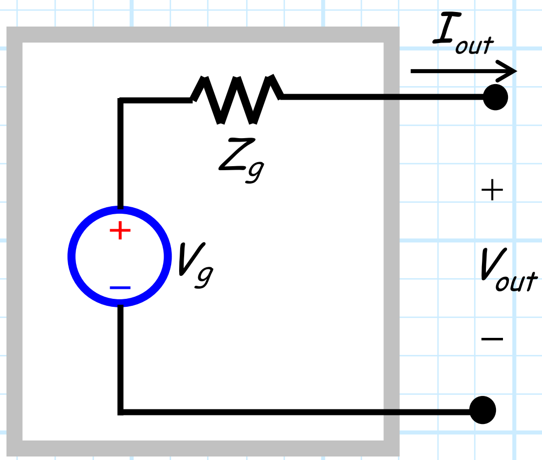


Thevenin's and Norton's Equivalent Source

From these two values (V_{out}^{oc} and I_{out}^{sc}) we can determine the **Thevenin's equivalent source**:

$$V_g = V_{out}^{oc}$$

$$Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$

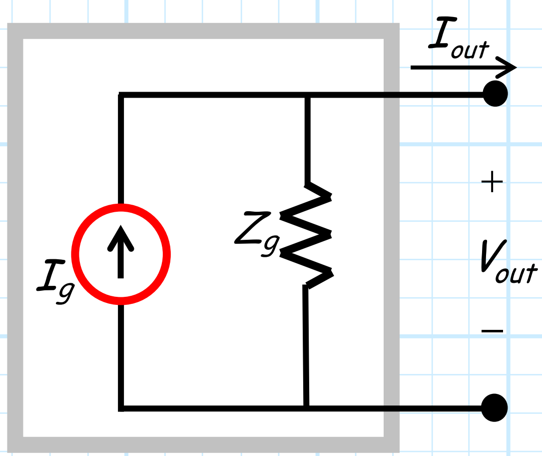


$$\boxed{\begin{aligned} V_{out} &= V_g - Z_g\, I_{out} \\ I_{out} &= \frac{V_g - V_{out}}{Z_g} \end{aligned}}$$

Or, we could use a **Norton's equivalent** circuit:

$$I_g = I_{out}^{sc}$$

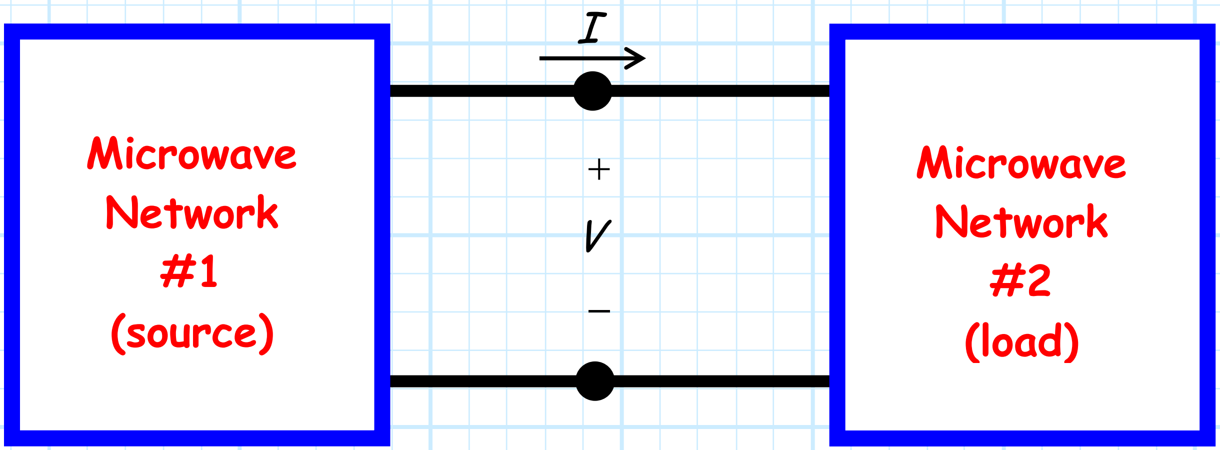
$$Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$



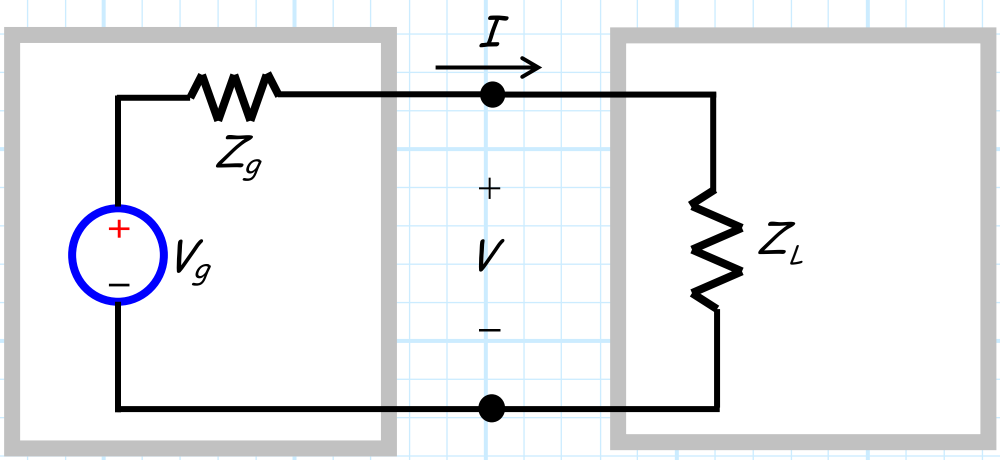
$$\boxed{\begin{aligned} I_{out} &= I_g - V_{out}/Z_g \\ V_{out} &= \left(I_g - I_{out}\right) Z_g \end{aligned}}$$

A Source and Load Equivalent

Thus, the **entire circuit**:



can be modeled with **equivalent circuits** as:



Power Absorbed by the Load…

Please note that we have assumed a **time harmonic** source, such that all the values in the circuit above (V_g, Z_g, I, V, Z_L) are **complex** (i.e., they have a **magnitude** and **phase**).

The **time-averaged power** absorbed (a **real** value!) by the **complex** load impedance is (remember??):

$$P_{abs} = \frac{1}{2}\text{Re}\{V\, I^*\}$$

Where * denotes the complex **conjugate** operator.

Analyzing the equivalent circuit, we find that the power **absorbed by the load** is:

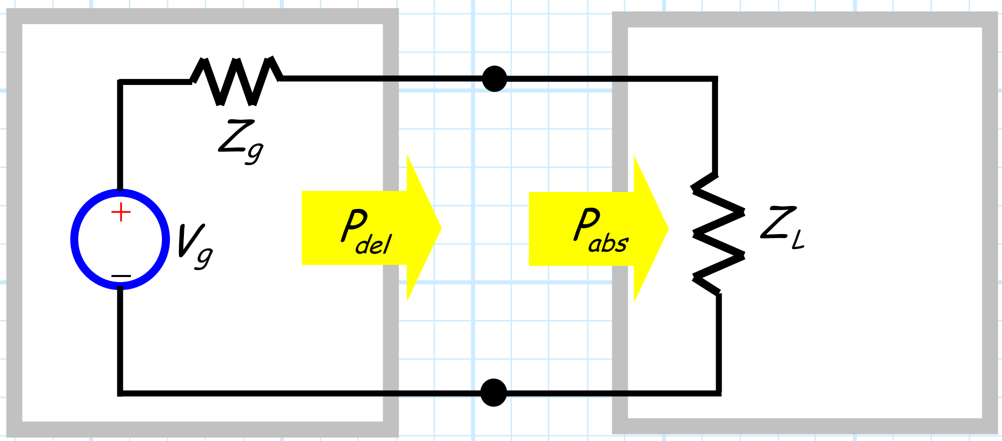
$$P_{abs} = \frac{1}{2}\text{Re}\{V\, I^*\} = \frac{|V_g|^2}{2} \frac{R_L}{|Z_g + Z_L|^2}$$

where R_L is the real (i.e., resistive) part of the load impedance:

$$\text{Re}\{Z_L\} = \text{Re}\{R_L + jX_L\} = R_L$$

…Equals Power Delivered by the Source

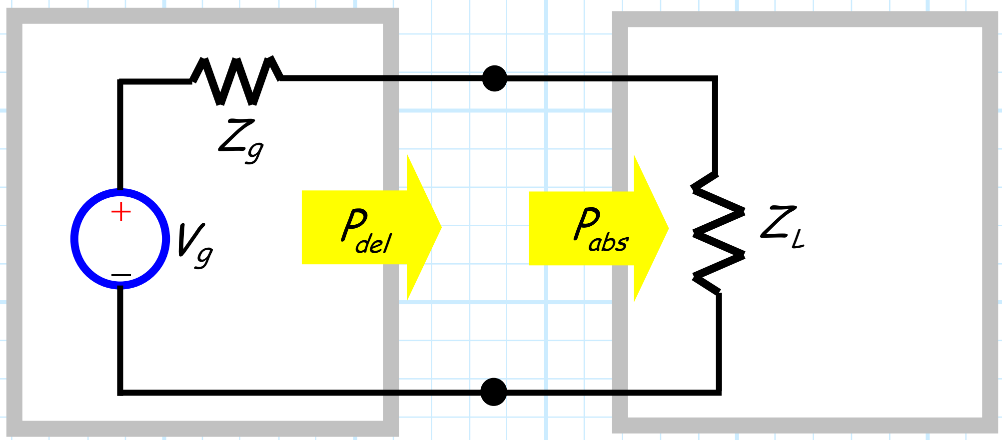
From **conservation of energy**, this absorbed power is likewise that of the power P_{del} **delivered by the source** (i.e., $P_{abs} = P_{del}$):



Special Cases of Source and Load Impedance

Consider again the power **absorbed** by the load (delivered by the source):

$$P_{del} = P_{abs} = \frac{|V_g|^2}{2} \frac{R_L}{|Z_g + Z_L|^2}$$



It is evident that this power transfer is dependent on **each** and **every** element of the equivalent circuit—the **source parameters** V_g and Z_g, as well as the **load impedance** Z_L.

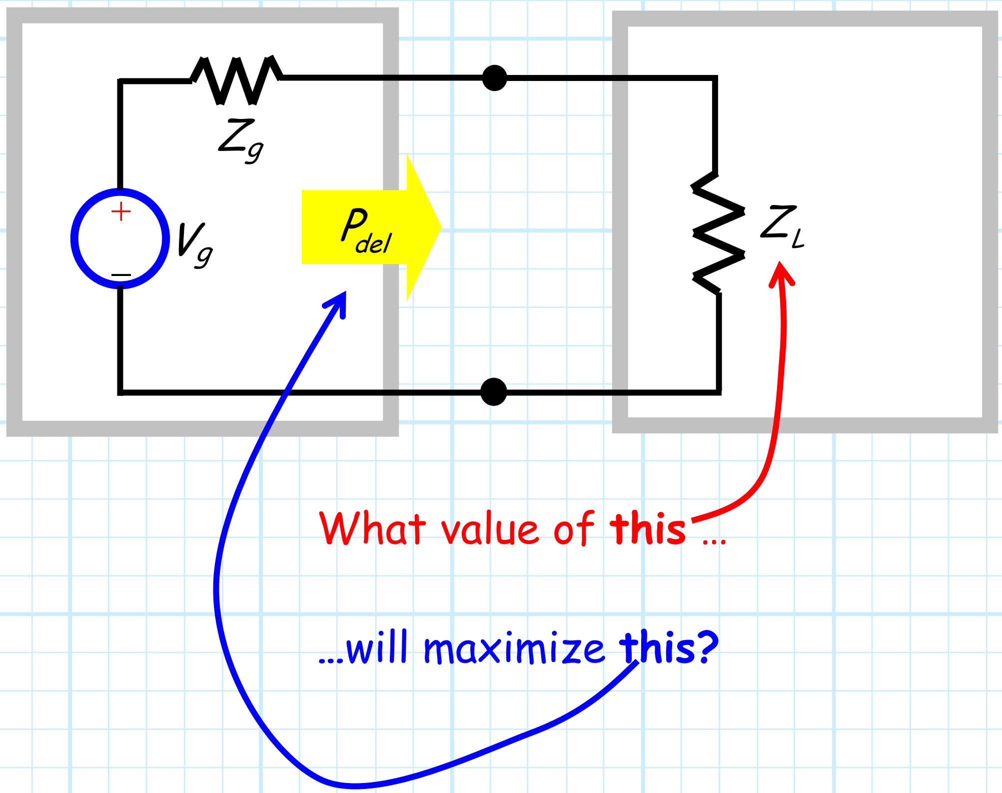
Q: *I assume that we want to **maximize** this power transfer. **How** can we maximize P_{abs}??*

A: The answer to that question is among the best known in electrical engineering. Unfortunately, it is also frequently misunderstood and misapplied—**so pay attention**!

Match the Load to the Source

First, let's ask **this** question:

Q1: *What **load** impedance Z_L will maximize the power delivered by the **source** (i.e., maximize P_{del})?*



A1: The **load** impedance $Z_L = Z_g^*$ will maximize the power delivered by the **source**.

The Available Power of the Source

We can likewise determine what the **value** of this maximum power is. For $Z_L = Z_g^*$, we find:

$$P_{del}\Big|_{Z_L=Z_g^*} = \frac{|V_g|^2}{2}\frac{R_g}{|Z_g + Z_g^*|^2} = \frac{|V_g|^2}{2}\frac{R_g}{|2R_g|^2} = \frac{|V_g|^2}{8R_g}$$

This maximum delivered power is **very important** and is dubbed the **available power** P_{avl} of the **source**:

$$\boxed{P_{avl} = \frac{|V_g|^2}{8R_g}}$$

Note the available power is dependent **just on source parameters** (i.e., V_g and R_g), and so P_{avl} is a parameter of the **source only**.

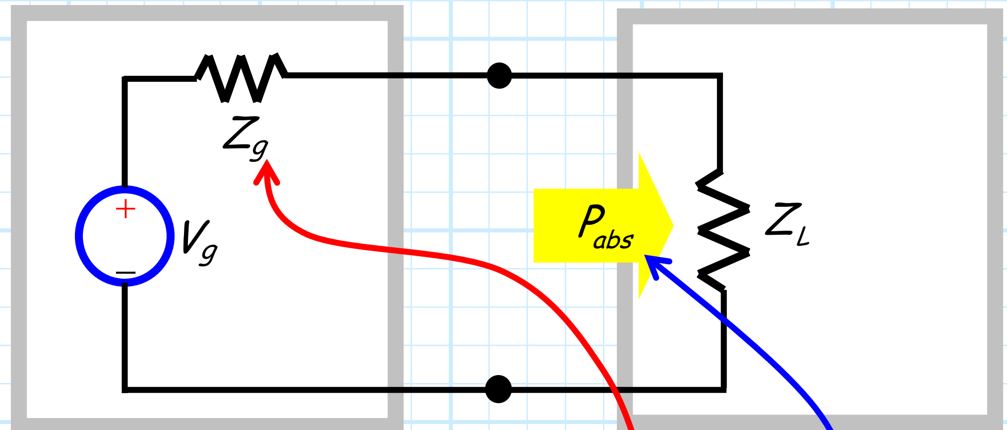
This available power is the **most** that can be **delivered** by the source(i.e., $P_{del} \le P_{avl}$), and this available source power can **only** be delivered if a load $Z_L = Z_g^*$ is connected:

$$P_{del} = P_{avl} = \frac{|V_g|^2}{8R_g} = \qquad \text{iff} \qquad Z_L = Z_g^*$$

A Completely Different Question!

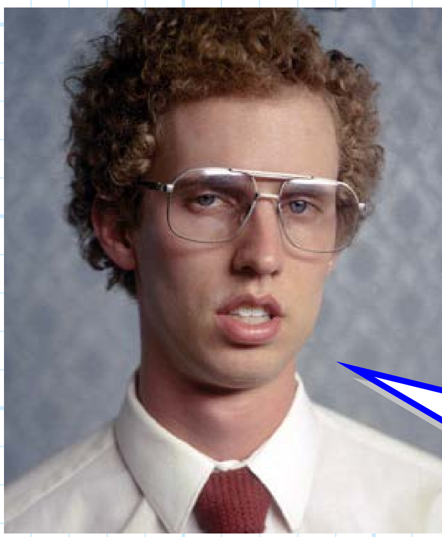
Now, let's ask **completely different** question:

Q2: *What **source** impedance Z_g will maximize the power absorbed by the **load** (i.e., maximize P_{abs})?*



What value of **this** ...

...will maximize **this?**



*Like, we already know the answer. Absorbed power is maximized by a **conjugate match**:*

$$Z_g = Z_L^* . \quad \textit{Gosh!}$$

NOT!

Not so fast! It can be shown that the value of Z_g that **maximizes** the absorbed power:

$$P_{abs} = \frac{|V_g|^2}{2} \frac{R_L}{|Z_g + Z_L|^2}$$

is:

$$\boxed{Z_g = -jX_L}$$

where X_L is the imaginary (i.e., reactive portion of the load ($X_L = \operatorname{Im}\{Z_L\}$).

Thus, we conclude:

Q2: *What **source** impedance Z_g will maximize the power absorbed by the **load** (i.e., maximize P_{abs})?*

A2: The **source** impedance $Z_g = -jX_L$ will maximize the power absorbed by the **load** (i.e., maximize P_{abs}).

Don't Make This Mistake!

Although it is **very** common for electrical engineers to incorrectly assume the answer to question **Q2** is the answer to **Q1** (i.e., $Z_g = Z_L^*$), and this is **far** from the correct answer!

Using the **correct** solution $Z_g = -jX_L$, we find:

$$P_{abs}\Big|_{Z_g=-jX_L} = \frac{|V_g|^2}{2}\frac{R_L}{|Z_g + Z_L|^2} = \frac{|V_g|^2}{2\,R_L}$$

Whereas if we enforce a **"conjugate match"** $Z_g = Z_L^*$ the load instead absorbs:

$$P_{abs}\Big|_{Z_g=Z_L^*} = \frac{|V_g|^2}{2}\frac{R_L}{|Z_L^* + Z_L|^2} = \frac{1}{4}\left(\frac{|V_g|^2}{2\,R_L}\right)$$

The power absorbed by the load when $Z_g = Z_L^*$ is just **25%** of the power absorbed if $Z_g = -jX_L$!

Dazed and Confused

Q: *But if Z_L is **not** equal to Z_g^* ($Z_L \neq Z_g^*$) isn't the absorbed power **less** than the available power ??*

A: You bet!

If $Z_g = -jX_L$, the absorbed power is **far less** than the **available power**.

Q: *I'm so **confused!***

I thought you said that setting $Z_g = -jX_L$ maximized the absorbed power??

A: See the next page!

ZL cannot alter available power— but Zg sure the heck can!

A: Here's the deal; altering the value of **load** impedance Z_L changes the delivered power P_{del} but does not alter the available power P_{avl} of the source.

The best we can do is set Z_L such that **all available power** is delivered to the load (i.e., set $Z_L = Z_g^*$).

Contrast this with altering the value of **source** impedance Z_g. Changing Z_g **will** alter the **available power** P_{avl} of the source!

Recall:

$$P_{avl} = \frac{|V_g|^2}{8\,R_g}$$

The ideal source impedance ($Z_g = -jX_L$) is purely reactive, so $R_g = 0$—the available power is therefore **infinite**!

Of course, achieving infinite available power is **not practical**—available power P_{avl} of any **realizable** source is finite.

Don't ever do this!

Still, engineers attempting to **maximize** the power **absorbed by a load** should:

1. Attempt to select/design/alter the **source** such that its **available power** P_{avl} **is maximized**.

2. Attach a **load** that is conjugate matched ($Z_L = Z_g^*$) to this source, such that **all available power is delivered** to the load.

A problem that often arises is a source with a **large available power** has a very **low source impedance**, such that it is **difficult/impractical** to provide a load where $Z_L = Z_g^*$.

Engineers sometimes alter/design/select **another source** that it **easier** to "match", but usually this results in a dramatic **decrease in available power**!

See what I mean?

For **example**, consider two cases:

Source	Available Power	Delivered Power
1	500 mW	200mW
2	100 mW	100 mW

For **which** source is "power transfer maximized"?

For source 2, **100%** of the **available power** is delivered to the load—clearly the **load is matched** to the source impedance.

For source 1, **only 40%** of the available power is delivered to the load—the load is most definitely **not matched** to source impedance.

Yet, the **mismatched load** absorbs **twice** the power of the mismatched case.

It does so because the available power of source 1 is **five times** larger than that of source 2.

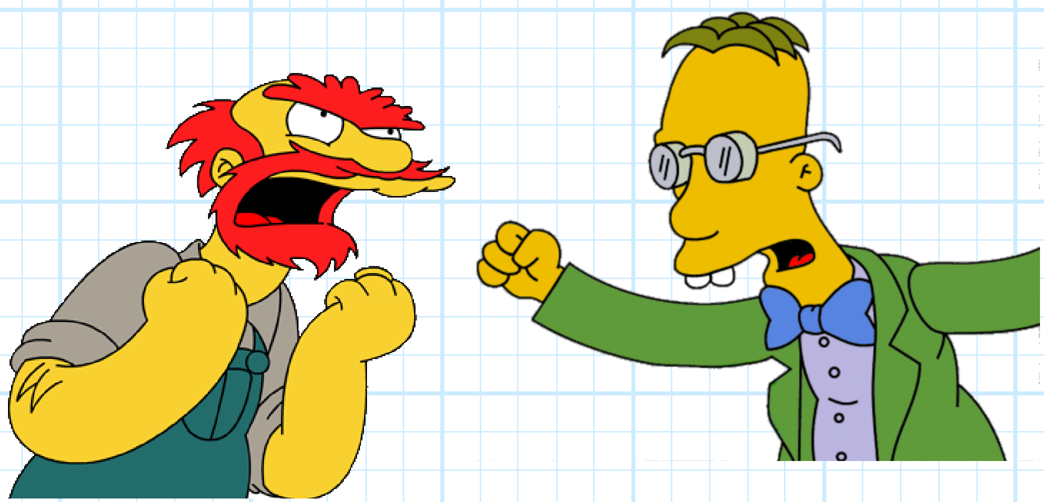
→ It's better to have **most** of **alot** than **all** of very **little**!!

Be careful!

Hence, we need to be **careful** when considering a conjugate match (e.g., what does "maximum power transfer" **really** mean?).

→ Selecting/altering the **load to match a source** is a **good** idea, but selecting/altering the **source to match a load** is typically **not**.

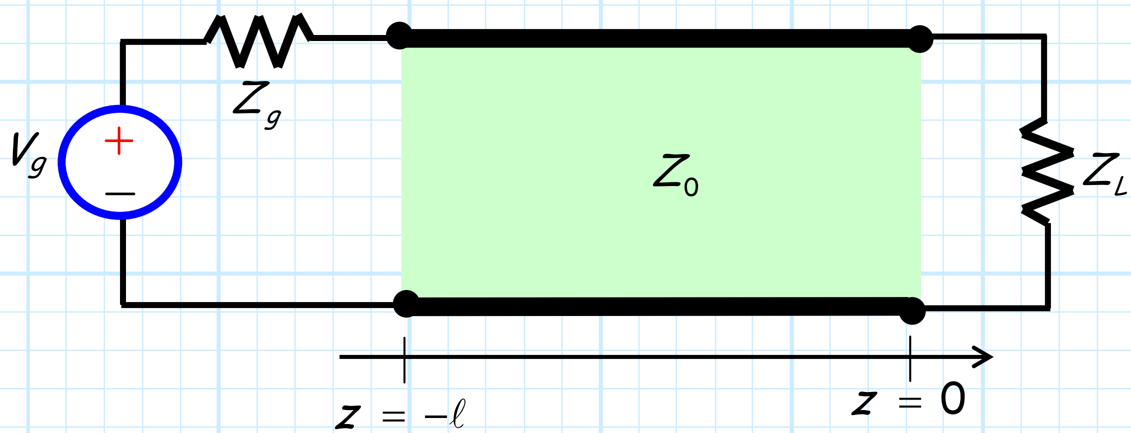
This question has—and continues to—spark many **arguments** among electrical engineers!!



A Transmission Line Connecting Source and Load

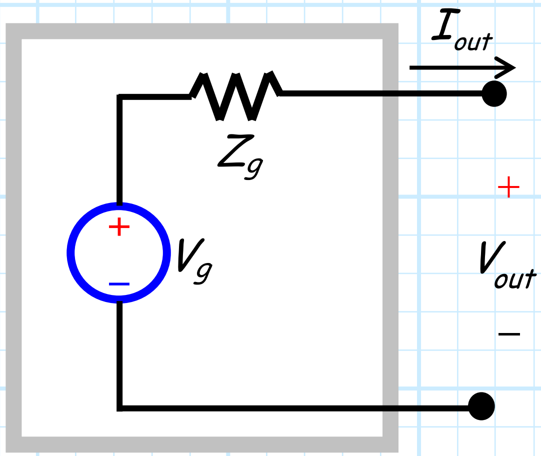
We can think of a transmission line as a conduit that allows **power** to flow **from** an **output** of one device/network **to** an **input** of another.

To simplify our analysis, we modeled the **input** of the device **receiving** the power with it input impedance (e.g., Z_L).



The sources are equivalent circuits

We can similarly model the device **delivering** the power with its Thevenin's or Norton's equivalent circuit.

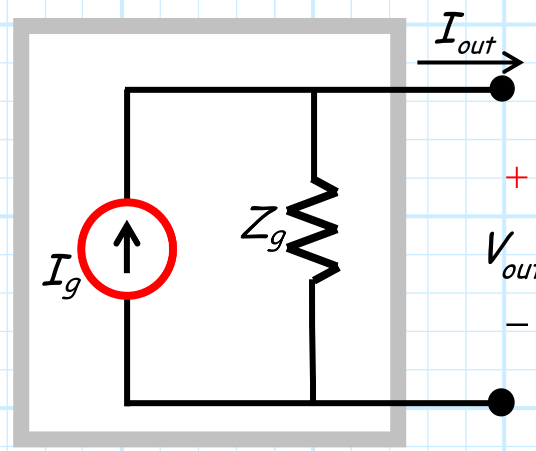


$$V_{out} = V_g - Z_g\, I_{out}$$

$$I_{out} = \frac{V_g - V_{out}}{Z_g}$$

$$I_{out} = I_g - V_{out}/Z_g$$

$$V_{out} = \left(I_g - I_{out}\right) Z_g$$



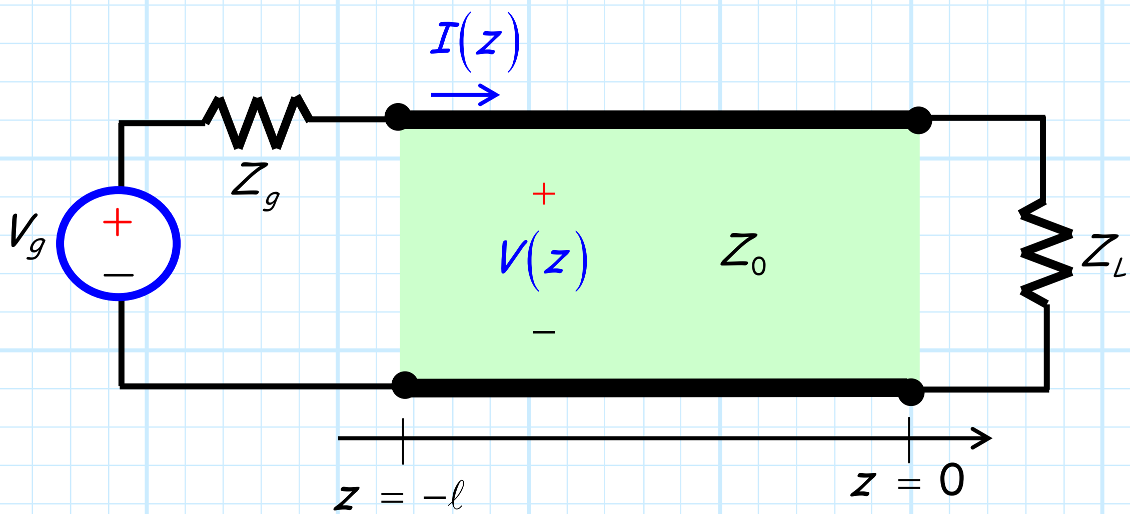
Typically, the power source is modeled with its **Thevenin's** equivalent.

However, we will find that the **Norton's** equivalent circuit is useful if we express the remainder of the circuit in terms of its **admittance** values (e.g., $Y_0, Y_L, Y(z)$).

We've already satisfied one boundary condition

Recall that we applied **boundary conditions** at location $z = 0$ to find that:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right] \qquad I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L e^{+j\beta z} \right]$$

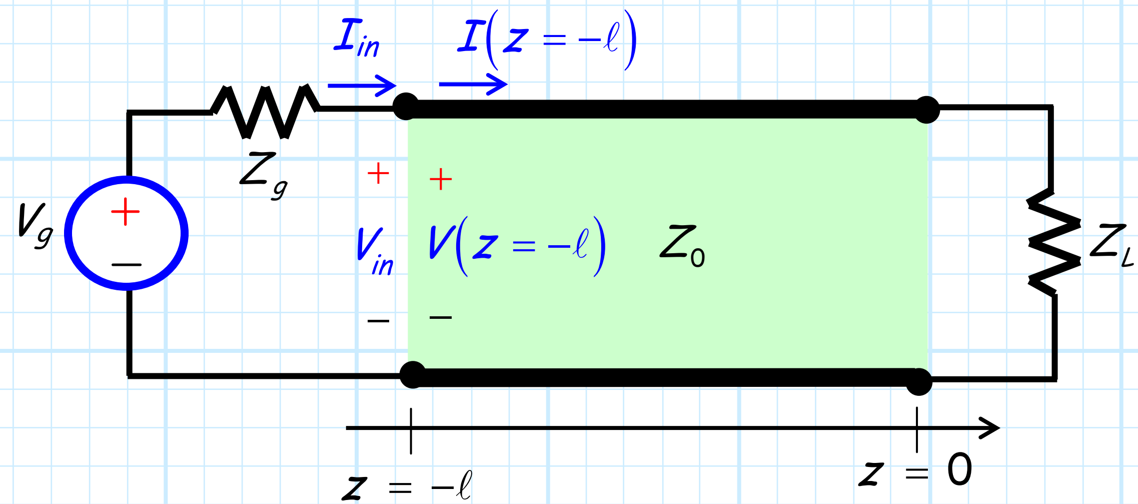


We are left with the **question**: just what is the **value** of complex constant V_0^+?!?

→**This** constant depends on the signal **source**!

We must now satisfy a second boundary condition

To determine its exact **value**, we must now apply **boundary conditions** at $z = -\ell$.



We know that at the **beginning** of the transmission line:

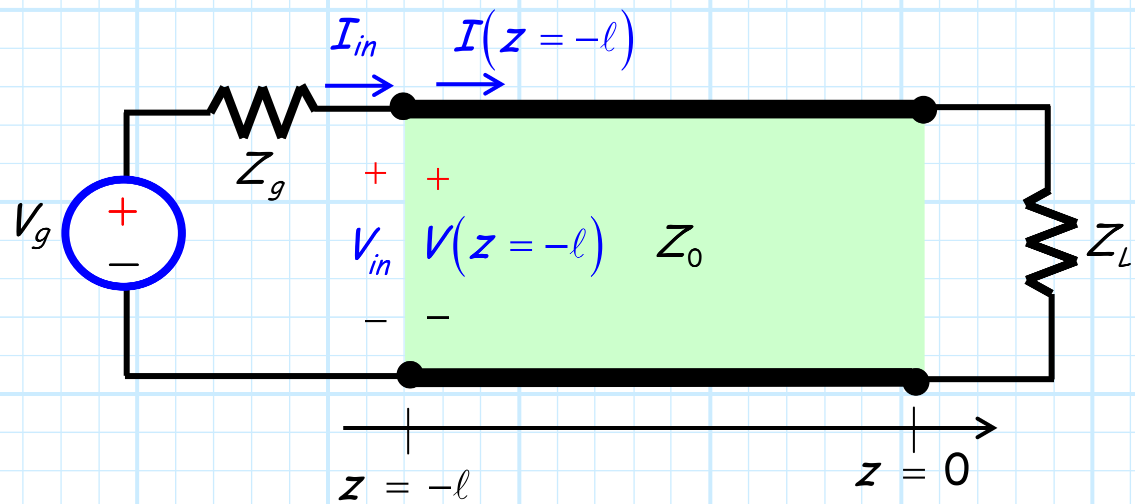
$$V(z=-\ell) = V_0^+\left[e^{+j\beta\ell} + \Gamma_L\, e^{-j\beta\ell}\right] \qquad I(z=-\ell) = \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_L\, e^{-j\beta\ell}\right]$$

Likewise, we know that the **source** must satisfy:

$$V_g = V_{in} + Z_g\, I_{in}$$

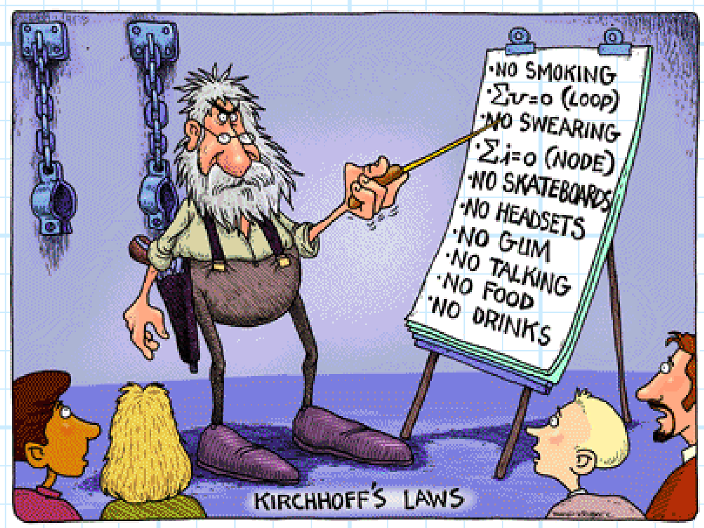
Good ol' Kirchoff!

To relate these **three** expressions, we need to apply **boundary conditions** at $z=-\ell$:



From **KVL** we find:

$$V_i = V(z=-\ell)$$



And from **KCL**:

$$I_i = I(z=-\ell)$$

The solution (yuck)!

Combining these equations, we get:

$$V_g = V_{in} + Z_g I_{in}$$

$$V_g = V_0^+ \left[e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell} \right] + Z_g \frac{V_0^+}{Z_0} \left[e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell} \right]$$

One equation → **one** unknown (V_0^+)!!

Solving, we find the value of V_0^+:

$$\boxed{V_0^+ = V_g e^{-j\beta\ell} \frac{Z_0}{Z_0 (1 + \Gamma_{in}) + Z_g (1 - \Gamma_{in})}}$$

where:

$$\Gamma_{in} = \Gamma(z = -\ell) = \Gamma_L e^{-j2\beta\ell}$$

The plus wave depends on the source *and* load

There is one **very important** point that must be made about the result: $V_0^+ = V_g\, e^{-j\beta\ell} \dfrac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_g\left(1-\Gamma_{in}\right)}$

And that is—the wave $V_0^+(z)$ **incident** on the load Z_L is actually dependent **on** the value of load Z_L !!!!!

$$\text{Remember:} \quad \Gamma_{in} = \Gamma\left(z=-\ell\right) = \Gamma_L\, e^{-j2\beta\ell}$$

We tend to think of the incident wave $V_0^+(z)$ being "**caused**" by the source, and it is certainly true that $V_0^+(z)$ **depends** on the source—after all, $V_0^+(z)=0$ if $V_g=0$.

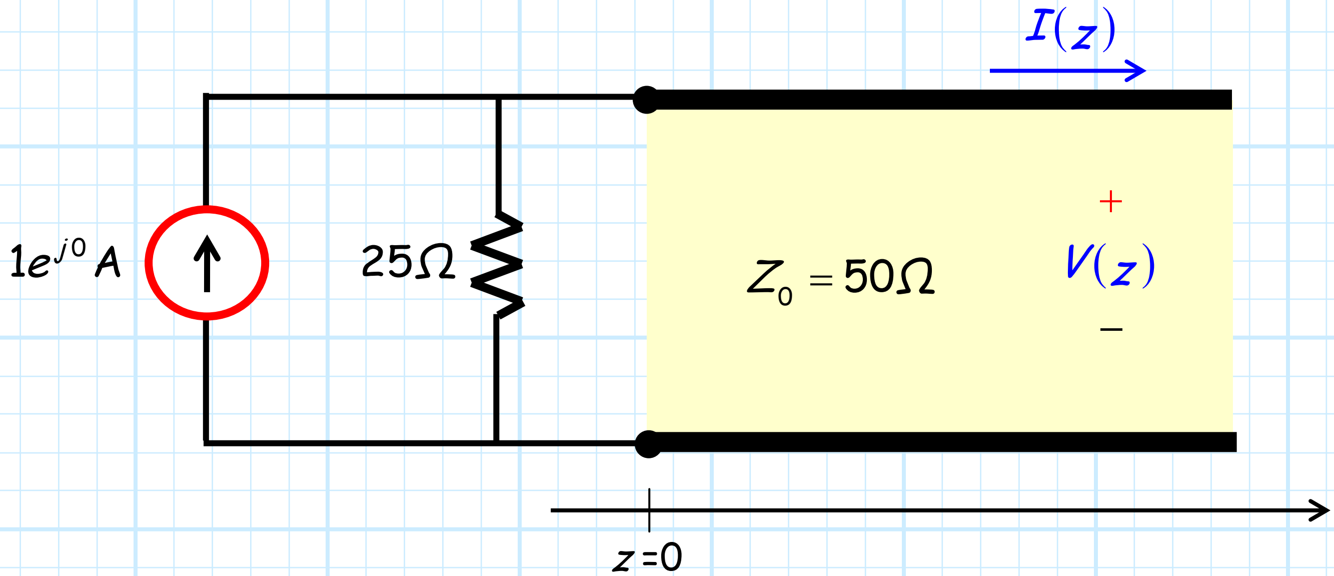
However, we find from the equation above that it **likewise** depends on the value of the **load**!

Thus we **cannot**—in general—consider the incident wave to be the "**cause**" and the reflected wave the "**effect**".

Instead, each wave must obtain the proper **amplitude** (e.g., V_0^+, V_0^-) so that the boundary conditions are satisfied at **both** the beginning and end of the transmission line.

Example: Boundary Conditions and Sources

Consider the circuit below:



It is known that the **current** along the transmission line is:

$$I(z) = 0.4\, e^{-j\beta z} - B\, e^{+j\beta z} \quad A \quad \text{for } z > 0$$

where B is some unknown **complex** value.

Determine the **value of** B.

Hint: $B \neq -0.6$

Solution

Since the **line current** is:

$$I(z) = 0.4\, e^{-j\beta z} - Be^{+j\beta z} = I_0^+\, e^{-j\beta z} + I_0^-\, e^{+j\beta z}$$

we conclude from inspection that:

$$I_0^+ = 0.4 \quad \text{and} \quad I_0^- = -B$$

and since:

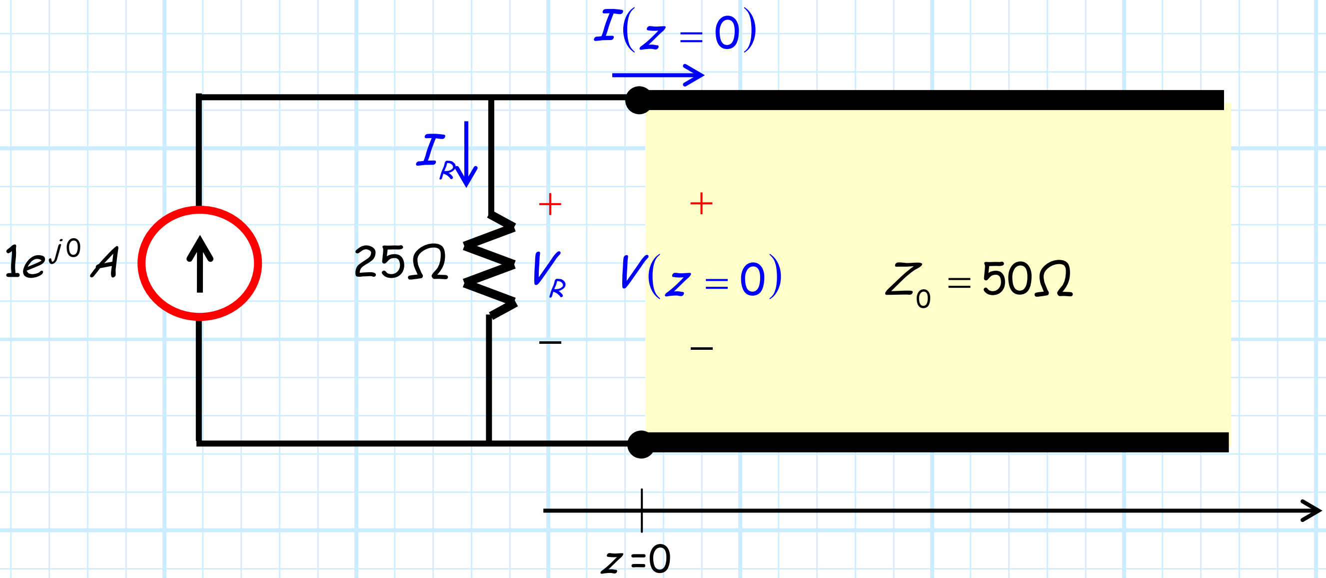
$$V_0^+ = Z_0\, I_0^+ \quad \text{and} \quad V_0^- = -Z_0\, I_0^-$$

we conclude:

$$V_0^+ = Z_0\, I_0^+ = 50(0.4) = 20.0 \quad \text{and} \quad V_0^- = -Z_0\, I_0^- = -50(-B) = 50B$$

Therefore, the voltage along this transmission line is:

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V_0^+\, e^{-j\beta z} + V_0^-\, e^{+j\beta z} \\ &= 20\, e^{-j\beta z} + 50B\, e^{+j\beta z} \end{aligned}$$



Now, from **KCL** we find the **boundary condition** imposed by the source:

$$1.0 - I_R = I(z=0)$$

And from **KVL:**

$$V_R = V(z=0)$$

Thus from Ohm's Law we find:

$$25 = \frac{V_R}{I_R} = \frac{V(z=0)}{e^{j0} - I(z=0)}$$

Therefore:

$$V(z=0) = 25\left(e^{j0} - I(z=0)\right)$$

where:

$$\begin{aligned} I(z=0) &= 0.4\, e^{-j\beta(0)} - Be^{+j\beta(0)} \\ &= 0.4 - B \end{aligned}$$

and:

$$V(z=0) = 20\, e^{-j\beta(0)} + 50B\, e^{+j\beta(0)}$$
$$= 20 + 50B$$

Inserting this into the previous equation:

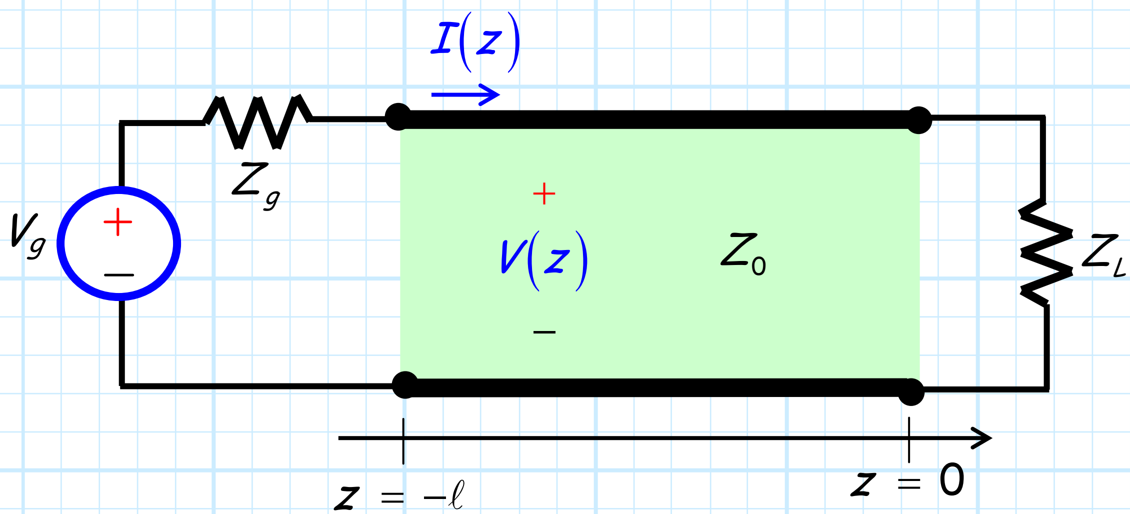
$$\begin{aligned}(20 + 50B) &= 25\left(e^{j0} - (0.4 - B)\right)\\ &= 25 - 0.4(25) + 25B\\ &= 15 + 25B\end{aligned}$$

One equation and **one** unknown! Solving for B:

$$\underline{\underline{B = -0.2}}$$

Delivered and Absorbed Power

Q: *If the purpose of a transmission line is to transfer* ***energy*** *from a source to a load, then exactly how at what rate is energy* ***absorbed*** *by load Z_L in the circuit shown below ??*



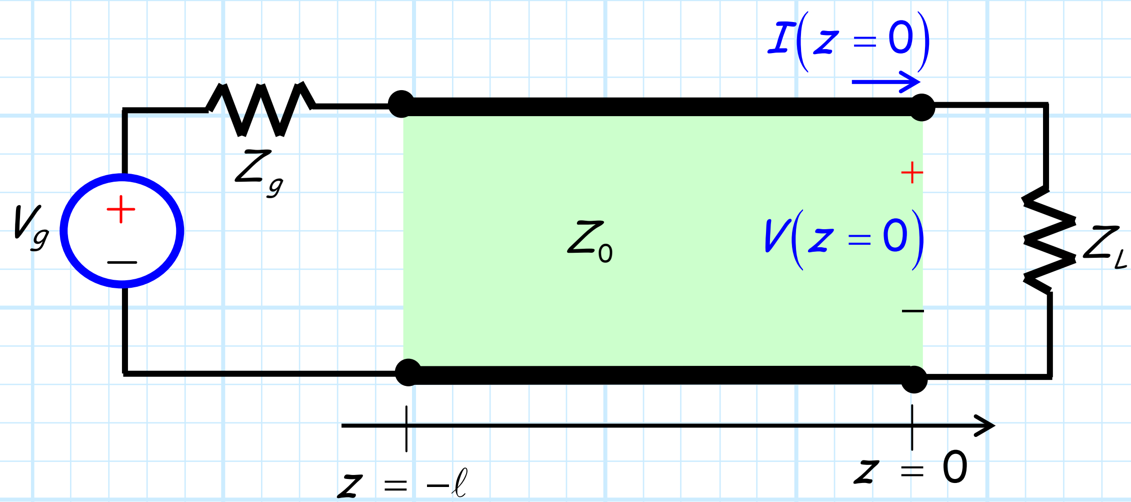
A: Now that we have a **complete** circuit (with **two** boundary conditions!), we of course can determine the numeric values of wave amplitudes V_0^+ and V_0^-.

$$V_0^+ = V_g \, e^{-j\beta\ell} \frac{Z_0}{Z_0(1+\Gamma_{in}) + Z_g(1-\Gamma_{in})} \qquad \text{and} \qquad V_0^- = \Gamma_L \, V_0^+$$

Finally! We can calculate numbers

With knowledge of V_0^+ and V_0^-, we likewise can determine the **total** voltage $V(z)$ (**finally**!) and **total** current $I(z)$ along the transmission line.

Of course, we can **evaluate** this current and voltage **at the load** (i.e., at $z = 0$), and then directly **compute** the rate at which energy is **absorbed by the load**.



$$P_{abs} = \frac{1}{2}\text{Re}\left\{V(z=0)\, I^*(z=0)\right\}$$

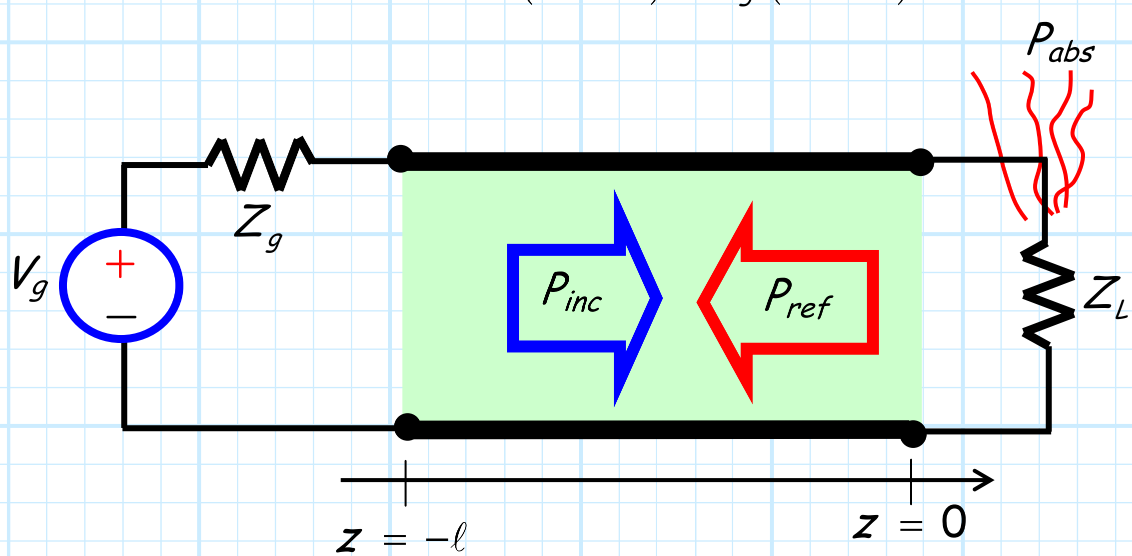
A truck with 30 cases of bananas…

Alternatively, we could likewise determine the absorbed power via the **"wave" viewpoint**, where we know that the absorbed power is simply the **difference** between the plus-wave (**incident**) and minus-wave (**reflected**).

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right)$$

But remember, this simple equation is useful **only** if we first compute the **plus-wave amplitude**:

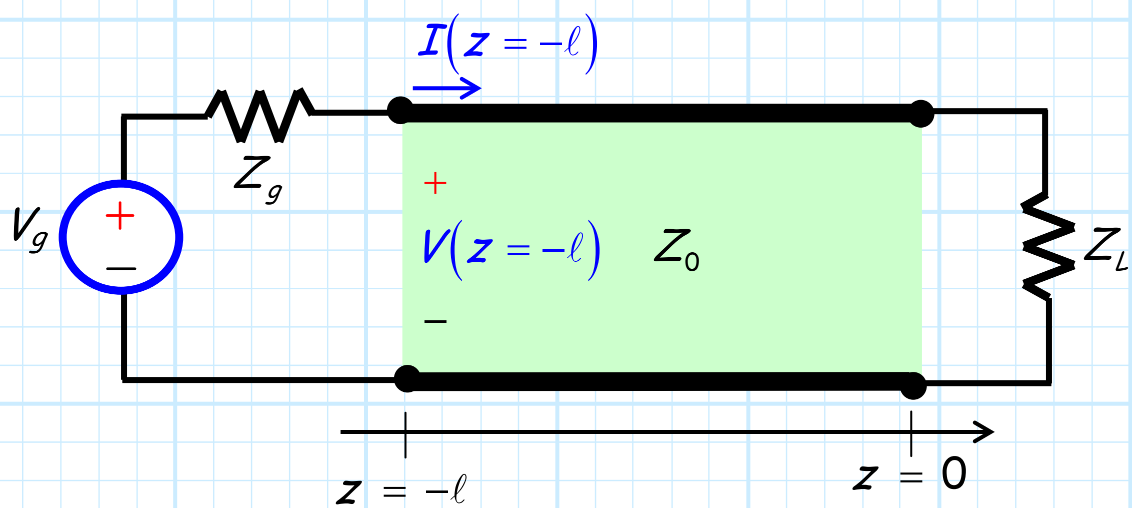
$$V_0^+ = V_g\, e^{-j\beta\ell}\,\frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)}$$



Move to the beginning of the line

Q: *What about the **source**? At what rate does **it** deliver energy **to the circuit**?*

A: Well, we could of course (since we know V_0^+ and V_0^-) determine the total voltage and current **at the source** (i.e., at $z = -\ell$), and then directly compute the rate at which energy is delivered **by the source**.



$$P_{del} = \frac{1}{2}\text{Re}\{V(z=-\ell)\, I^*(z=-\ell)\}$$

Conservation of energy—
it just makes thing so easy

However, since the transmission line is **lossless**, conservation of energy allows us to correctly conclude that this power **delivered by the source** is likewise the rate at which energy is **absorbed by the load**:

$$\boxed{P_{abs} = P_{del}}$$

Thus, all of the **three** calculations we just discussed will provide the **same correct value** for delivered/absorbed power!

$$P_{del} = \frac{1}{2}\mathrm{Re}\left\{V\left(z=-\ell\right)\mathcal{I}^*\left(z=-\ell\right)\right\} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right) = P_{abs} = \frac{1}{2}\mathrm{Re}\left\{V\left(z=0\right)\mathcal{I}^*\left(z=0\right)\right\} = P_{abs}$$

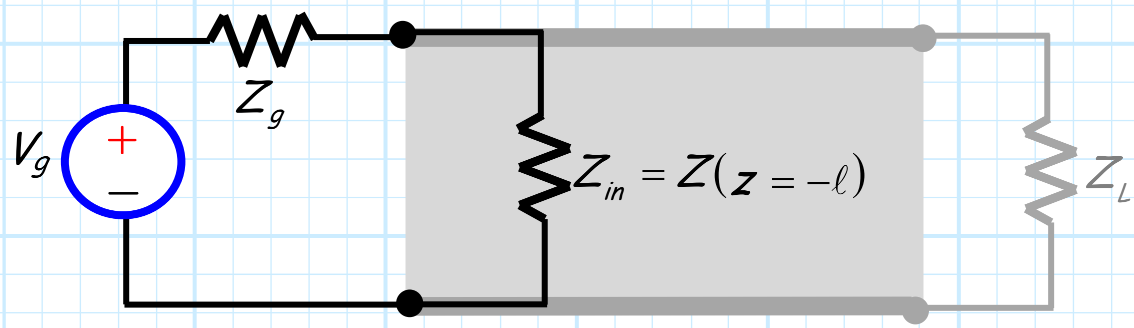
Again, for all three results, we **must** determine V_0^+ and V_0^-.

$$V_0^+ = V_g\, e^{-j\beta\ell}\,\frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)} \qquad \text{and} \qquad V_0^- = \Gamma_L\, V_0^+$$

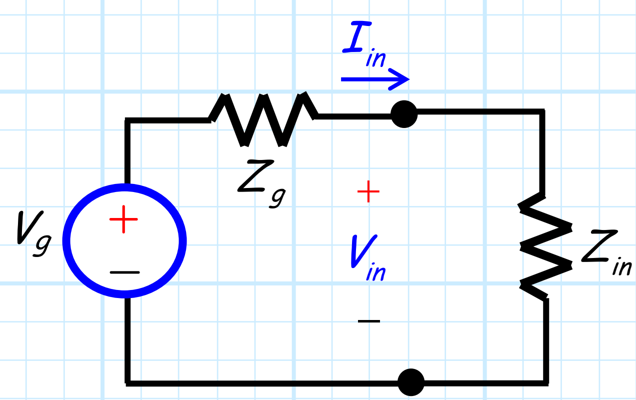
A 4th way—a way that is even easier

However, there is one method for determining the delivered/absorbed power that does **not** require the calculation of V_0^+!

Say we replace the transmission line and load with its equivalent circuit—its **input impedance**:



Recall that we do **not** need to know V_0^+ to determine input impedance—line impedance $Z(z)$ is independent of either V_0^+ or V_0^-. The **equivalent circuit** is thus:



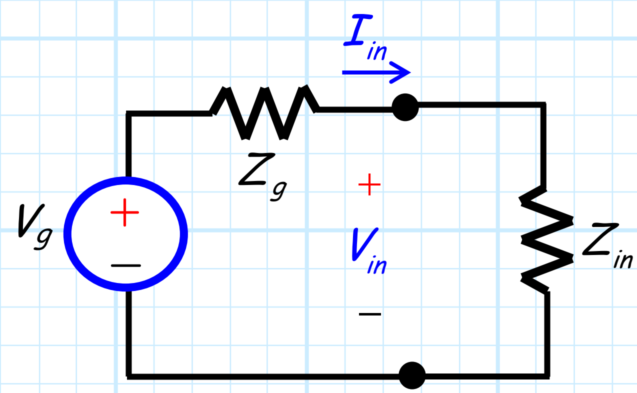
Dust off you introductory circuits text

Now we can easily find the power **delivered** by the source—it's simply **equal** to the rate at which energy is **absorbed** by the input impedance!

We simply have to apply beginning **circuit theory**.

Note by **voltage division** we can determine:

$$V(z=-\ell) = V_g \frac{Z_{in}}{Z_g + Z_{in}}$$



And from **Ohm's Law** we conclude:

$$I(z=-\ell) = \frac{V_g}{Z_g + Z_{in}}$$

A power equation without V_0^+

And thus, the **power** P_{in} absorbed by Z_{in} (and thus the **power** P_{del} delivered by the source) is:

$$\begin{aligned} P_{del} = P_{in} &= \frac{1}{2} \operatorname{Re}\left\{ V_{in}\, I_{in}^* \right\} \\ &= \frac{1}{2} \operatorname{Re}\left\{ V_g \frac{Z_{in}}{Z_g + Z_{in}} \frac{V_g^*}{\left(Z_g + Z_{in}\right)^*} \right\} \\ &= \frac{1}{2} \frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2} \operatorname{Re}\left\{ Z_{in} \right\} \end{aligned}$$

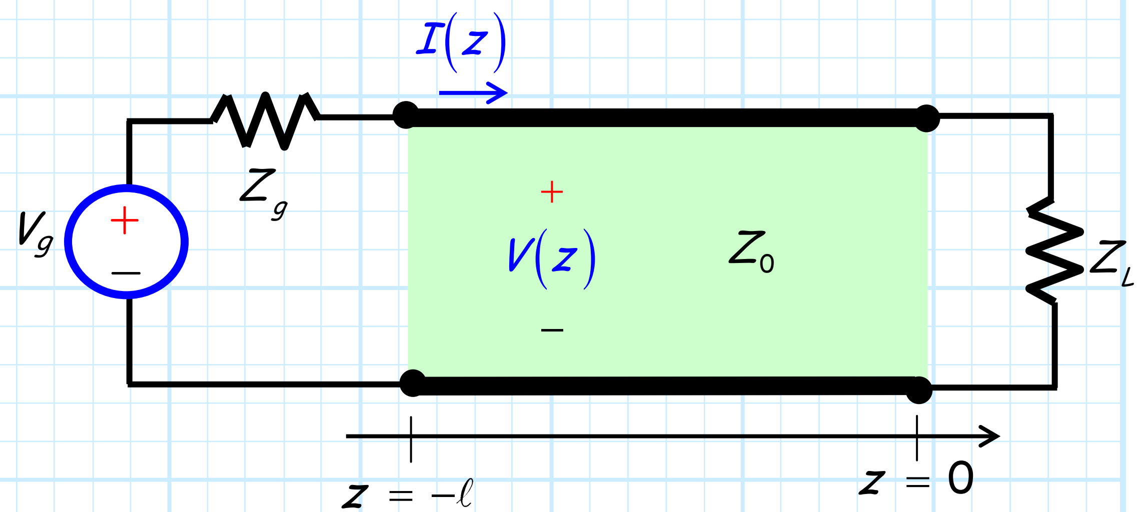
By conservation of energy, we know that this is likewise the rate at which the **load** Z_L **absorbs energy**:

$$P_{del} = \frac{1}{2} \frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2} \operatorname{Re}\left\{ Z_{in} \right\} = P_{abs}$$

Special Cases of Source and Load Impedance

Let's look at **specific cases** of:

1. Z_g
2. and Z_L,

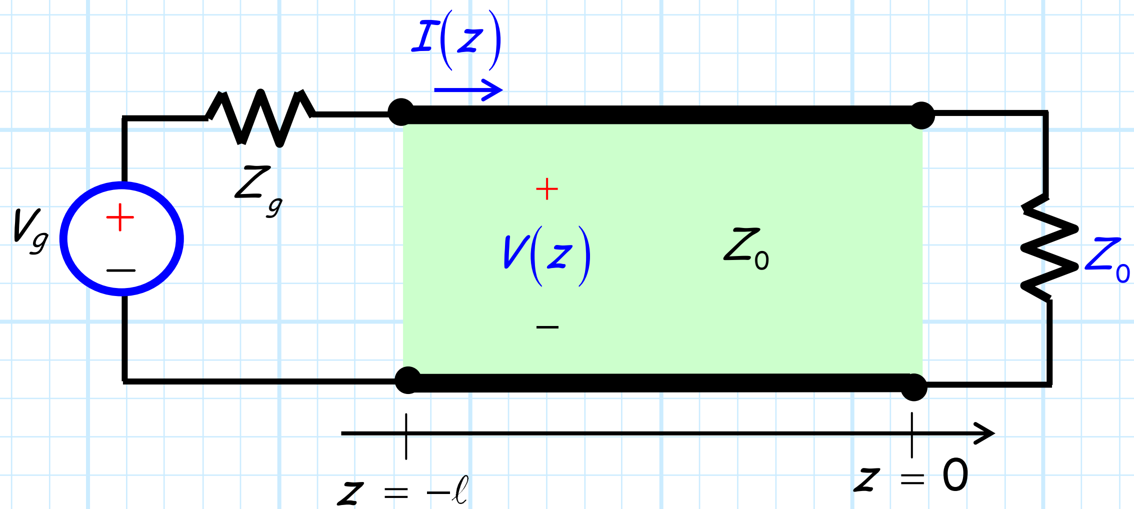


and then determine how they **affect**:

1. V_0^+
2. and P_{abs}.

The first special case: the matched load

$$Z_L = Z_0 + j0$$



Of course for this case—where the load is "**matched**" to the transmission line—we find that $\Gamma_L = 0$ and thus $\Gamma_{in} = 0$.

As a result, the complex amplitude of the plus-wave **simplifies** to:

$$\begin{aligned} V_0^+ &= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_g\left(1-\Gamma_{in}\right)} \\ &= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0 + Z_g} \end{aligned}$$

The absorbed power is just the incident power

Likewise, the absorbed/delivered power is simply that of the **incident** wave:

$$P_{abs} = \frac{|V_0^+|^2}{2\,Z_0}\left(1-|\Gamma_L|^2\right) = \frac{|V_0^+|^2}{2\,Z_0}$$

as the **matched** condition causes the reflected power to be **zero** ($P^- = 0$)!

Now, recognizing that the **input impedance** of the transmission line for this case will likewise be $Z_{in} = Z_0$, we can **alternatively** express the **absorbed power** (without determining V_0^+!) :

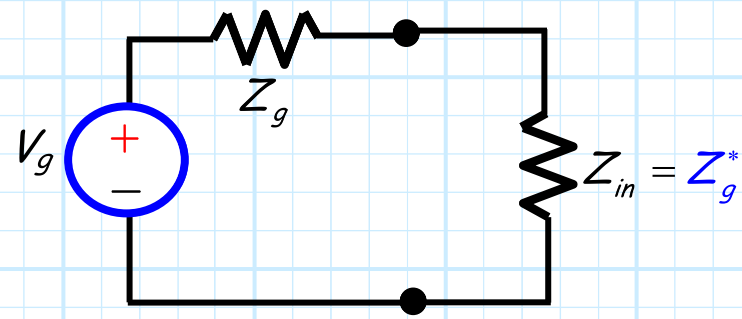
$$P_{abs} = \frac{1}{2}\frac{|V_g|^2}{|Z_g + Z_{in}|^2}\,\text{Re}\{Z_{in}\} = \frac{1}{2}\frac{|V_g|^2}{|Z_g + Z_0|^2}\,Z_0 = \frac{|V_g|^2}{2}\,\frac{Z_0}{|Z_g + Z_0|^2}$$

The 2nd special case: the conjugate match

$$\boxed{Z_{in} = Z_g^*}$$

For this case, we find the load Z_L takes on **whatever value** required to make $Z_{in} = Z_g^*$.

→ The input impedance is a **conjugate match** to the source impedance.



This is of course is a **very** important case!

First, using the fact that:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_g^* - Z_0}{Z_g^* + Z_0}$$

We **can** show that (trust me!) that the plus-wave amplitude **simplifies** to:

$$V_0^+ = V_g\, e^{-j\beta\ell}\, \frac{Z_g^* + Z_0}{4\,\mathrm{Re}\{Z_g\}}$$

Yawn: the load absorbs the available power

Not a particularly interesting result, but now let's look at the **absorbed power**.

$$\begin{aligned} P_{abs} &= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2}\operatorname{Re}\left\{Z_{in}\right\} \\ &= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_g^*\right|^2}\operatorname{Re}\left\{Z_g^*\right\} \\ &= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|2\operatorname{Re}\left\{Z_g^*\right\}\right|^2}\operatorname{Re}\left\{Z_g^*\right\} \\ &= \frac{1}{2}\left|V_g\right|^2\frac{1}{4\operatorname{Re}\left\{Z_g^*\right\}} \\ &= \frac{\left|V_g\right|^2}{8\,R_g} = P_{avl} \end{aligned}$$

The absorbed power is **of course** the **available power** of the source.

Since the input impedance is a **conjugate match** to the source, the source is delivering energy at the **highest rate** it possibly **can**!

But, the load still reflects power!

Q: *But if the power **delivered** is at a maximum, so too is the power **absorbed** by the load. I.E.:*

$$P_{abs} = P_{del} = P_{avl}$$

*If the power absorbed from the load is at a **maximum**, then the power **reflected** from the load must be at a **minimum**.*

*The load must be absorbing **all** the incident power; **right**?*

A: You might think so—and many engineers in fact **do** think so.

→ But those engineers are **incorrect**!

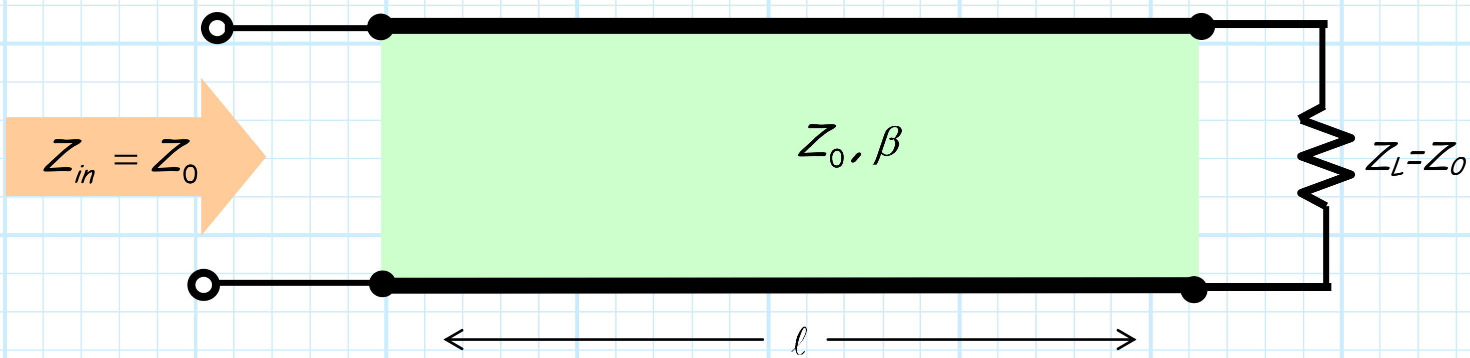
To see why, consider the load Z_L that **minimizes** the reflected power.

We know this load **must** have one specific value—the **matched** load $Z_L = Z_0$.

If the load is Z_0 then so is Z_{in}

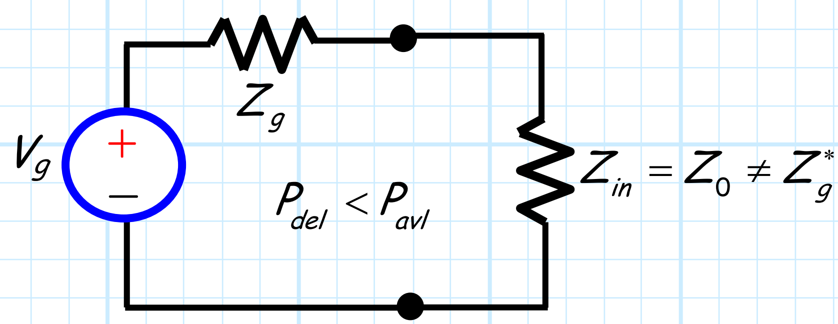
But, **if** the load is matched to the transmission line (i.e., $Z_L = Z_0$, then the **input** impedance will likewise be equal to Z_0:

$$Z_{in} = Z_0$$



And, **generally** speaking, an input impedance of $Z_{in} = Z_0$ will **not** be equal to the complex conjugate of the source impedance:

$$Z_0 \neq Z_g^* \quad !!!!$$



Look closer at this result

Recall that we **just** determined that for $Z_L = Z_0$, the absorbed power is:

$$P_{abs} = \frac{|V_g|^2}{2} \frac{Z_0}{|Z_0 + Z_g|^2}$$

It can be shown that this value is **less** than the available power of the source:

$$\frac{|V_g|^2}{2} \frac{Z_0}{|Z_0 + Z_g|^2} \le \frac{|V_g|^2}{8\,R_g} = P_{avl}$$

Q: *Huh!? This makes **no** sense!*

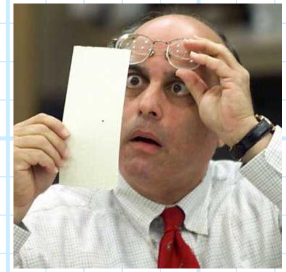
*After all, just **look** at the expression for absorbed power:*

$$P_{abs} = \frac{|V_0^+|^2}{2\,Z_0}\left(1 - |\Gamma_L|^2\right)$$

***Clearly**, this value is maximized when $\Gamma_L = 0$ (i.e., when $Z_L = Z_0$)!!!*

Remember, the load affects V^+

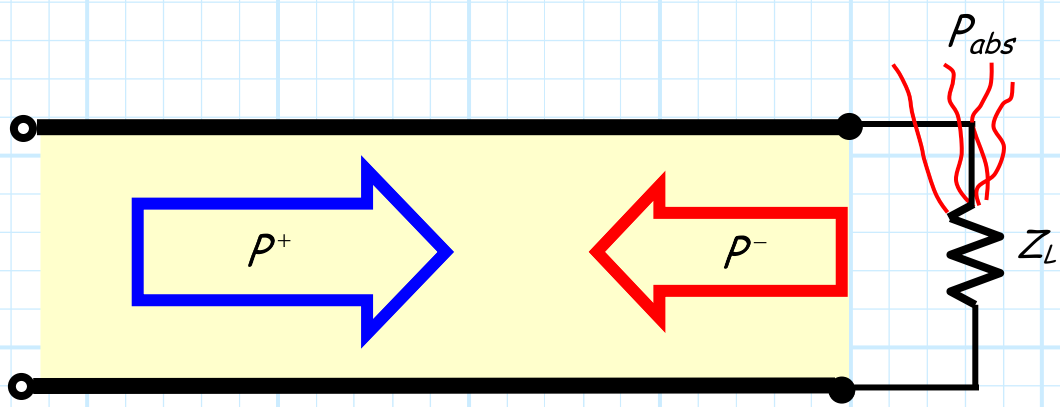
A: Let's **look closer** at this result:



$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right)$$

Remember, we can rewrite this to show that the **absorbed** power is simply the **difference** between the power of the **incident** and **reflected** waves:

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right) = \frac{\left|V_0^+\right|^2}{2\,Z_0} - \frac{\left|V_0^+\Gamma_L\right|^2}{2\,Z_0} = \frac{\left|V_0^+\right|^2}{2\,Z_0} - \frac{\left|V_0^-\right|^2}{2\,Z_0} = P^+ - P^-$$



Absorbed power is maximized if the *difference* between incident and reflected is maximized!

Of course, it **is** true that the load impedance Z_L affects the **minus-wave** amplitude V_0^-, and thus affects likewise the **reflected** wave power P^-.

Remember however, that the value of Z_L **likewise** affects the **plus**-wave amplitude V_0^+, and thus affects likewise the **incident** wave power P^-!

* Thus, the value of Z_L that minimizes P^- will **not** generally maximize P^+!

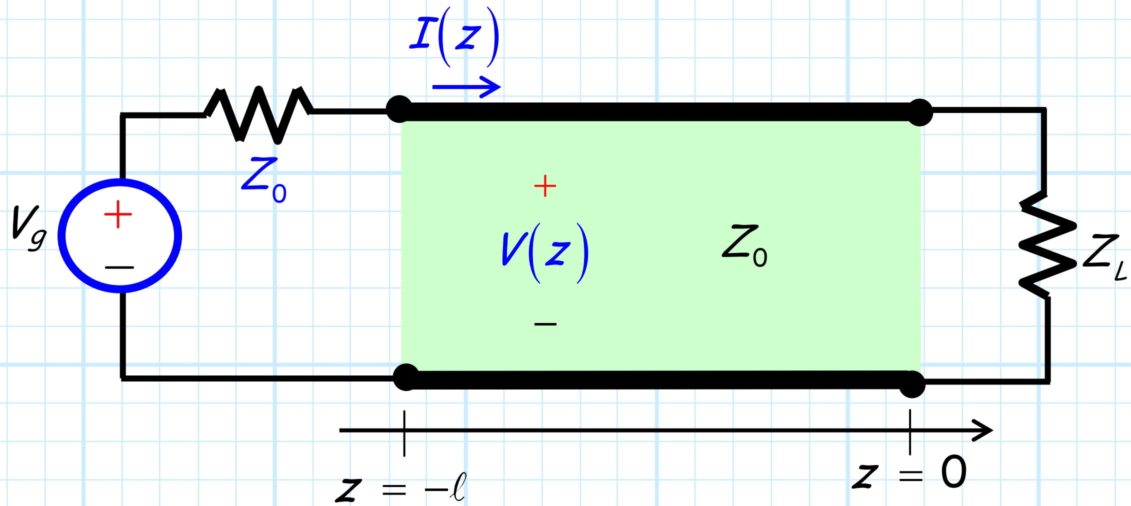
* **Likewise** the value of Z_L that maximizes P^+ will **not** generally minimize P^-.

* Instead, the value of Z_L that maximizes the **absorbed** power P_{abs} is, by definition, the value that **maximizes** the **difference** $P^+ - P^-$.

We find that this **ideal** impedance Z_L is the value that results in the **ideal** case of $Z_{in} = Z_g^*$!

The third special case: the matched source

$$\boxed{Z_g = Z_0 + j0}$$



For this case, we find that V_0^+ **simplifies** greatly:

$$
\begin{aligned}
V_0^+ &= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\,(1+\Gamma_{in}) + Z_g\,(1-\Gamma_{in})} \\
&= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\,(1+\Gamma_{in}) + Z_0\,(1-\Gamma_{in})} \\
&= V_g\, e^{-j\beta\ell} \frac{1}{1+\Gamma_{in}+1-\Gamma_{in}} \\
&= \frac{1}{2} V_g\, e^{-j\beta\ell}
\end{aligned}
$$

Look at what **this** says!

This is the answer—regardless of the load!

It says that the incident wave in this case is **independent** of the load attached at the other end (the value Γ_{in} is nowhere to be found)!

Thus, for the **one** case $Z_g = Z_0$, we in fact **can** consider $V^+(z)$ as being the **source wave**.

And then the reflected wave $V^-(z)$ is the **causal result** of this stimulus.

Specifically, for this case we can write **directly** the plus-wave—without knowing **anything** about the load Z_L:

$$V^+(z) = V_0^+ e^{-j\beta z} = \left(\frac{1}{2} V_g\, e^{-j\beta\ell}\right) e^{-j\beta z} = \frac{V_g}{2}\, e^{-j\beta(z+\ell)}$$

This blue box equation will come in quite handy!

Therefore, the value at the **load** (i.e., $z=0$) is:

$$V^{+}(z=0)=\frac{V_g}{2}\,e^{-j\beta\ell}$$

And the value of the **plus**-wave at the **source** (i.e., $z=-\ell$) is:

$$\boxed{V^{+}(z=-\ell)=\frac{V_g}{2}}$$

This last result is **very** important.

Remember this result, it will come in **quite handy** later in the course!

All I can say is: wow!

Now consider another **really** important result.

Recall the power associated with the plus-wave is:

$$P^{+} = \frac{\left|V_0^{+}\right|^2}{2\,Z_0}$$

Inserting the **simplified** expression for V_0^{+} (i.e., when $Z_g = Z_0$):

$$P^{+} = \frac{\left|V_0^{+}\right|^2}{2\,Z_0} = \frac{\left|V_g\, e^{-j\beta\ell}\right|^2}{8\,Z_0} = \frac{\left|V_g\right|^2}{8\,Z_0} \quad \leftarrow \text{Wow!}$$

Q: *I don't see* ***why*** *this is "wow". Am I* ***missing*** *something?*

A: Apparently you are. Look **closer** at the above result.

Do I have to explain everything?

Remember, this result is for the **special case** where the source impedance is a **real value**, numerically equal to Z_0:

$$Z_g = Z_0 + j0$$

Therefore:

$$R_g = \text{Re}\{Z_g\} = Z_0$$

The **source** resistance is **numerically equal** to transmission line characteristic impedance Z_0.

Thus, the **power** of the plus-wave can be alternatively written as:

$$P^+ = \frac{|V_g|^2}{8\,Z_0} = \frac{|V_g|^2}{8\,R_g} \qquad \leftarrow \text{Wow!}$$

Q: *This result looks vaguely **familiar**; haven't we seen this **before**?*

A: Sigh. This result is the **available power of the source**!!!!!!!

The incident power is the available power (wow)!

Therefore:

$$P^{+} = \frac{|V_g|^2}{8\, Z_0} = \frac{|V_g|^2}{8\, R_g} = P_{avl} \qquad \leftarrow \text{Wow!}$$

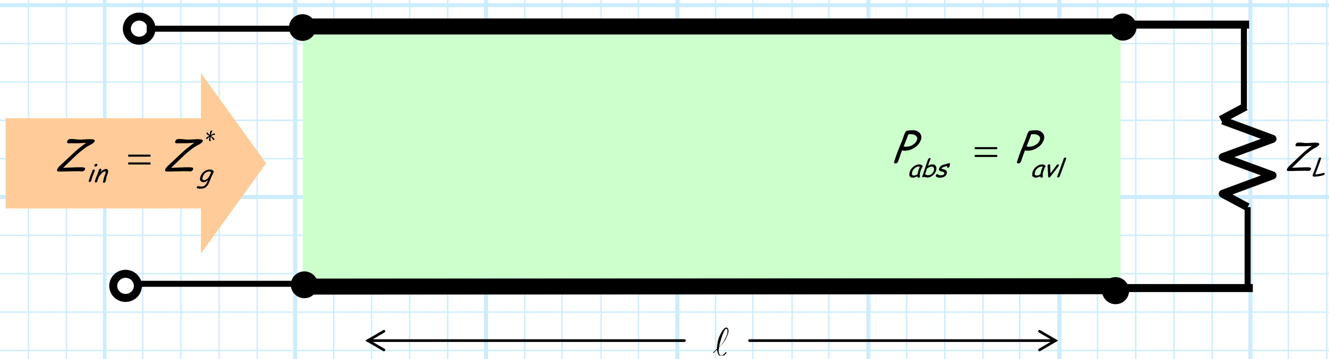
For the special case $Z_g = Z_0 + j0$, the power P^{+} associated with the plus-wave (i.e., the **incident** wave) is **equal** to the **available** power P_{avl} of the **source**.

Q: ***Wow!*** *Does this mean that the power* ***absorbed*** *by the load is* ***equal*** *to the* ***available*** *power* P_{avl} *of the source?*

A: **Absolutely not!**

But some of this incident power is reflected (Doh!)

Remember, **only** if $Z_{in} = Z_g^*$ does the power absorbed by the load **equal** the **available** power.



* Making $Z_g = Z_0 + j0$ causes the power of the **incident** wave to be equal to the **available** power.

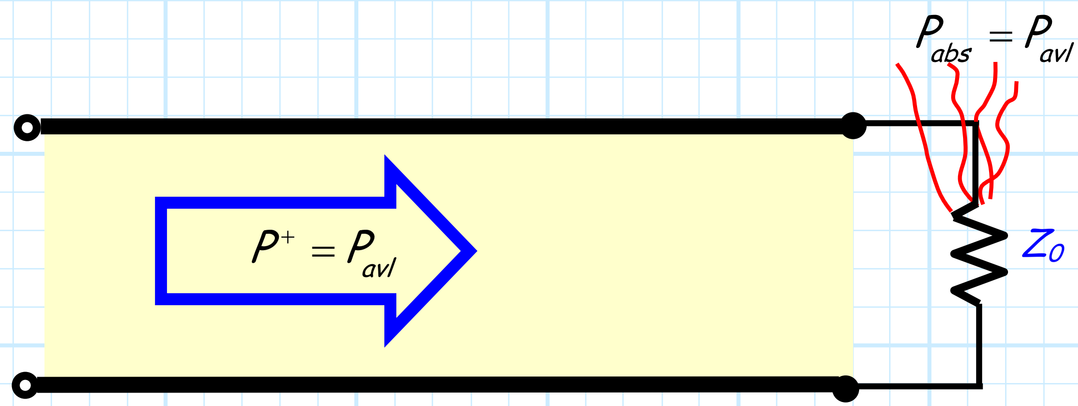
* But, the value Z_L of the **load impedance** determines how much of that incident power is absorbed—and how much gets **reflected**.

The *really* special case!

Q: *Hey, that's right!*

If *the load is matched (i.e., $Z_L = Z_0 + j0$), then wouldn't* ***none*** *of the incident power be* ***reflected*** *(i.e., $P^- = 0$).*

And so wouldn't ***all*** *the incident (i.e., available!) power be* ***absorbed*** *by the load?*



A: That's **exactly correct**!

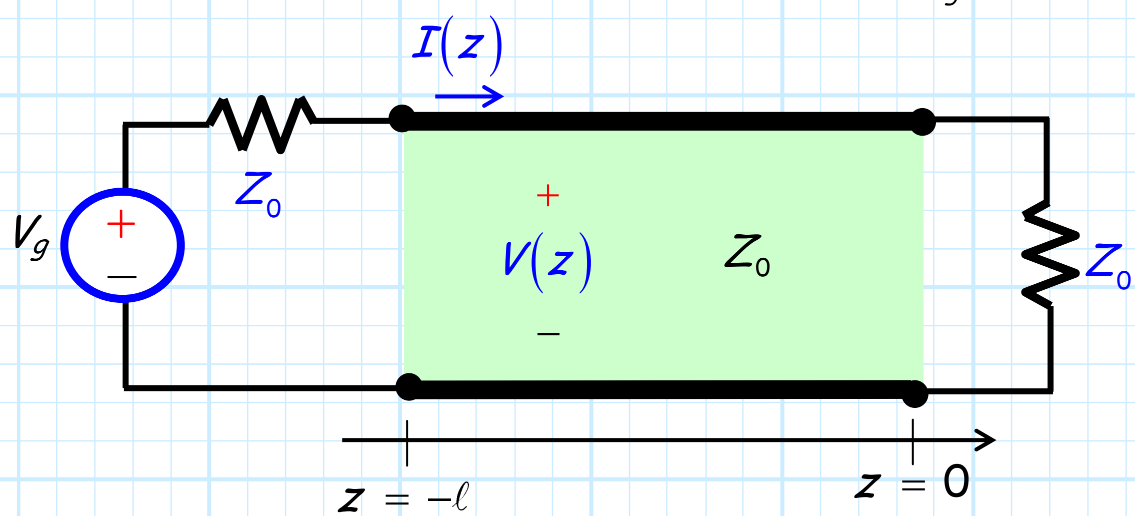
Q: *But you said earlier that* ***minimizing*** *the* ***reflected*** *power would* ***not*** *result in* ***maximizing*** *the* ***absorbed*** *power. Aren't you* ***contradicting*** *yourself?*

A: Not at all.

A conjugate match!

Generally speaking, minimizing the **reflected** power (i.e., $Z_L = Z_0$) will **not** result in maximizing the **absorbed** power.

However, we **are** considering the **special case** where also $Z_g = Z_0$!



Thus, maximum power transfer will occur **if** there is a conjugate match $Z_{in} = Z_g^*$.

Note for this **special** case $Z_g^* = Z_0$, meaning that **input impedance** Z_{in} must be equal to Z_0 for **maximum** power delivery.

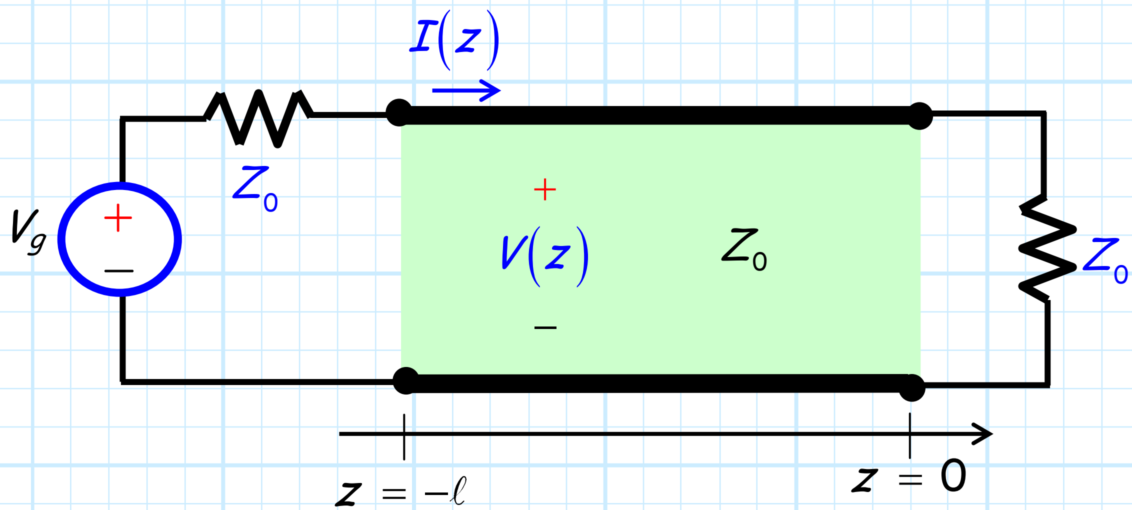
Of course, the input impedance Z_{in} **will** be equal to Z_0 when the **load impedance is matched** (i.e., $Z_L = Z_0$)!

"Zee-zeros" everywhere

Thus, in many ways, the case:

$$Z_g = Z_0 = Z_L$$

(i.e., **both** source and load impedances are numerically equal to Z_0) is **ideal**.



Let's summarize this "ideal case":

* A **conjugate match** occurs ($Z_{in} = Z_g^*$), and so **all** the **available power** from the source is **absorbed** by the load:

$$P_{del} = P_{avl} = P_{abs}$$

It's just *so* simple; you hardly deserve credit for this class

* The **incident** (plus) wave is **independent** of the **load** impedance:

$$V^{+}(z)=\frac{V_g}{2}\,e^{-j\beta(z+\ell)} \qquad V^{+}(z=0)=\frac{V_g}{2}\,e^{-j\beta\ell} \qquad V^{+}(z=-\ell)=\frac{V_g}{2}$$

* And the **power** associated with this **incident** wave is **equal** to the **available power** of the source:

$$P^{+}=\frac{\left|V_g\right|^2}{8\,Z_0}=P_{avl}$$

* The **reflected** (minus) wave is **zero**:

$$V^{-}(z)=0$$

* **So** of course, the **power** associated with the reflected wave is likewise **zero**.

$$P^{-}=0$$

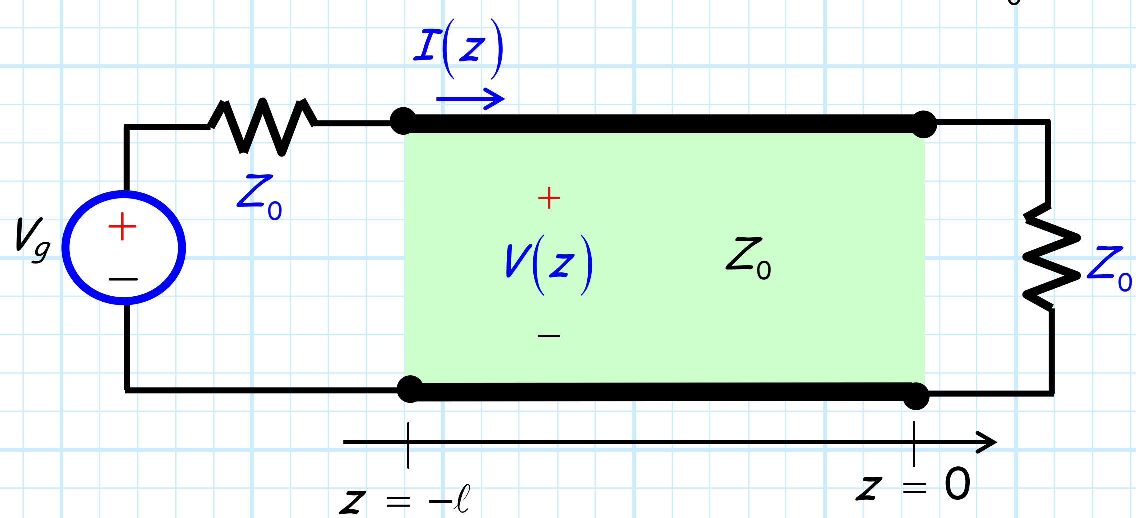
Load AND source is matched— it's a very good thing

* This **again** allows us to verify that **all available power** is absorbed by the **load**:

$$P_{abs} = P^{+} - P^{-} = P_{avl} - 0 = P_{avl}$$

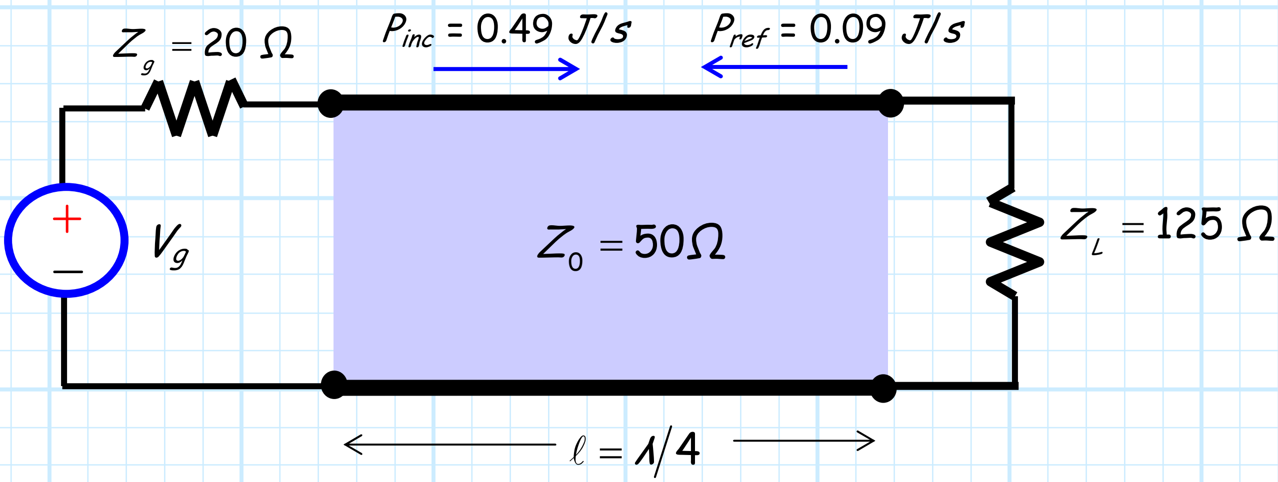
* Finally, the **total** current and voltage simplifies **nicely**:

$$V(z) = V^{+}(z) + V^{-}(z) = \frac{V_g}{2}\, e^{-j\beta(z+\ell)} \qquad I(z) = \frac{V^{+}(z) - V^{-}(z)}{Z_0} = \frac{V_g}{2Z_0}\, e^{-j\beta(z+\ell)}$$



Example: Conservation of Energy and You

Consider this circuit, where the transmission line is **lossless** and has length $\ell = \lambda/4$:



The wave **incident** on the load Z_L has power of $P_{inc} = 0.49\ J/s$.

The wave **reflected** from the load Z_L has power of $P_{ref} = 0.09\ J/s$.

Determine the **magnitude** of source voltage V_g (i.e., determine $|V_g|$).

Hint: This is **not** a boundary condition problem. Do **not** attempt to find $V(z)$ and/or $I(z)$!

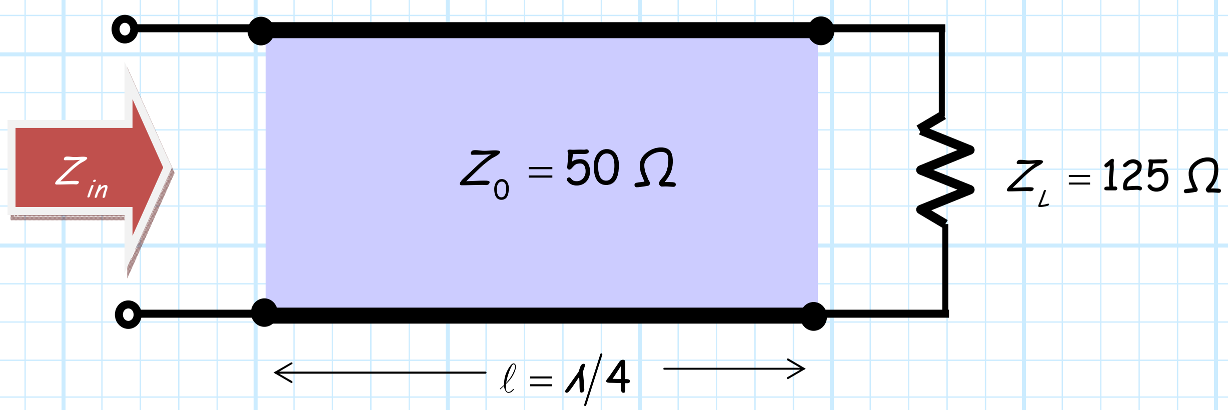
Solution

From **conservation of energy**, we find the load absorbes energy at a rate of:

$$\begin{aligned} P_{abs} &= P_{inc} - P_{ref} \\ &= 0.49 - 0.09 \\ &= 0.4\ W \end{aligned}$$

Since the transmission line is **lossless**, this absorbed power **must** likewise be the power **delivered** by the source to the input of the transmission line (i.e., the power absorbed by input impedance Z_{in}).

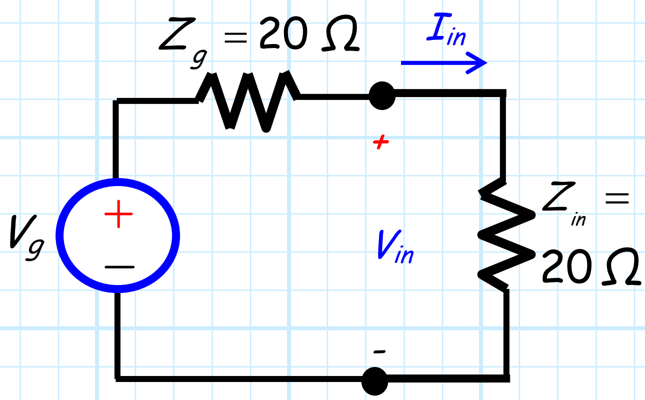
$$P_{in} = P_{abs} = 0.40 \quad J/s$$



Note the transmission line length has the **special case** $\ell = \lambda/4$, therefore the input impedance is easily computed:

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{50^2}{125} = 20\Omega = Z_g$$

A **conjugate match** ($Z_{in} = Z_g^*$)!



Thus, the power **delivered** by the source is:

$$
\begin{aligned}
P_{del} &= \frac{1}{2} Re\left\{V_{in} I_{in}^*\right\} \\
&= \frac{1}{2} Re\left\{\left(\frac{V_g}{2}\right)\left(\frac{V_g^*}{20+20}\right)\right\} \\
&= \frac{1}{8} Re\left\{\left|V_g\right|^2\right\}\frac{1}{20} \\
&= \frac{\left|V_g\right|^2}{160}
\end{aligned}
$$

And since we know that $P_{abs} = P_{del} = 0.4\ W$, we can conclude:

$$
P_{del} = 0.4 = \frac{\left|V_g\right|^2}{160} \qquad \Rightarrow \qquad \underline{\underline{\left|V_g\right| = \sqrt{160(0.4)} = 8.0 \quad V}}
$$