2.6 – Generator and Load Mismatches

Reading Assignment: ***pp. 77-79***

Q: *How is the incident wave $V^+(z)$ generated on a transmission line?*

A: With a power source (i.e., generator)!

HO: A TRANSMISSION LINE CONNECTING SOURCE AND LOAD

EXAMPLE: BOUNDARY CONDITIONS AND SOURCES

Q: *So, how can we determine the power delivered by a source?*

A: HO: DELIVERED POWER

Q: *So how do we insure that the delivered power is as large as possible?*

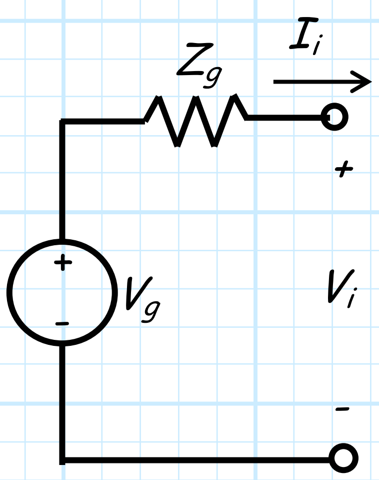
A: HO: SPECIAL CASES OF SOURCE AND LOAD IMPEDANCE

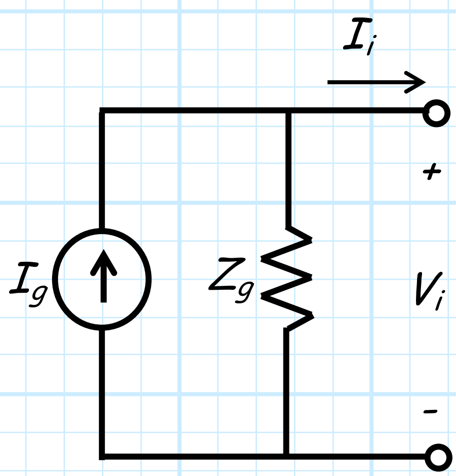
EXAMPLE: CONSERVATION OF ENERGY AND YOU

A Transmission Line Connecting Source & Load

We can think of a transmission line as a conduit that allows **power** to flow **from** an **output** of one device/network **to** an **input** of another.

To simplify our analysis, we can model the **input** of the device **receiving** the power with it input impedance (e.g., Z_L), while we can model the device **output delivering** the power with its Thevenin's or Norton's equivalent circuit.

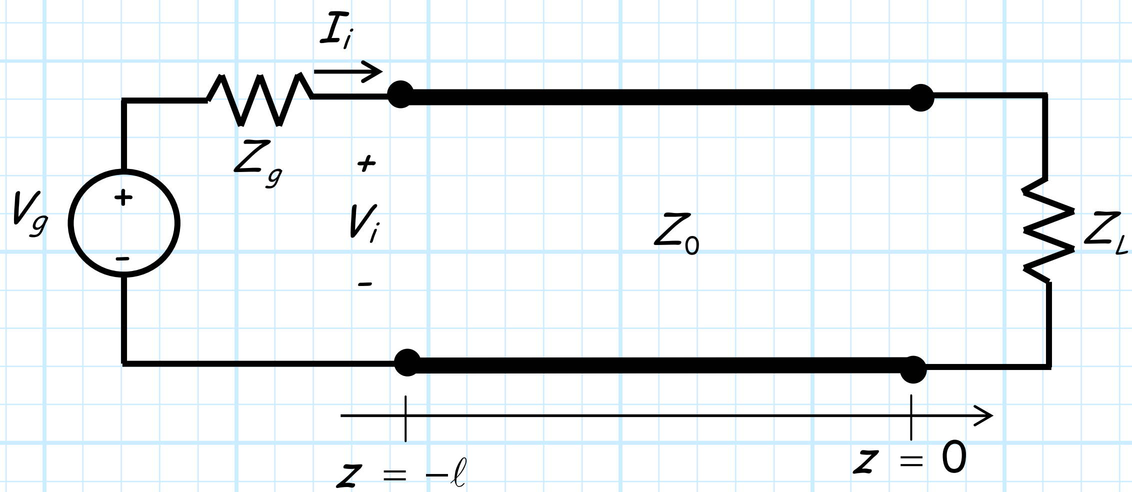




$$V_g = V_i + Z_g I_i$$

$$I_g = \frac{V_i}{Z_g} + I_i$$

Typically, the power source is modeled with its **Thevenin's** equivalent; however, we will find that the **Norton's** equivalent circuit is useful if we express the remainder of the circuit in terms of its **admittance** values (e.g., $Y_0, Y_L, Y(z)$).



Recall from the telegrapher's equations that the current and voltage along the transmission line are:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

At $z = 0$, we enforced the **boundary condition** resulting from Ohm's Law:

$$Z_L = \frac{V_L}{I_L} = \frac{V(z=0)}{I(z=0)} = \frac{\left(V_0^+ + V_0^-\right)}{\left(\dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}\right)}$$

Which resulted in:

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \doteq \Gamma_L$$

So therefore:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L \, e^{+j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L \, e^{+j\beta z} \right]$$

We are left with the **question**: just what is the **value** of complex constant V_0^+?!?

This constant depends on the signal **source**! To determine its exact **value**, we must now apply **boundary conditions** at $z = -\ell$.

We know that at the **beginning** of the transmission line:

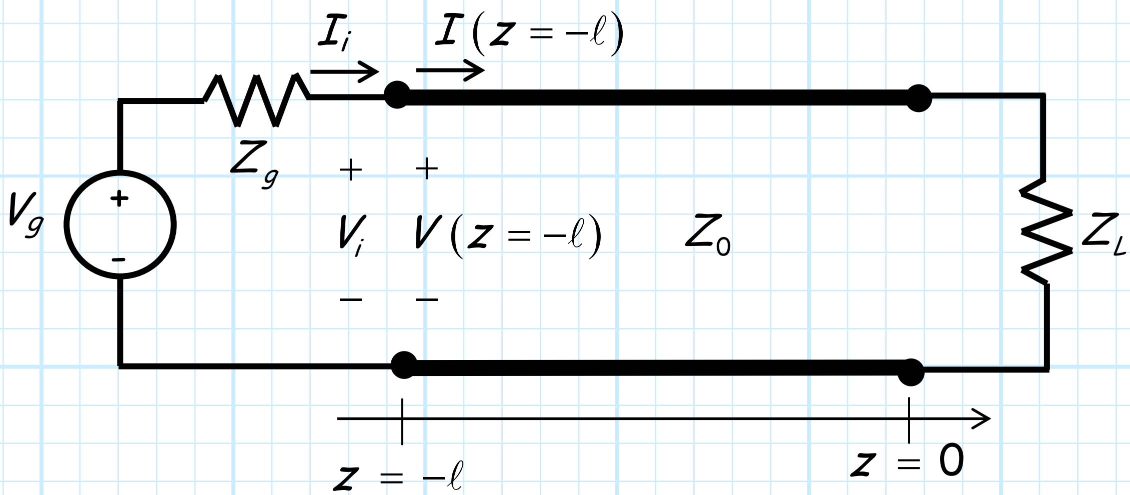
$$V(z = -\ell) = V_0^+ \left[e^{+j\beta\ell} + \Gamma_L \, e^{-j\beta\ell} \right]$$

$$I(z = -\ell) = \frac{V_0^+}{Z_0} \left[e^{+j\beta\ell} - \Gamma_L \, e^{-j\beta\ell} \right]$$

Likewise, we know that the **source** must satisfy:

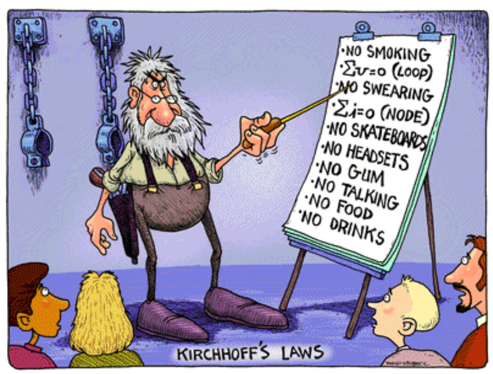
$$V_g = V_i + Z_g \, I_i$$

To relate these **three** expressions, we need to apply **boundary conditions** at $z=-\ell$:



From **KVL** we find:

$$V_i = V(z=-\ell)$$



And from **KCL**:

$$I_i = I(z=-\ell)$$

Combining these equations, we find:

$$V_g = V_i + Z_g I_i$$

$$V_g = V_0^+\left[e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right] + Z_g \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right]$$

One equation → **one** unknown (V_0^+)!!

Solving, we find the value of V_0^+:

$$V_0^+ = V_g\, e^{-j\beta\ell}\, \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)}$$

where:

$$\Gamma_{in} = \Gamma\left(z=-\ell\right) = \Gamma_L\, e^{-j2\beta\ell}$$

Note this result looks different than the equation in your textbook (eq. 2.71):

$$V_0^+ = V_g\, \frac{Z_0}{Z_0+Z_g}\, \frac{e^{-j\beta\ell}}{\left(1-\Gamma_L\Gamma_g e^{-j2\beta\ell}\right)}$$

where:

$$\Gamma_g \doteq \frac{Z_g - Z_0}{Z_g + Z_0}$$

I like **my** expression better.

Although the two equations are equivalent, **my** expression is explicitly written in terms of $\Gamma_{in} = \Gamma\left(z=-\ell\right)$ (a very **useful**, **precise**, and **unambiguous** value), while the book's expression is written in terms of this **so-called** "source reflection coefficient" Γ_g (a **misleading**, **confusing**, **ambiguous**, and mostly **useless** value).

Specifically, we might be **tempted** to equate Γ_g with the value $\Gamma\left(z=-\ell\right)=\Gamma_{in}$, but it is **not** ($\Gamma_g \neq \Gamma\left(z=-\ell\right)$)!

There is one **very important** point that must be made about the result:

$$V_0^+ = V_g \, e^{-j\beta\ell} \frac{Z_0}{Z_0 \left(1+\Gamma_{in}\right) + Z_g \left(1-\Gamma_{in}\right)}$$

And that is—the wave $V_0^+(z)$ **incident** on the load Z_L is actually dependent **on** the value of load Z_L !!!!!

Remember:

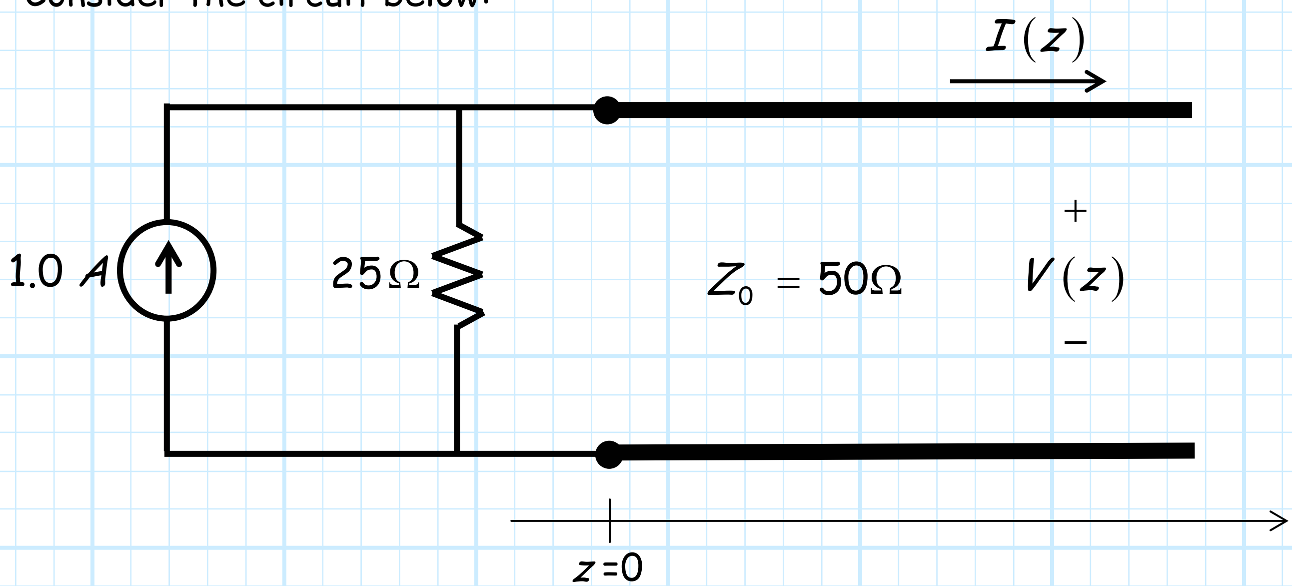
$$\Gamma_{in} = \Gamma\left(z=-\ell\right) = \Gamma_L \, e^{-j2\beta\ell}$$

We tend to think of the incident wave $V_0^+(z)$ being "**caused**" by the source, and it is certainly true that $V_0^+(z)$ **depends** on the source—after all, $V_0^+(z)=0$ if $V_g=0$. However, we find from the equation above that it **likewise** depends on the value of the load!

Thus we **cannot**—in general—consider the incident wave to be the "**cause**" and the reflected wave the "**effect**". Instead, each wave must obtain the proper **amplitude** (e.g., V_0^+, V_0^-) so that the boundary conditions are satisfied at **both** the beginning and end of the transmission line.

Example: Boundary Conditions and Sources

Consider the circuit below:



It is known that the **current** along the transmission line is:

$$I(z) = 0.4\, e^{-j\beta z} - B e^{+j\beta z} \quad A \quad \text{for } z > 0$$

where B is some unknown complex value.

Determine the value of B.

Hint: $B \neq -0.6$

Solution

Since the line current is:

$$I(z) = 0.4\, e^{-j\beta z} - Be^{+j\beta z} = I_0^+\, e^{-j\beta z} + I_0^- e^{+j\beta z}$$

we conclude that:

$$I_0^+ = 0.4 \quad \text{and} \quad I_0^- = -B$$

and since:

$$V_0^+ = Z_0\, I_0^+ \quad \text{and} \quad V_0^- = -Z_0\, I_0^-$$

we conclude:

$$V_0^+ = Z_0\, I_0^+ = 50(0.4) = 20.0 \quad \text{and} \quad V_0^- = -Z_0\, I_0^- = -50(-B) = 50B$$

Therefore, the voltage along this transmission line is:

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \\ &= 20\, e^{-j\beta z} + 50B\, e^{+j\beta z} \end{aligned}$$

Now, from KCL we find the **boundary condition** imposed by the source:

$$1.0 - \frac{V(z=0)}{25} = I(z=0)$$

where:

$$\begin{aligned} I(z=0) &= 0.4\, e^{-j\beta(0)} - Be^{+j\beta(0)} \\ &= 0.4 - B \end{aligned}$$

and:

$$V(z=0) = 20\, e^{-j\beta(0)} + 50B\, e^{+j\beta(0)}$$
$$= 20 + 50B$$

Thus combining the three previous equations:

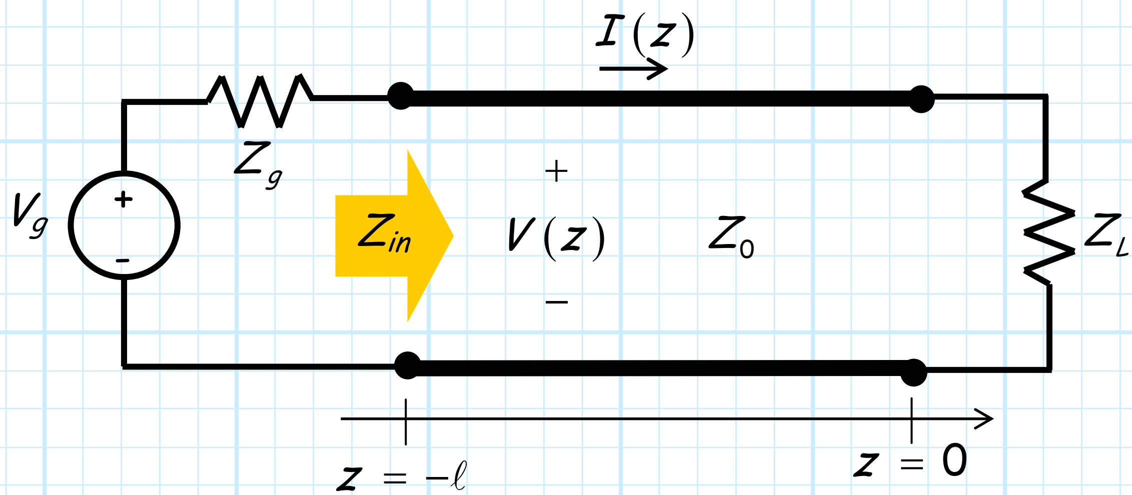
$$1.0 - \frac{20+50B}{25} = 0.4 - B$$

One equation and one unknown! Solving for B:

$$\underline{B = 1.0 - 0.8 - 0.4 = -0.2}$$

Delivered Power

Q: *If the purpose of a transmission line is to transfer **power** from a source to a load, then exactly how much power is **delivered** to Z_L for the circuit shown below ??*



A: We of course **could** determine V_0^+ and V_0^-, and then determine the power absorbed by the load (P_{abs}) as:

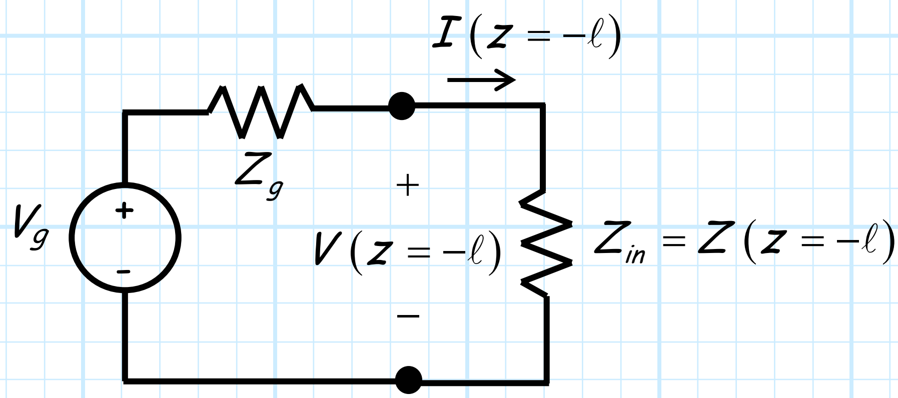
$$P_{abs} = \frac{1}{2}\text{Re}\{V(z=0)\, I^*(z=0)\}$$

However, if the transmission line is **lossless**, then we know that the power delivered to the load must be **equal** to the power "delivered" to the **input** (P_{in}) of the transmission line:

$$P_{abs} = P_{in} = \frac{1}{2}\text{Re}\{V(z=-\ell)\, I^*(z=-\ell)\}$$

However, we can determine this power **without** having to solve for V_0^+ and V_0^- (i.e., $V(z)$ and $I(z)$). We can simply use our knowledge of **circuit theory**!

We can **transform** load Z_L to the beginning of the transmission line, so that we can replace the transmission line with its **input impedance** Z_{in}:



Note by **voltage division** we can determine:

$$V(z=-\ell) = V_g \frac{Z_{in}}{Z_g + Z_{in}}$$

And from **Ohm's Law** we conclude:

$$I(z=-\ell) = \frac{V_g}{Z_g + Z_{in}}$$

And thus, the **power** P_{in} delivered to Z_{in} (and thus the **power** P_{abs} delivered to the load Z_L) is:

$$P_{abs} = P_{in} = \frac{1}{2}\text{Re}\left\{V(z=-\ell)\, I^*(z=-\ell)\right\}$$

$$= \frac{1}{2}\text{Re}\left\{V_g \frac{Z_{in}}{Z_g + Z_{in}} \frac{V_g^*}{\left(Z_g + Z_{in}\right)^*}\right\}$$

$$= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2}\text{Re}\left\{Z_{in}\right\}$$

$$= \frac{1}{2}\left|V_g\right|^2 \frac{\left|Z_{in}\right|^2}{\left|Z_g + Z_{in}\right|^2}\text{Re}\left\{Y_{in}\right\}$$

Note that we could **also** determine P_{abs} from our **earlier** expression:

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1 - \left|\Gamma_L\right|^2\right)$$

But we would of course have to **first** determine $V_0^+(\ell)$:

$$V_0^+ = V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_g\left(1-\Gamma_{in}\right)}$$

Special Cases of Source and Load Impedance

Let's look at **specific cases** of Z_g and Z_L, and determine how they affect V_0^+ and P_{abs}.

$$Z_g = Z_0$$

For this case, we find that V_0^+ **simplifies** greatly:

$$\begin{aligned}
V_0^+ &= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)} \\
&= V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_0\left(1-\Gamma_{in}\right)} \\
&= V_g\, e^{-j\beta\ell} \frac{1}{1+\Gamma_{in}+1-\Gamma_{in}} \\
&= \frac{1}{2} V_g\, e^{-j\beta\ell}
\end{aligned}$$

Look at what **this** says!

It says that the incident wave in this case is **independent** of the load attached at the other end!

Thus, for the **one** case $Z_g = Z_0$, we in fact can consider $V^+(z)$ as being the source wave, and then the reflected wave $V^-(z)$ as being the result of this stimulus.

Remember, the complex value V_0^+ is the value of the incident wave evaluated at the end of the transmission line ($V_0^+ = V^+(z=0)$). We can likewise determine the value of the incident wave at the **beginning** of the transmission line (i.e., $V^+(z=-\ell)$). For this case, where $Z_g = Z_0$, we find that this value can be very simply stated (!):

$$
\begin{aligned}
V^+(z=-\ell) &= V_0^+ e^{-j\beta(z=-\ell)} \\
&= \left(\frac{1}{2} V_g \, e^{-j\beta\ell}\right) e^{+j\beta\ell} \\
&= \frac{V_g}{2}
\end{aligned}
$$

Likewise, we find that the delivered power for this case can be simply stated as:

$$
\begin{aligned}
P_{abs} &= \frac{\left|V_0^+\right|^2}{2\, Z_0}\left(1-\left|\Gamma_L\right|^2\right) \\
&= \frac{\left|V_g\right|^2}{8\, Z_0}\left(1-\left|\Gamma_L\right|^2\right)
\end{aligned}
$$

$$Z_L = Z_0$$

In this case, we find that $\Gamma_L = 0$, and thus $\Gamma_{in} = 0$. As a result:

$$\begin{aligned} V_0^+ &= V_g\, e^{-j\beta\ell}\, \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_g\left(1-\Gamma_{in}\right)} \\ &= V_g\, e^{-j\beta\ell}\, \frac{Z_0}{Z_0 + Z_g} \end{aligned}$$

Likewise, we find that:

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right) = \frac{\left|V_0^+\right|^2}{2\,Z_0}$$

Here the delivered power P_{abs} is simply that of the incident wave (P^+), as the matched condition causes the reflected power to be zero ($P^- = 0$)!

Inserting the value of V_0^+, we find:

$$
\begin{aligned}
P_{abs} &= \frac{\left|V_0^+\right|^2}{2\,Z_0} \\
&= \frac{\left|V_g\right|^2}{2\,Z_0}\frac{\left(Z_0\right)^2}{\left|Z_0+Z_g\right|^2} \\
&= \frac{\left|V_g\right|^2}{2}\frac{Z_0}{\left|Z_0+Z_g\right|^2}
\end{aligned}
$$

Note that this result can likewise be found by recognizing that $Z_{in}=Z_0$ when $Z_L=Z_0$:

$$
\begin{aligned}
P_{abs} &= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g+Z_{in}\right|^2}\,\text{Re}\{Z_{in}\} \\
&= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g+Z_0\right|^2}\,Z_0 \\
&= \frac{\left|V_g\right|^2}{2}\frac{Z_0}{\left|Z_g+Z_0\right|^2}
\end{aligned}
$$

$$Z_{in}=Z_g^*$$

For this case, we find Z_L takes on whatever value required to make $Z_{in}=Z_g^*$. This is a **very** important case!

First, using the fact that:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_g^* - Z_0}{Z_g^* + Z_0}$$

We can show that (trust me!):

$$V_0^+ = V_g \, e^{-j\beta\ell} \, \frac{Z_g^* + Z_0}{4\,\text{Re}\{Z_g\}}$$

Not a particularly interesting result, but let's look at the absorbed power.

$$\begin{aligned} P_{abs} &= \frac{1}{2} \frac{|V_g|^2}{|Z_g + Z_{in}|^2} \text{Re}\{Z_{in}\} \\ &= \frac{1}{2} \frac{|V_g|^2}{|Z_g + Z_g^*|^2} \text{Re}\{Z_g^*\} \\ &= \frac{1}{2} \frac{|V_g|^2}{|2\text{Re}\{Z_g^*\}|^2} \text{Re}\{Z_g^*\} \\ &= \frac{1}{2} |V_g|^2 \frac{1}{4\text{Re}\{Z_g^*\}} \doteq P_{avl} \end{aligned}$$

Although this result does not look particularly interesting **either**, we find the result is **very** important!

It can be shown that—for a **given** V_g and Z_g —the value of input impedance Z_{in} that will absorb the **largest possible** amount of power is the value $Z_{in} = Z_g^*$.

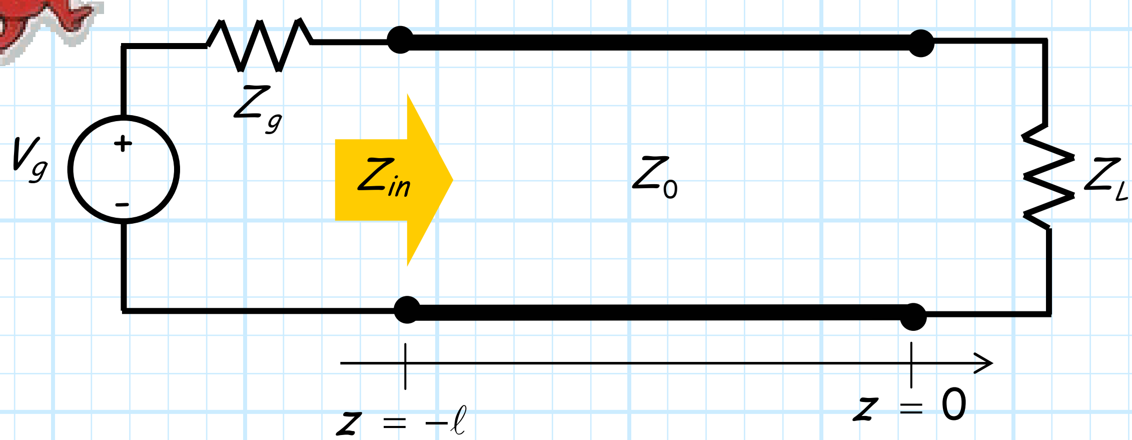
This case is known as the **conjugate match**, and is essentially the goal of every transmission line problem—to deliver the largest possible power to Z_{in}, and thus to Z_L as well!

This maximum delivered power is known as the **available power** (P_{avl}) of the source.

There are **two** very important things to understand about this result!

Very Important Thing #1

Consider again the terminated transmission line:



Recall that if $Z_L = Z_0$, the **reflected** wave will be **zero**, and the absorbed power will be:

$$P_{abs} = \frac{|V_g|^2}{2} \frac{Z_0}{|Z_0 + Z_g|^2} \leq P_{avl}$$

But note if $Z_L = Z_0$, the input impedance $Z_{in} = Z_0$—but then $Z_{in} \neq Z_g^*$ (generally)! In other words, $Z_L = Z_0$ does **not** (generally) result in a **conjugate match**, and thus setting $Z_L = Z_0$ does **not** result in maximum power absorption!

Q: *Huh!? This makes **no** sense! A load value of $Z_L = Z_0$ will **minimize** the reflected wave ($P^- = 0$)—**all** of the incident power will be absorbed.*

*Any other value of $Z_L = Z_0$ will result in **some** of the incident wave being reflected—how in the world could this **increase** absorbed power?*

*After all, just **look** at the expression for absorbed power:*

$$P_{abs} = \frac{|V_0^+|^2}{2\,Z_0}\left(1 - |\Gamma_L|^2\right)$$

***Clearly**, this value is maximized when $\Gamma_L = 0$ (i.e., when $Z_L = Z_0$)!!!*

A: You are forgetting one very important fact! Although it **is** true that the load impedance Z_L affects the **reflected** wave power P^-, the value of Z_L—as we have shown in this handout—**likewise** helps determine the value of the **incident** wave (i.e., the value of P^+) as well.

* Thus, the value of Z_L that minimizes P^- will **not** generally maximize P^+!

* **Likewise** the value of Z_L that maximizes P^+ will not generally minimize P^-.

* Instead, the value of Z_L that maximizes the **absorbed** power P_{abs} is, by definition, the value that maximizes the **difference** $P^+ - P^-$.

We find that this impedance Z_L is the value that results in the **ideal** case of $Z_{in} = Z_g^*$.

Q: *Yes, but what about the case where $Z_g = Z_0$? For **that** case, we determined that the incident wave **is** independent of Z_L. Thus, it would seem that at least for that case, the **delivered** power would be maximized when the **reflected** power was minimized (i.e., $Z_L = Z_0$).*

A: **True!** But think about what the **input** impedance would be in that case—$Z_{in} = Z_0$. Oh by the way, that provides a **conjugate match** ($Z_{in} = Z_0 = Z_g^*$)!

Thus, in some ways, the case $Z_g = Z_0 = Z_L$ (i.e., **both** source and load impedances are numerically equal to Z_0) is **ideal**. A **conjugate match** occurs, the incident wave is **independent** of Z_L, there is **no** reflected wave, and all the math **simplifies** quite nicely:

$$V_0^+ = \frac{1}{2} V_g \, e^{-j\beta\ell} \qquad\qquad P_{abs} = P_{avl} = \frac{|V_g|^2}{8\, Z_0}$$

Very Important Thing #2

Note the conjugate match criteria **says**:

Given *source impedance Z_g, maximum power transfer occurs when the input impedance is set at value $Z_{in} = Z_g^*$.*

It does **NOT** say:

Given *input impedance Z_{in}, maximum power transfer occurs when the source impedance is set at value $Z_g = Z_{in}^*$.*

This last statement is in fact **false!**

A **factual** statement is this:

Given *input impedance Z_{in}, maximum power transfer occurs when the source impedance is set at value $Z_g = 0 - jX_{in}$ (i.e., $R_g = 0$).*

Q: *Huh??*

A: Remember, the value of source impedance Z_g affects the available power P_{avl} of the source. To maximize P_{avl}, the real (resistive) component of the source impedance should be as small as possible (regardless of Z_{in}!), a fact that is **evident** when observing the expression for **available power**:

$$P_{avl} = \frac{1}{2}\left|V_g\right|^2 \frac{1}{4\,\text{Re}\{Z_g^*\}} = \frac{\left|V_g\right|^2}{8R_g}$$

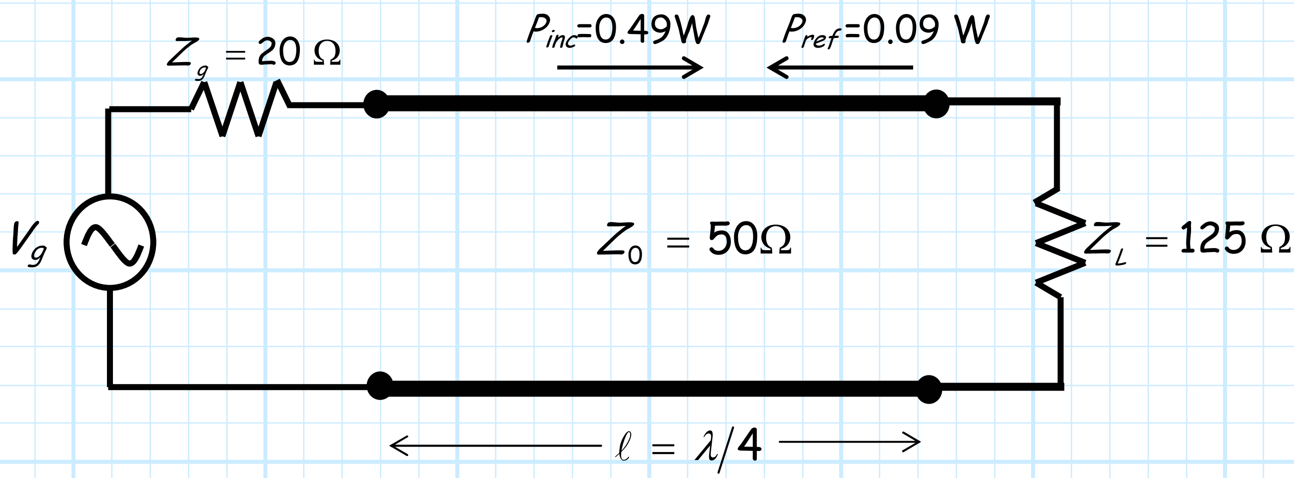
Thus, **maximizing** the power delivered **to** a load (P_{abs}), **from** a source, has **two** components:

1. Maximize the **power available** (P_{avl}) from a source (e.g., minimize R_g).

2. **Extract** all of this available power by setting the input impedance Z_{in} to a value $Z_{in} = Z_g^*$ (thus $P_{abs} = P_{avl}$).

Example: Conservation of Energy and You

Consider this circuit, where the transmission line is **lossless** and has length $\ell = \lambda/4$:



The wave incident on the load Z_L has power of P_{inc}=0.49 W.

The wave reflected from the load Z_L has power of P_{ref}= 0.09 W.

Determine the magnitude of source voltage V_g (i.e., determine $|V_g|$).

Hint: This is **not** a boundary condition problem. Do **not** attempt to find $V(z)$ and/or $I(z)$!

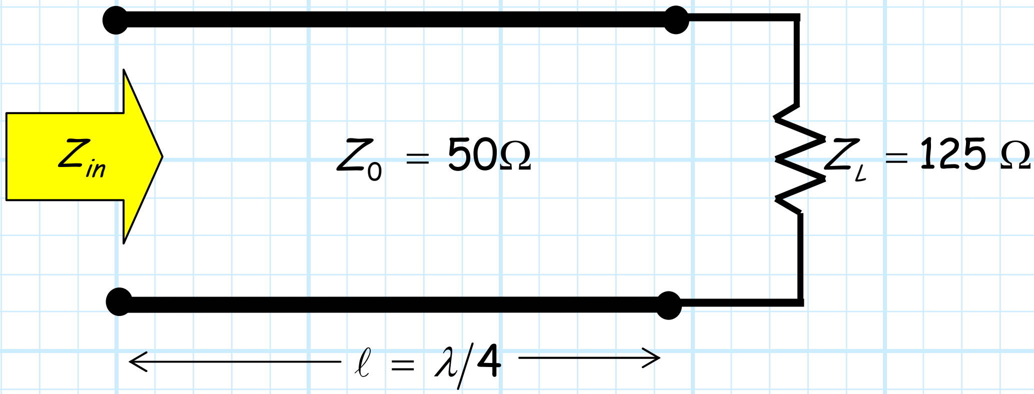
Solution

From **conservation of energy**, we find the power absorbed by the load must be:

$$\begin{aligned} P_{abs} &= P_{inc} - P_{ref} \\ &= 0.49 - 0.09 \\ &= 0.4\ W \end{aligned}$$

Since the transmission line is **lossless**, this absorbed power **must** likewise be the power delivered to the input of the transmission line (i.e., the power absorbed by input impedance Z_{in}).

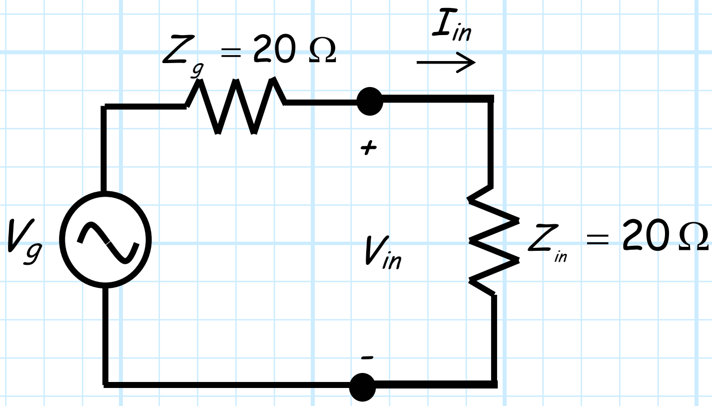
$$P_{in} = P_{abs} = 0.40\ W$$



Note the transmission line length has the **special case** $\ell = \lambda/4$, therefore the input impedance is easily computed:

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{50^2}{125} = 20\Omega$$

A **conjugate match** ($Z_{in} = Z_g^*$)!



Thus, the power absorbed by Z_{in} (i.e., P_{in}) is:

$$\begin{aligned} P_{in} &= \frac{1}{2} Re\left\{ V_{in} I_{in}^{*} \right\} \\ &= \frac{1}{2} Re\left\{ \left(\frac{V_g}{2}\right)\left(\frac{V_g^{*}}{20+20}\right) \right\} \\ &= \frac{1}{8} Re\left\{ \left|V_g\right|^2 \right\} \frac{1}{20} \\ &= \frac{\left|V_g\right|^2}{160} \end{aligned}$$

And since we know that P_{in} = 0.4 W, we can conclude:

$$P_{in} = 0.4 = \frac{\left|V_g\right|^2}{160} \qquad \Rightarrow \qquad \left|V_g\right| = \sqrt{160(0.4)} = 8.0 \quad V$$