

4.3 - The Scattering Matrix

Reading Assignment: pp. 174-183

Admittance and Impedance matrices use the quantities $I(z)$, $V(z)$, and $Z(z)$ (or $Y(z)$).

Q: Is there an equivalent matrix for transmission line activity expressed in terms of $V^+(z)$, $V^-(z)$, and $\Gamma(z)$?

A: Yes! Its called the scattering matrix.

HO: THE SCATTERING MATRIX

Q: Can we likewise determine something physical about our device or network by simply looking at its scattering matrix?

A: HO: MATCHED, RECIPROCAL, LOSSLESS

EXAMPLE: A LOSSLESS, RECIPROCAL DEVICE

Q: Isn't all this linear algebra a bit academic? I mean, it can't help us design components, can it?

A: It sure can! An analysis of the scattering matrix can tell us if a certain device is even possible to construct, and if so, what the form of the device must be.

HO: THE MATCHED, LOSSLESS, RECIPROCAL 3-PORT NETWORK

HO: THE MATCHED, LOSSLESS, RECIPROCAL 4-PORT NETWORK

Q: But how are scattering parameters useful? How do we use them to solve or analyze real microwave circuit problems?

A: Study the examples provided below!

EXAMPLE: THE SCATTERING MATRIX

EXAMPLE: SCATTERING PARAMETERS

Q: OK, but how can we determine the scattering matrix of a device?

A: We must carefully apply our transmission line theory!

EXAMPLE: DETERMINING THE SCATTERING MATRIX

Q: Determining the Scattering Matrix of a multi-port device would seem to be particularly laborious. Is there any way to simplify the process?

A: Many (if not most) of the useful devices made by us humans exhibit a high degree of symmetry. This can greatly simplify circuit analysis—if we know how to exploit it!

HO: CIRCUIT SYMMETRY

EXAMPLE: USING SYMMETRY TO DETERMINING A SCATTERING MATRIX

Q: Is there any other way to use circuit symmetry to our advantage?

A: Absolutely! One of the most powerful tools in circuit analysis is **Odd-Even Mode** analysis.

HO: SYMMETRIC CIRCUIT ANALYSIS

HO: ODD-EVEN MODE ANALYSIS

EXAMPLE: ODD-EVEN MODE CIRCUIT ANALYSIS

Q: Aren't you finished with this section yet?

A: Just one more very important thing.

HO: GENERALIZED SCATTERING PARAMETERS

EXAMPLE: THE SCATTERING MATRIX OF A CONNECTOR

The Scattering Matrix

At "low" frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

But, at microwave frequencies, it is **difficult** to measure total currents and voltages!



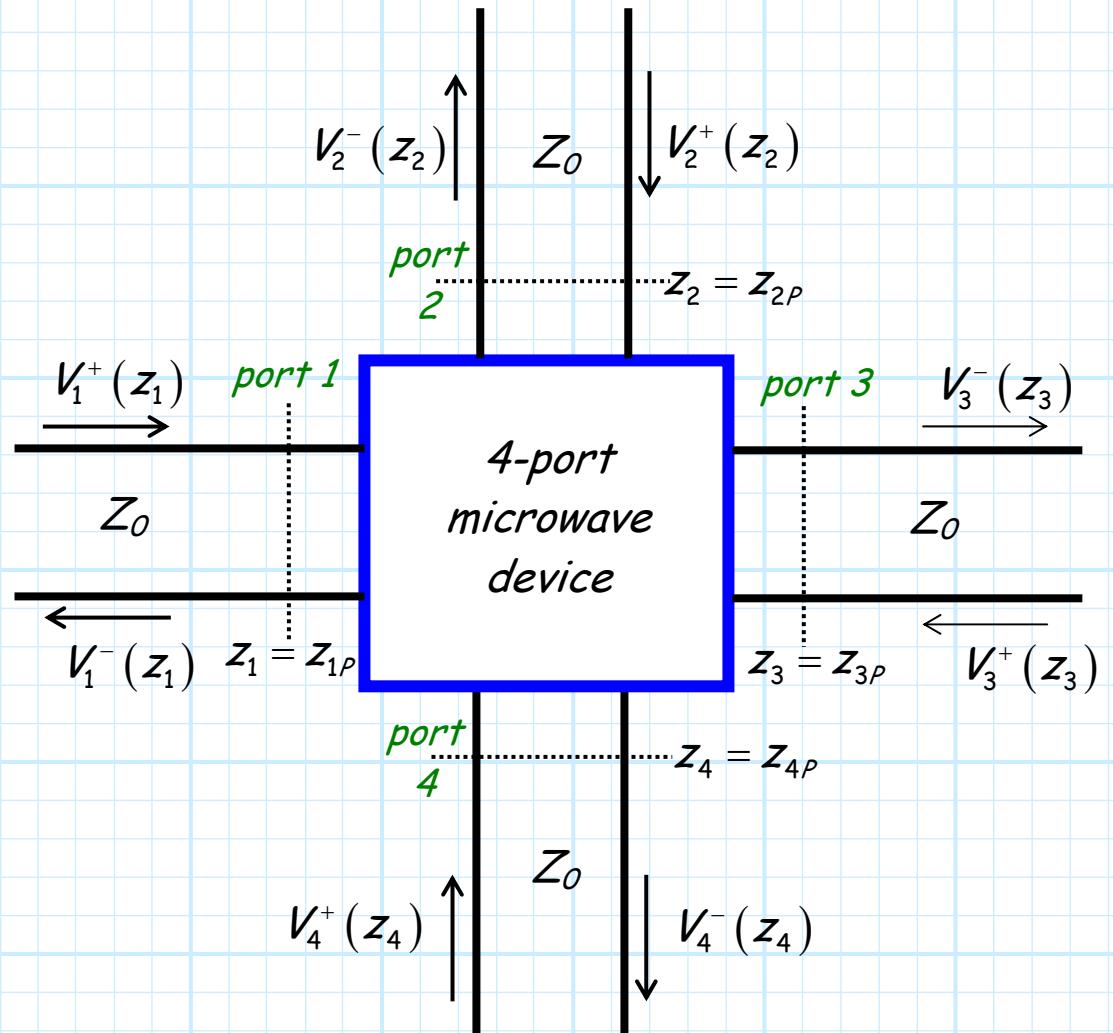
- * Instead, we can measure the **magnitude** and **phase** of each of the two transmission line waves $V^+(z)$ and $V^-(z)$.
- * In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at all other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a given frequency ω , and a given line impedance Z_0 .

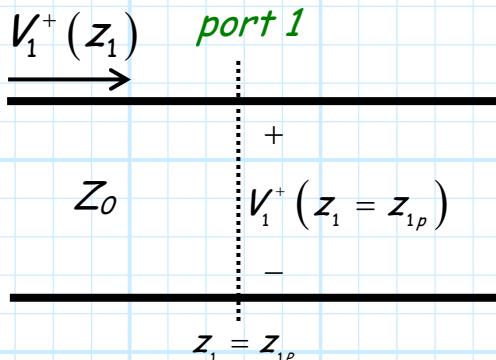
Consider now the 4-port microwave device shown below:

Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. Note the negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.

→ Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters!**

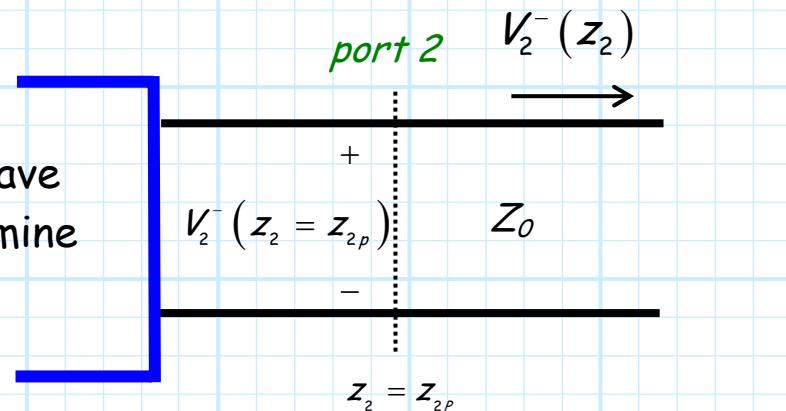


Say there exists an **incident wave** on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing into **port 1**, at the port 1 plane (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing out of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is known as the **scattering parameter** S_{21} :

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_{02}^- e^{+j\beta z_{2P}}}{V_{01}^+ e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} e^{+j\beta(z_{2P} + z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \quad \text{and} \quad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

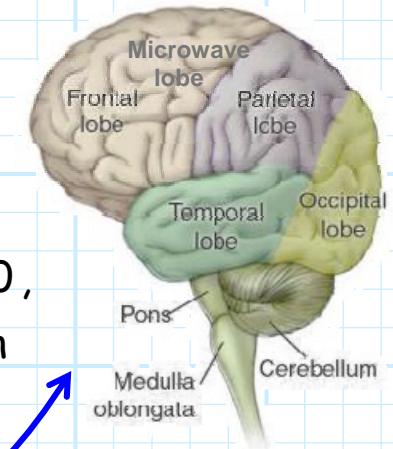
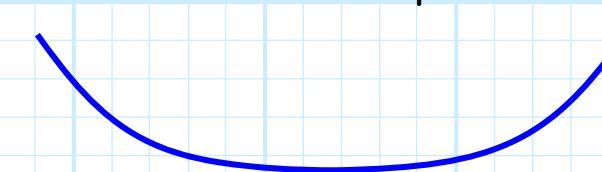
Thus, more **generally**, the ratio of the wave incident on port n to the wave emerging from port m is:

$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \quad (\text{given that } V_k^+(z_k) = 0 \text{ for all } k \neq n)$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- e^{+j\beta 0}}{V_{0n}^+ e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

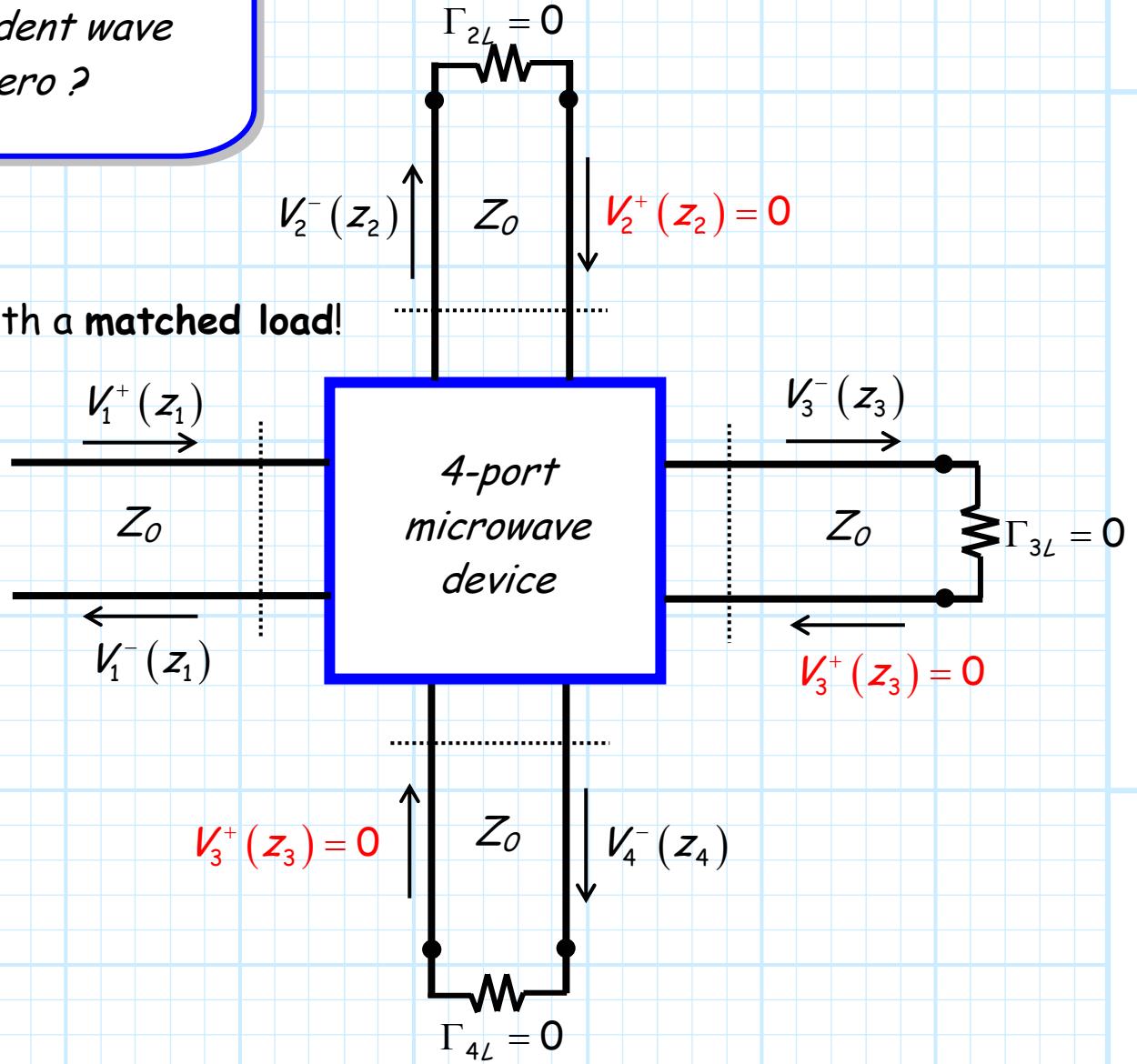
We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!





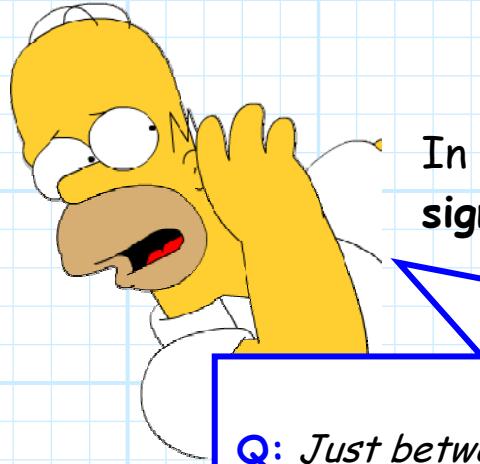
Q: But how do we ensure that **only one** incident wave is non-zero?

A: Terminate all other ports with a matched load!



Note that if the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:

$$V_n^+(z_n) = 0$$



In other words, terminating a port ensures that there will be no signal incident on that port!

Q: Just between you and me, I think you've messed this up! In all previous handouts you said that if $\Gamma_L = 0$, the wave in the **minus** direction would be zero:

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

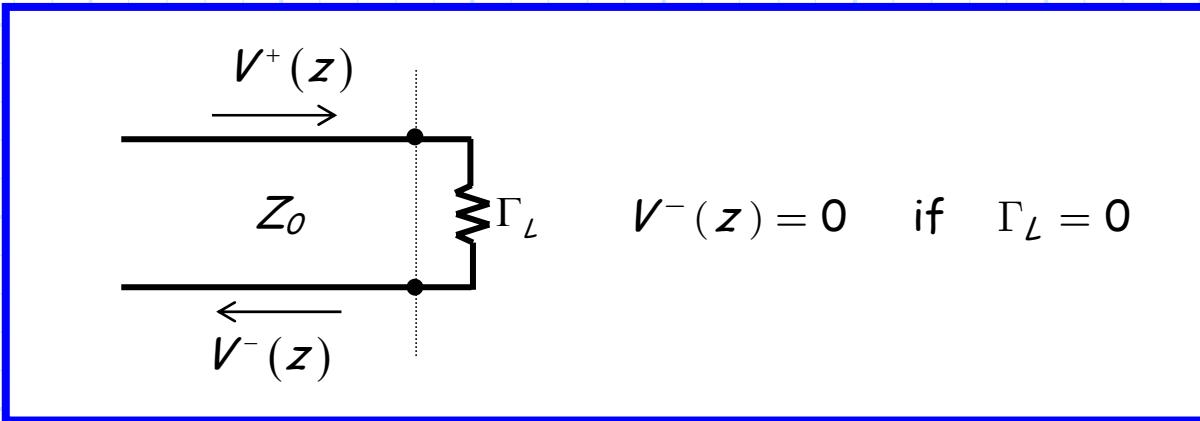
but just now you said that the wave in the **positive** direction would be zero:

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

Of course, there is **no way** that **both** statements can be correct!

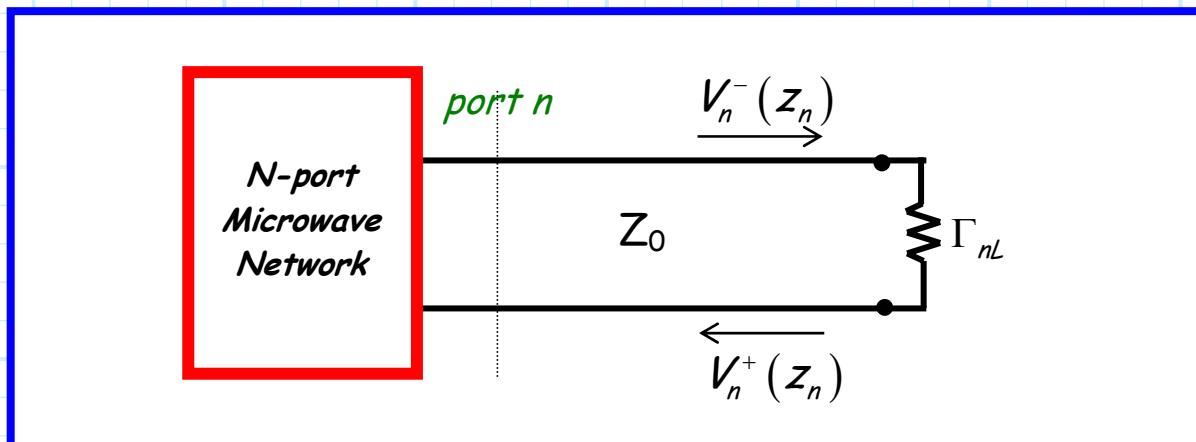
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we originally analyzed this case:



In this original case, the wave incident on the load is $V^+(z)$ (plus direction), while the reflected wave is $V^-(z)$ (minus direction).

Contrast this with the case we are now considering:



For this current case, the situation is **reversed**. The wave incident on the load is now denoted as $V_n^-(z_n)$ (coming **out** of port n), while the wave reflected off the load is now denoted as $V_n^+(z_n)$ (going **into** port n).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that for some load Γ_L :

$$V^{\text{reflected}}(z = z_L) = \Gamma_L V^{\text{incident}}(z = z_L)$$



*For each case, you must be able to correctly identify the mathematical statement describing the wave **incident** on, and **reflected** from, some passive load.*

*Like most equations in engineering, the **variable names** can change, but the **physics** described by the mathematics will **not**!*

Now, back to our discussion of **S-parameters**. We found that if $z_{np} = 0$ for all ports n , the scattering parameters could be directly written in terms of wave amplitudes V_{0n}^+ and V_{0n}^- .

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \quad (\text{when } V_k^+(z_k) = 0 \text{ for all } k \neq n)$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \quad (\text{when all ports, except port } n, \text{ are terminated in matched loads})$$

One more important note—notice that for the ports terminated in matched loads (i.e., those ports with no incident wave), the voltage of the exiting wave is also the **total voltage**!

$$\begin{aligned} V_m(z_m) &= V_{0m}^+ e^{-j\beta z_m} + V_{0m}^- e^{+j\beta z_m} \\ &= 0 + V_{0m}^- e^{+j\beta z_m} \\ &= V_{0m}^- e^{+j\beta z_m} \quad (\text{for all terminated ports}) \end{aligned}$$

Thus, the value of the exiting wave at each terminated port is likewise the value of the total voltage at those ports:

$$\begin{aligned} V_m(0) &= V_{0m}^+ + V_{0m}^- \\ &= 0 + V_{0m}^- \\ &= V_{0m}^- \quad (\text{for all terminated ports}) \end{aligned}$$

And so, we can express some of the scattering parameters equivalently as:

$$S_{mn} = \frac{V_m(0)}{V_{0n}^+} \quad (\text{for terminated port } m, \text{ i.e., for } m \neq n)$$

You might find this result helpful if attempting to determine scattering parameters where $m \neq n$ (e.g., S_{21}, S_{43}, S_{13}), as we can often use traditional circuit theory to easily determine the total port voltage $V_m(0)$.

However, we cannot use the expression above to determine the scattering parameters when $m = n$ (e.g., S_{11}, S_{22}, S_{33}).



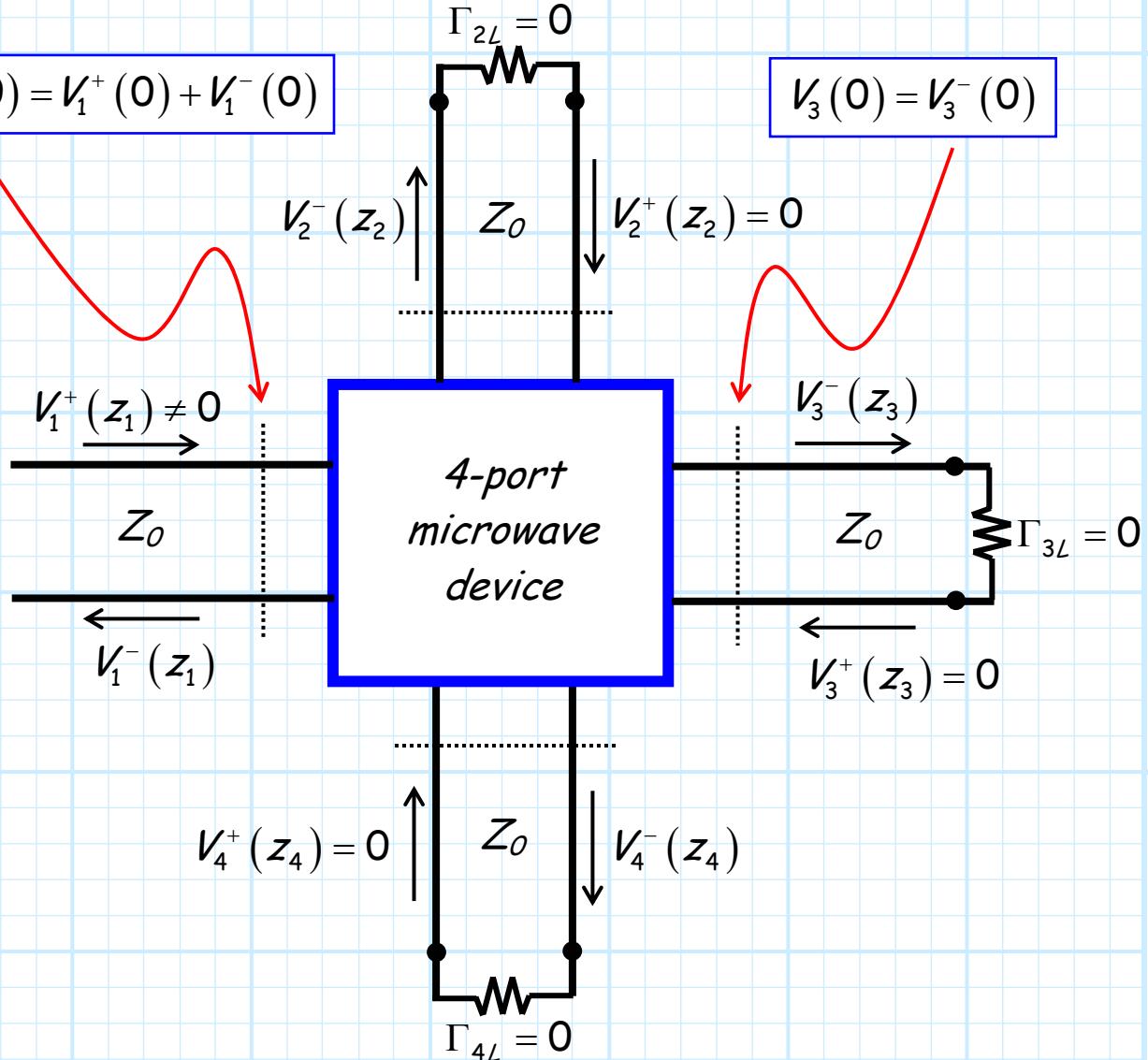
Think about this! The scattering parameters for these cases are:

$$S_{nn} = \frac{V_{0n}^-}{V_{0n}^+}$$

$$V_1(0) = V_1^+(0) + V_1^-(0)$$

Therefore, port n is a port where there actually is some incident wave V_{0n}^+ (port n is not terminated in a matched load!).

And thus, the total voltage is not simply the value of the exiting wave, as both an incident wave and exiting wave exists at port n .



Typically, it is much more difficult to determine/measure the scattering parameters of the form S_{nn} , as opposed to scattering parameters of the form S_{mn} (where $m \neq n$) where there is only an exiting wave from port m !

We can use the scattering matrix to determine the solution for a more general circuit—one where the ports are not terminated in matched loads!



Q: I'm not understanding the importance scattering parameters. How are they useful to us microwave engineers?

A: Since the device is linear, we can apply superposition. The output at any port due to all the incident waves is simply the coherent sum of the output at that port due to each wave!

For example, the output wave at port 3 can be determined by (assuming $z_{np} = 0$):

$$V_{03}^- = S_{34} V_{04}^+ + S_{33} V_{03}^+ + S_{32} V_{02}^+ + S_{31} V_{01}^+$$

More generally, the output at port m of an N -port device is:

$$V_{0m}^- = \sum_{n=1}^N S_{mn} V_{0n}^+ \quad (z_{np} = 0)$$

This expression can be written in **matrix** form as:

$$\mathbf{V}^- = \mathcal{S} \mathbf{V}^+$$

Where \mathbf{V}^- is the **vector**:

$$\mathbf{V}^- = [V_{01}^-, V_{02}^-, V_{03}^-, \dots, V_{0N}^-]^T$$

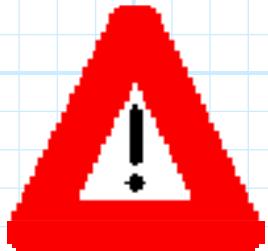
and \mathbf{V}^+ is the **vector**:

$$\mathbf{V}^+ = [V_{01}^+, V_{02}^+, V_{03}^+, \dots, V_{0N}^+]^T$$

Therefore \mathcal{S} is the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} S_{11} & \dots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \dots & S_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N -port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!



But **beware!** The values of the scattering matrix for a particular device or network, just like Γ_L , are **frequency dependent!** Thus, it may be more instructive to **explicitly** write:

$$\mathcal{S}(\omega) = \begin{bmatrix} S_{11}(\omega) & \dots & S_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega) & \dots & S_{mn}(\omega) \end{bmatrix}$$

Also realize that—also just like Γ_L —the scattering matrix is dependent on **both the device/network and the Z_0 value of the transmission lines connected to it.**

Thus, a device connected to transmission lines with $Z_0 = 50\Omega$ will have a **completely different scattering matrix** than that same device connected to transmission lines with $Z_0 = 100\Omega$!!!

Matched, Lossless, Reciprocal Devices

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify it as being **matched**. Let's examine **each** of these three characteristics, and how they relate to the scattering matrix.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when all other ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (but **only** that port!).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m \text{ if matched}$$

We find therefore that a matched device will exhibit a scattering matrix where all diagonal elements are zero.

Therefore:

$$\mathcal{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched**, three port device.

Lossless

For a lossless device, all of the power that delivered to each device port must eventually find its way out!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Recall the **power incident** on some port m is related to the amplitude of the **incident wave** (V_{0m}^+) as:

$$P_m^+ = \frac{|V_{0m}^+|^2}{2Z_0}$$

While power of the **wave exiting** the port is:

$$P_m^- = \frac{|V_{0m}^-|^2}{2Z_0}$$

Thus, the power **delivered to** (absorbed by) that port is the **difference** of the two:

$$\Delta P_m = P_m^+ - P_m^- = \frac{|V_{0m}^+|^2}{2Z_0} - \frac{|V_{0m}^-|^2}{2Z_0}$$

Thus, the **total power incident** on an N -port device is:

$$P^+ = \sum_{m=1}^N P_m^+ = \frac{1}{2Z_0} \sum_{m=1}^N |V_{0m}^+|^2$$

Note that:

$$\sum_{m=1}^N |V_{0m}^+|^2 = (\mathbf{V}^+)^H \mathbf{V}^+$$

where operator H indicates the **conjugate transpose** (i.e., Hermitian transpose) operation, so that $(\mathbf{V}^+)^H \mathbf{V}^+$ is the **inner product** (i.e., dot product, or scalar product) of complex vector \mathbf{V}^+ with itself.

Thus, we can write the **total power incident** on the device as:

$$P^+ = \frac{1}{2Z_0} \sum_{m=1}^N |V_{0m}^+|^2 = \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0}$$

Similarly, we can express the total power of the waves exiting our M -port network to be:

$$P^- = \frac{1}{2Z_0} \sum_{m=1}^N |V_{0m}^-|^2 = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0}$$

Now, recalling that the incident and exiting wave amplitudes are related by the scattering matrix of the device:

$$\mathbf{V}^- = \mathcal{S} \mathbf{V}^+$$

Thus we find:

$$P^- = \frac{(\mathbf{V}^-)^H \mathbf{V}^-}{2Z_0} = \frac{(\mathbf{V}^+)^H \mathcal{S}^H \mathcal{S} \mathbf{V}^+}{2Z_0}$$

Now, the total power delivered to the network is:

$$\Delta P = \sum_{m=1}^M \Delta P_m = P^+ - P^-$$

Or explicitly:

$$\begin{aligned}\Delta P &= P^+ - P^- \\ &= \frac{(\mathbf{V}^+)^H \mathbf{V}^+}{2Z_0} - \frac{(\mathbf{V}^+)^H \mathcal{S}^H \mathcal{S} \mathbf{V}^+}{2Z_0} \\ &= \frac{1}{2Z_0} (\mathbf{V}^+)^H (\mathcal{I} - \mathcal{S}^H \mathcal{S}) \mathbf{V}^+\end{aligned}$$

where \mathcal{I} is the identity matrix.

Q: Is there actually some point to this long, rambling, complex presentation?

A: Absolutely! If our M-port device is lossless then the total power exiting the device must always be equal to the total power incident on it.

If network is lossless, then $P^+ = P^-$.

Or stated another way, the total power delivered to the device (i.e., the power absorbed by the device) must always be zero if the device is lossless!

If network is lossless, then $\Delta P = 0$

Thus, we can conclude from our math that for a lossless device:

$$\Delta P = \frac{1}{2Z_0} (\mathbf{V}^+)^H (\mathcal{I} - \mathcal{S}^H \mathcal{S}) \mathbf{V}^+ = 0 \quad \text{for all } \mathbf{V}^+$$

This is true only if:

$$\mathcal{I} - \mathcal{S}^H \mathcal{S} = 0 \quad \Rightarrow \quad \mathcal{S}^H \mathcal{S} = \mathcal{I}$$

Thus, we can conclude that the scattering matrix of a lossless device has the characteristic:

If a network is lossless, then $\mathcal{S}^H \mathcal{S} = \mathcal{I}$

Q: Huh? What exactly is this supposed to tell us?

A: A matrix that satisfies $\mathcal{S}^H \mathcal{S} = \mathcal{I}$ is a special kind of matrix known as a unitary matrix.

If a network is lossless, then its scattering matrix \mathcal{S} is unitary.

Q: How do I recognize a unitary matrix if I see one?

A: The columns of a unitary matrix form an orthonormal set!

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

The diagram shows a 4x4 matrix labeled \mathcal{S} . The columns are labeled "matrix" and "columns". Red arrows point from the first three columns to the word "matrix", and a blue arrow points from the fourth column to the word "columns".

In other words, each **column** of the scattering matrix will have a **magnitude equal to one**:

$$\sum_{m=1}^N |S_{mn}|^2 = 1 \quad \text{for all } n$$

while the inner product (i.e., dot product) of **dissimilar columns** must be zero.

$$\sum_{n=1}^N S_{ni} S_{nj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \dots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

In other words, dissimilar columns are **orthogonal**.

Consider, for example, a lossless **three-port** device. Say a signal is incident on port 1, and that all other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{|V_{01}^+|^2}{2Z_0}$$

while the power **exiting** the device at each port is:

$$P_m^- = \frac{|V_{0m}^-|^2}{2Z_0} = \frac{|S_{m1}V_{01}^-|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

The **total** power exiting the device is therefore:

$$\begin{aligned} P^- &= P_1^- + P_2^- + P_3^- \\ &= |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ \\ &= (|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2) P_1^+ \end{aligned}$$

Since this device is **lossless**, then the **incident power (only on port 1)** is **equal to exiting power** (i.e., $P^- = P_1^+$). This is true **only** if:

$$|\textcolor{red}{S_{11}}|^2 + |\textcolor{red}{S_{21}}|^2 + |\textcolor{red}{S_{31}}|^2 = 1$$

Of course, this will likewise be true if the incident wave is placed on **any of the other ports of this lossless device**:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

We can state in general then that:

$$\sum_{m=1}^3 |S_{mn}|^2 = 1 \quad \text{for all } n$$

In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$S = \begin{bmatrix} 0 & \frac{1}{2} & j\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & j\frac{\sqrt{3}}{2} \\ j\frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & j\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\mathcal{S}^T = \mathcal{S} \quad \text{if reciprocal}$$

where T indicates (non-conjugate) transpose.

An example of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\mathbf{S}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10 - j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example: A Lossless, Reciprocal Network

A lossless, reciprocal 3-port device has S -parameters of $S_{11} = 1/2$, $S_{31} = 1/\sqrt{2}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.



→ Find the remaining 6 scattering parameters.

Q: *This problem is clearly impossible—you have not provided us with sufficient information!*

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Start with what we **currently** know:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{2} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Because the device is **reciprocal**, we then also know:

$$S_{21} = S_{12}$$

$$S_{13} = S_{31} = \frac{1}{\sqrt{2}}$$

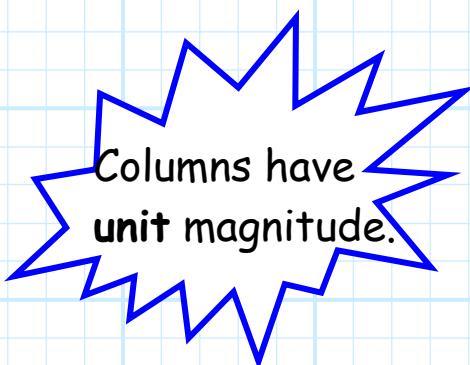
$$S_{32} = S_{23}$$

And therefore:

$$\mathcal{S} = \begin{bmatrix} \frac{1}{2} & \mathcal{S}_{21} & \frac{1}{\sqrt{2}} \\ \mathcal{S}_{21} & \mathcal{S}_{22} & \mathcal{S}_{32} \\ \frac{1}{\sqrt{2}} & \mathcal{S}_{32} & 0 \end{bmatrix}$$

Now, since the device is **lossless**, we know that:

$$\begin{aligned} 1 &= |\mathcal{S}_{11}|^2 + |\mathcal{S}_{21}|^2 + |\mathcal{S}_{31}|^2 \\ &= \left(\frac{1}{2}\right)^2 + |\mathcal{S}_{21}|^2 + \left(\frac{1}{\sqrt{2}}\right)^2 \end{aligned}$$



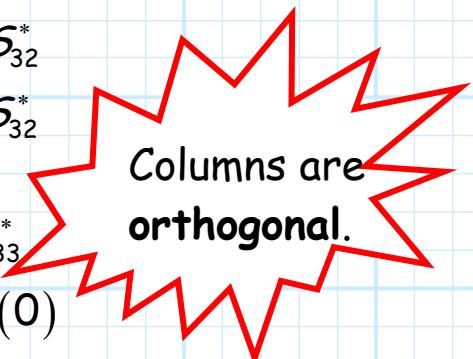
$$\begin{aligned} 1 &= |\mathcal{S}_{12}|^2 + |\mathcal{S}_{22}|^2 + |\mathcal{S}_{32}|^2 \\ &= |\mathcal{S}_{21}|^2 + |\mathcal{S}_{22}|^2 + |\mathcal{S}_{32}|^2 \end{aligned}$$

$$\begin{aligned} 1 &= |\mathcal{S}_{13}|^2 + |\mathcal{S}_{23}|^2 + |\mathcal{S}_{33}|^2 \\ &= \left(\frac{1}{2}\right)^2 + |\mathcal{S}_{32}|^2 + \left(\frac{1}{\sqrt{2}}\right)^2 \end{aligned}$$

and:

$$\begin{aligned} 0 &= \mathcal{S}_{11}\mathcal{S}_{12}^* + \mathcal{S}_{21}\mathcal{S}_{22}^* + \mathcal{S}_{31}\mathcal{S}_{32}^* \\ &= \frac{1}{2}\mathcal{S}_{21}^* + \mathcal{S}_{21}\mathcal{S}_{22}^* + \frac{1}{\sqrt{2}}\mathcal{S}_{32}^* \end{aligned}$$

$$\begin{aligned} 0 &= \mathcal{S}_{11}\mathcal{S}_{13}^* + \mathcal{S}_{21}\mathcal{S}_{23}^* + \mathcal{S}_{31}\mathcal{S}_{33}^* \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \mathcal{S}_{21}\mathcal{S}_{32}^* + \frac{1}{\sqrt{2}}(0) \end{aligned}$$



$$\begin{aligned} 0 &= \mathcal{S}_{12}\mathcal{S}_{13}^* + \mathcal{S}_{22}\mathcal{S}_{23}^* + \mathcal{S}_{32}\mathcal{S}_{33}^* \\ &= \mathcal{S}_{21}\left(\frac{1}{\sqrt{2}}\right) + \mathcal{S}_{22}\mathcal{S}_{32}^* + \mathcal{S}_{32}(0) \end{aligned}$$

These six expressions simplify to:

$$|S_{21}| = \frac{1}{2}$$

$$1 = |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2$$

$$|S_{32}| = \frac{1}{\sqrt{2}}$$

$$0 = \frac{1}{2} S_{21} + S_{21} S_{22} + \frac{1}{\sqrt{2}} S_{32}$$

$$0 = \frac{1}{(2\sqrt{2})} + S_{21} S_{32}$$

$$0 = S_{21} \left(\frac{1}{\sqrt{2}} \right) + S_{22} S_{32}$$

where we have used the fact that since the elements are all real, then $S_{21}^* = S_{21}$ (etc.).



Q: I count the expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is no solution!

A: Actually, we have six real equations and six real unknowns, since scattering element has a magnitude and phase. In this case we know the values are real, and thus the phase is either 0° or 180° (i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

From the first three equations, we can find the magnitudes:

$$|S_{21}| = \frac{1}{2}$$

$$|S_{22}| = \frac{1}{2}$$

$$|S_{32}| = \frac{1}{\sqrt{2}}$$

and from the last three equations we find the phase:

$$S_{21} = \frac{1}{2}$$

$$S_{22} = \frac{1}{2}$$

$$S_{32} = -\frac{1}{\sqrt{2}}$$

Thus, the scattering matrix for this lossless, reciprocal device is:

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

A Matched, Lossless, Reciprocal 3-Port Network

Consider a 3-port device. Such a device would have a scattering matrix :

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Assuming the device is passive and made of simple (isotropic) materials, the device will be reciprocal, so that:

$$S_{21} = S_{12} \quad S_{31} = S_{13} \quad S_{23} = S_{32}$$

Likewise, if it is matched, we know that $S_{11} = S_{22} = S_{33} = 0$.

As a result, a lossless, reciprocal device would have a scattering matrix of the form:

$$\mathcal{S} = \begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{21} & 0 & S_{32} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

Just 3 non-zero scattering parameters define the entire matrix!

Likewise, if we wish for this network to be **lossless**, the scattering matrix must be **unitary**, and therefore:

$$\begin{array}{ll} |S_{21}|^2 + |S_{31}|^2 = 1 & S_{31}^* S_{32} = 0 \\ |S_{21}|^2 + |S_{32}|^2 = 1 & S_{21}^* S_{32} = 0 \\ |S_{31}|^2 + |S_{32}|^2 = 1 & S_{21}^* S_{31} = 0 \end{array}$$

Since each complex value S is represented by **two real numbers** (i.e., real and imaginary parts), the equations above result in **9 real equations**. The problem is, the 3 complex values S_{21} , S_{31} and S_{32} are represented by only **6 real unknowns**.



We have **over constrained** our problem ! There are **no solutions** to these equations !

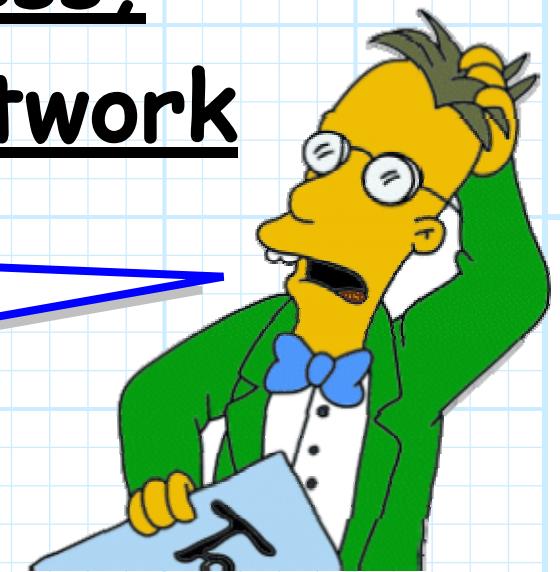
As unlikely as it might seem, this means that a matched, lossless, reciprocal 3-port device of any kind is a physical impossibility!

You can make a lossless reciprocal 3-port device, or a matched reciprocal 3-port device, or even a matched, lossless (but non-reciprocal) 3-port network.

*But try as you might, you **cannot** make a lossless, matched, and reciprocal three port component!*

The Matched, Lossless, Reciprocal, 4-Port Network

Guess what! I have determined that—unlike a 3-port device—a matched, lossless, reciprocal 4-port device is physically possible! In fact, I've found two general solutions!



The first solution is referred to as the **symmetric** solution:

$$S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Note for this symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α , $j\beta$, and two zeros)!

The second solution is referred to as the **anti-symmetric** solution:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for this anti-symmetric solution, **two rows and two columns have the same four values** (i.e., α , β , and two zeros), while the **other two row and columns have (slightly) different values** (α , $-\beta$, and two zeros)

It is **quite** evident that each of these solutions are **matched and reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional constraint** on the values of α, β . Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^N |S_{mn}|^2 = 1 \quad \text{for all } n$$

Applying this to the **symmetric** case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

Likewise, for the anti-symmetric case, we also get

$$|\alpha|^2 + |\beta|^2 = 1$$

It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must **related** as:

$$|\alpha|^2 + |\beta|^2 = 1$$

Generally speaking, we will find that $|\alpha| \geq |\beta|$. Given the constraint on these two values, we can thus conclude that:

$$0 \leq |\beta| \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} \leq |\alpha| \leq 1$$

Example: Scattering Parameters

Consider a **two-port device** with a scattering matrix (at some specific frequency ω_0):

$$\mathcal{S}(\omega = \omega_0) = \begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

and $Z_0 = 50\Omega$.

Say that the transmission line connected to **port 2** of this device is terminated in a **matched load**, and that the wave incident on **port 1** is:

$$V_1^+(z_1) = -j2 e^{-j\beta z_1}$$

where $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$.
2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$.
3. the net power flowing into port 1

1. Since the incident wave on port 1 is:

$$V_1^+(z_1) = -j2 e^{-j\beta z_1}$$

we can conclude (since $z_{1P} = 0$):

$$\begin{aligned} V_1^+(z_1 = z_{1P}) &= -j2 e^{-j\beta z_{1P}} \\ &= -j2 e^{-j\beta(0)} \\ &= -j2 \end{aligned}$$

and since port 2 is matched (and only because its matched!), we find:

$$\begin{aligned} V_1^-(z_1 = z_{1P}) &= S_{11} V_1^+(z_1 = z_{1P}) \\ &= 0.1(-j2) \\ &= -j0.2 \end{aligned}$$

The voltage at port 1 is thus:

$$\begin{aligned} V_1(z_1 = z_{1P}) &= V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) \\ &= -j2.0 - j0.2 \\ &= -j2.2 \\ &= 2.2 e^{-j\pi/2} \end{aligned}$$

Likewise, since port 2 is matched:

$$V_2^+(z_2 = z_{2P}) = 0$$

And also:

$$\begin{aligned} V_2^- (z_2 = z_{2P}) &= S_{21} V_1^+ (z_1 = z_{1P}) \\ &= j0.7 (-j2) \\ &= 1.4 \end{aligned}$$

Therefore:

$$\begin{aligned} V_2 (z_2 = z_{2P}) &= V_2^+ (z_2 = z_{2P}) + V_2^- (z_2 = z_{2P}) \\ &= 0 + 1.4 \\ &= 1.4 \\ &= 1.4 e^{-j0} \end{aligned}$$

2. The port currents can be easily determined from the results of the previous section.

$$\begin{aligned} I_1 (z_1 = z_{1P}) &= I_1^+ (z_1 = z_{1P}) - I_1^- (z_1 = z_{1P}) \\ &= \frac{V_1^+ (z_1 = z_{1P})}{Z_0} - \frac{V_1^- (z_1 = z_{1P})}{Z_0} \\ &= -j \frac{2.0}{50} + j \frac{0.2}{50} \\ &= -j \frac{1.8}{50} \\ &= -j0.036 \\ &= 0.036 e^{-j\pi/2} \end{aligned}$$

and:

$$\begin{aligned}
 I_2(z_2 = z_{2P}) &= I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) \\
 &= \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0} \\
 &= \frac{0}{50} - \frac{1.4}{50} \\
 &= -0.028 \\
 &= 0.028 e^{+j\pi}
 \end{aligned}$$

3. The net power flowing into port 1 is:

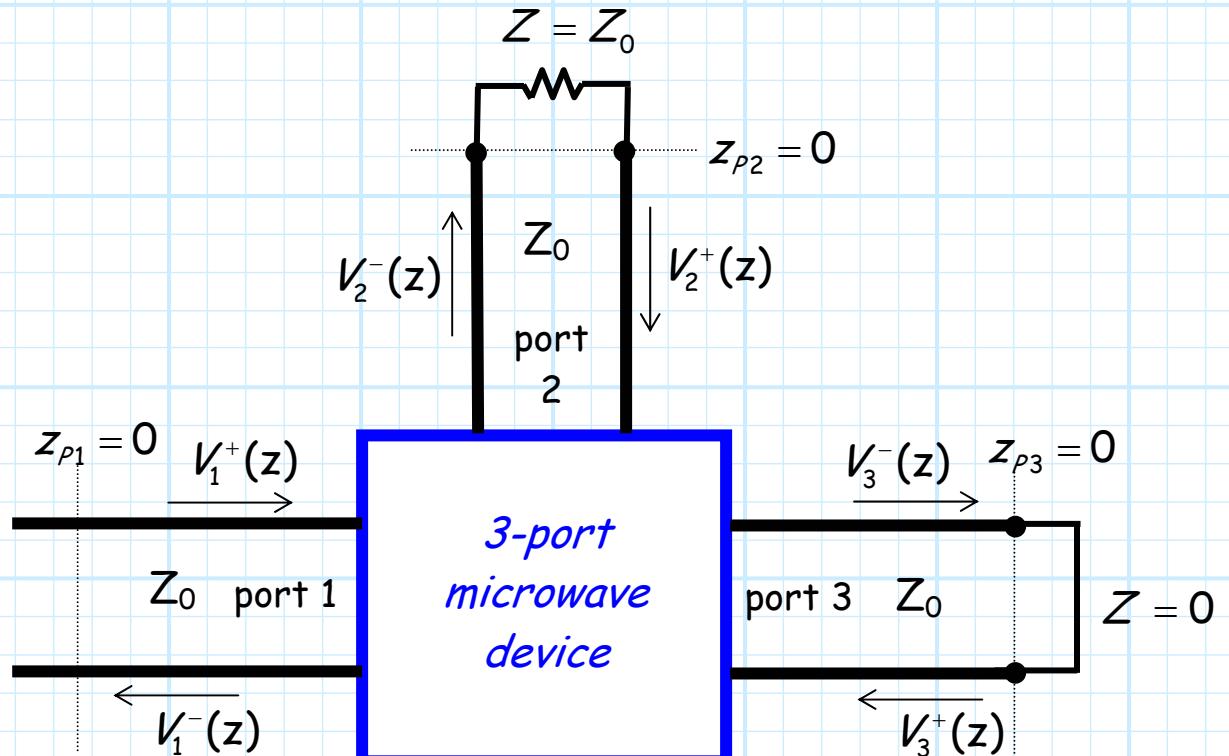
$$\begin{aligned}
 \Delta P_1 &= P_1^+ - P_1^- \\
 &= \frac{|V_{01}|^2}{2Z_0} - \frac{|V_{01}^-|^2}{2Z_0} \\
 &= \frac{(2)^2 - (0.2)^2}{2(50)} \\
 &= 0.0396 \text{ Watts}
 \end{aligned}$$

Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the scattering matrix:

$$\mathcal{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



Because of the matched load at port 2 (i.e., $\Gamma_L = 0$), we know that:

$$\frac{V_2^+(z_2 = 0)}{V_2^-(z_2 = 0)} = \frac{V_{02}^+}{V_{02}^-} = 0$$

and therefore:

$$V_{02}^+ = 0$$



*You've made a terrible mistake!
Fortunately, I was here to
correct it for you—since $\Gamma_L = 0$,
the constant V_{02}^- (not V_{02}^+) is
equal to zero.*

NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on port 2). Therefore, $V_{02}^+ = 0$ is **correct!**

Likewise, because of the short circuit at port 3 ($\Gamma_L = -1$):

$$\frac{V_3^+(z_3 = 0)}{V_3^-(z_3 = 0)} = \frac{V_{03}^+}{V_{03}^-} = -1$$

and therefore:

$$V_{03}^+ = -V_{03}^-$$

Problem:

- a) Find the reflection coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

- b) Find the transmission coefficient from port 1 to port 2, i.e.,

$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

I am amused by the trivial problems that you apparently find so difficult. I know that:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$



NO!!! The above statement is not correct!



Remember, $V_{01}^-/V_{01}^+ = S_{11}$ only if ports 2 and 3 are terminated in **matched loads**! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1 , we must start with the three equations provided by the scattering matrix:

$$V_{01}^- = 0.2 V_{02}^+ + 0.5 V_{03}^+$$

$$V_{02}^- = 0.5 V_{01}^+ + 0.2 V_{03}^+$$

$$V_{03}^- = 0.5 V_{01}^+ + 0.5 V_{02}^+$$

and the two equations provided by the attached loads:

$$V_{02}^+ = 0$$

$$V_{03}^+ = -V_{03}^-$$

We can divide all of these equations by V_{01}^+ , resulting in:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = 0.2 \frac{V_{02}^+}{V_{01}^+} + 0.5 \frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 + 0.2 \frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5 + 0.5 \frac{V_{02}^+}{V_{01}^+}$$

$$\frac{V_{02}^+}{V_{01}^+} = 0$$

$$\frac{V_{03}^+}{V_{01}^+} = -\frac{V_{03}^-}{V_{01}^+}$$

Look what we have—5 equations and 5 unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = -0.5 \frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 - 0.2 \frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5$$

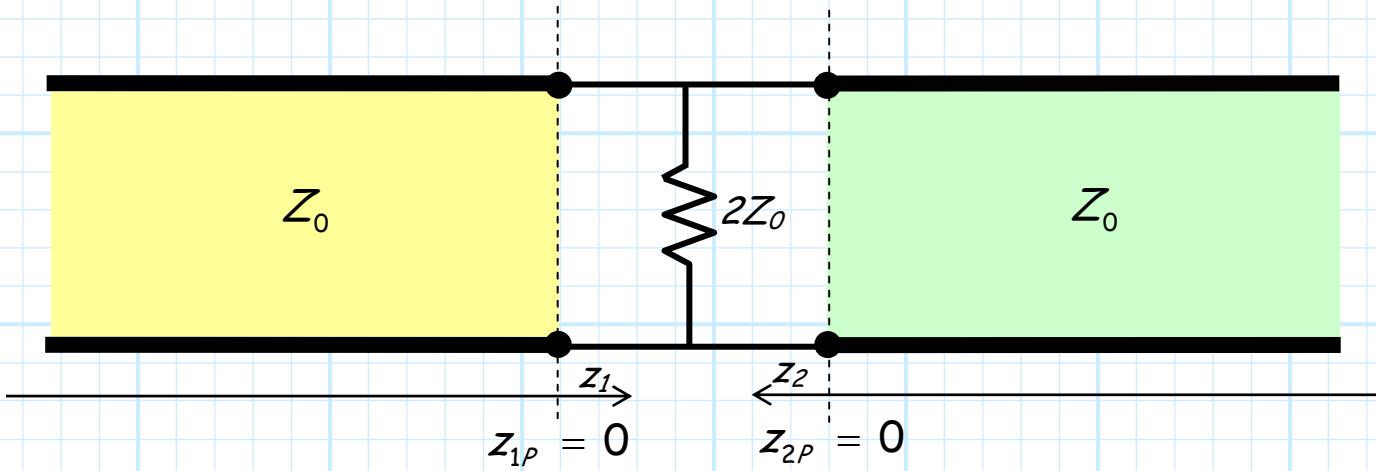
Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$T_{21} = 0.5 - 0.2(0.5) = 0.4$$

Example: Determining the Scattering Matrix

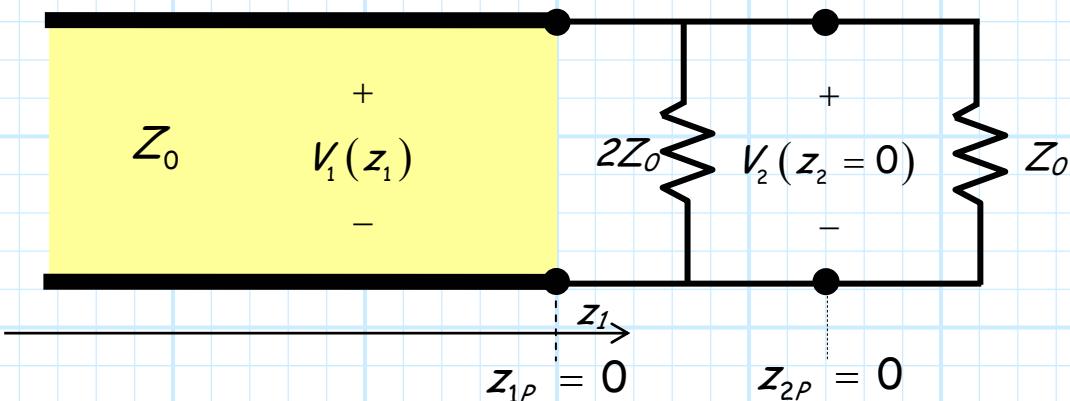
Let's determine the scattering matrix of this two-port device:



The first step is to terminate port 2 with a matched load, and then determine the values:

$$V_1^-(z_1 = z_{P1}) \quad \text{and} \quad V_2^-(z_2 = z_{P2})$$

in terms of $V_1^+(z_1 = z_{P1})$.



Recall that since port 2 is matched, we know that:

$$V_2^+(z_2 = z_{2P}) = 0$$

And thus:

$$\begin{aligned} V_2(z_2 = 0) &= V_2^+(z_2 = 0) + V_2^-(z_2 = 0) \\ &= 0 + V_2^-(z_2 = 0) \\ &= V_2^-(z_2 = 0) \end{aligned}$$

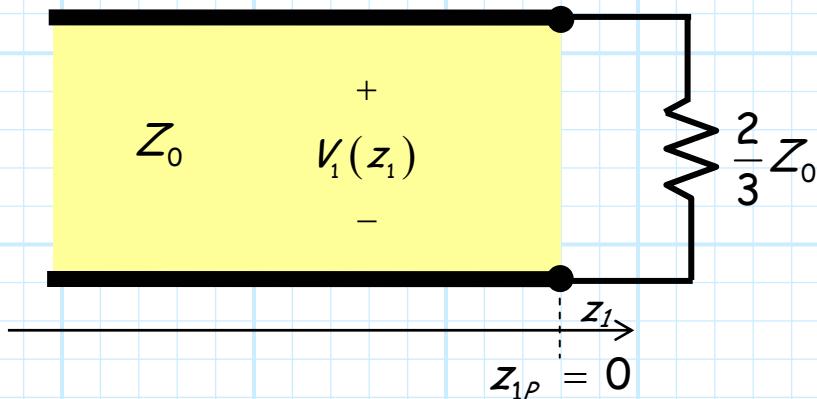
In other words, we simply need to determine $V_2^-(z_2 = 0)$ in order to find $V_2^-(z_2 = 0)$!

However, determining $V_1^-(z_1 = 0)$ is a bit trickier. Recall that:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$

Therefore we find $V_1(z_1 = 0) \neq V_1^-(z_1 = 0)$!

Now, we can simplify this circuit:



And we know from the telegraphers equations:

$$\begin{aligned} V_1(z_1) &= V_1^+(z_1) + V_1^-(z_1) \\ &= V_{01}^+ e^{-j\beta z_1} + V_{01}^- e^{+j\beta z_1} \end{aligned}$$

Since the load $2Z_0/3$ is located at $z_1 = 0$, we know that the boundary condition leads to:

$$V_1(z_1) = V_{01}^+ (e^{-j\beta z_1} + \Gamma_L e^{+j\beta z_1})$$

where:

$$\begin{aligned} \Gamma_L &= \frac{\left(\frac{2}{3}\right)Z_0 - Z_0}{\left(\frac{2}{3}\right)Z_0 + Z_0} \\ &= \frac{\left(\frac{2}{3}\right) - 1}{\left(\frac{2}{3}\right) + 1} \\ &= \frac{-\frac{1}{3}}{\frac{5}{3}} \\ &= -0.2 \end{aligned}$$

Therefore:

$$V_1^+(z_1) = V_{01}^+ e^{-j\beta z_1} \quad \text{and} \quad V_1^-(z_1) = V_{01}^+ (-0.2) e^{+j\beta z_1}$$

and thus:

$$V_1^+(z_1 = 0) = V_{01}^+ e^{-j\beta(0)} = V_{01}^+$$

$$V_1^-(z_1 = 0) = V_{01}^+ (-0.2) e^{+j\beta(0)} = -0.2 V_{01}^+$$

We can now determine S_{11} !

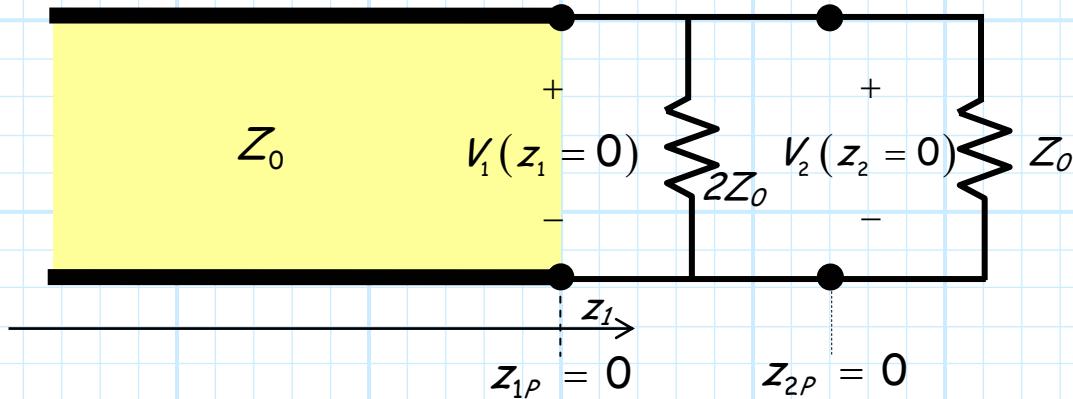
$$S_{11} = \frac{V_1^-(z_1 = 0)}{V_1^+(z_1 = 0)} = \frac{-0.2 V_{01}^+}{V_{01}^+} = -0.2$$

Now its time to find $V_2^- (z_2 = 0)$!

Again, since port 2 is terminated, the incident wave on port 2 must be zero, and thus the value of the exiting wave at port 2 is equal to the total voltage at port 2:

$$V_2^- (z_2 = 0) = V_2 (z_2 = 0)$$

This total voltage is relatively easy to determine. Examining the circuit, it is evident that $V_1(z_1 = 0) = V_2(z_2 = 0)$.



Therefore:

$$\begin{aligned} V_2 (z_2 = 0) &= V_1 (z_1 = 0) \\ &= V_{01}^+ (e^{-j\beta(0)} - 0.2 e^{+j\beta(0)}) \\ &= V_{01}^+ (1 - 0.2) \\ &= V_{01}^+ (0.8) \end{aligned}$$

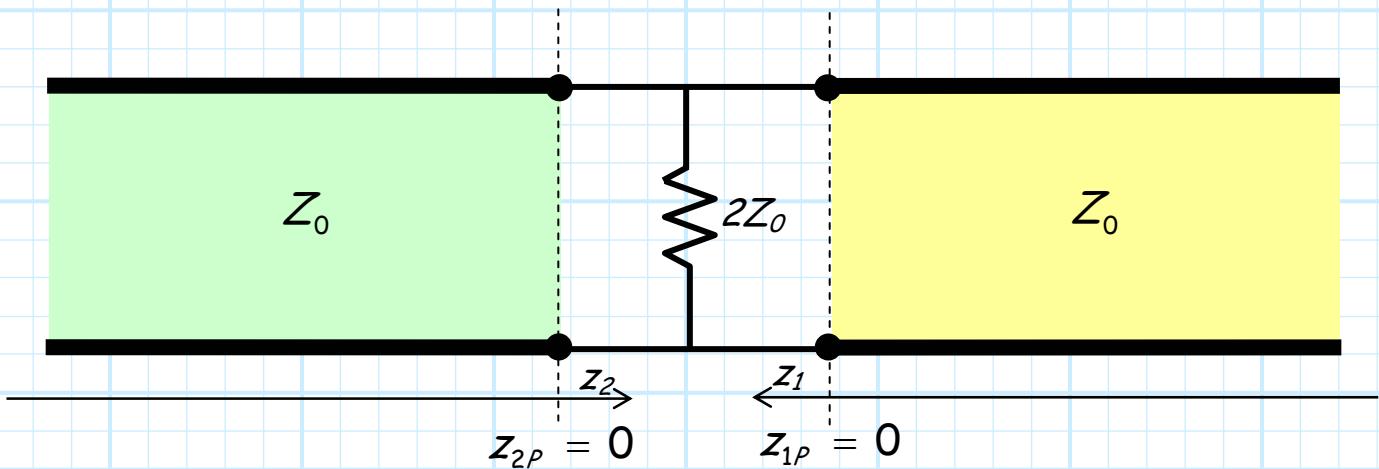
And thus the scattering parameter S_{21} is:

$$S_{21} = \frac{V_2^- (z_2 = 0)}{V_1^+ (z_1 = 0)} = \frac{0.8 V_{01}^+}{V_{01}^+} = 0.8$$

Now we just need to find S_{12} and S_{22} .

Q: Yikes! This has been an awful lot of work, and you mean that we are only half-way done!?

A: Actually, we are nearly finished! Note that this circuit is symmetric—there is really no difference between port 1 and port 2. If we “flip” the circuit, it remains unchanged!



Thus, we can conclude due to this symmetry that:

$$S_{11} = S_{22} = -0.2$$

and:

$$S_{21} = S_{12} = 0.8$$

Note this last equation is likewise a result of reciprocity.

Thus, the scattering matrix for this two port network is:

$$\mathcal{S} = \begin{bmatrix} -0.2 & 0.8 \\ 0.8 & -0.2 \end{bmatrix}$$

Circuit Symmetry

One of the most powerful concepts in for evaluating circuits is that of symmetry. Normal humans have a **conceptual** understanding of symmetry, based on an **esthetic** perception of structures and figures.

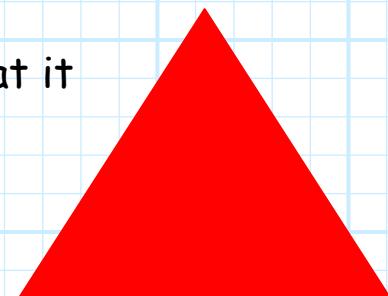


Évariste Galois

On the other hand, **mathematicians** (as they are wont to do) have defined symmetry in a very precise and unambiguous way. Using a branch of mathematics called **Group Theory**, first developed by the young genius **Évariste Galois** (1811-1832), **symmetry** is defined by a set of operations (a group) that leaves an object **unchanged**.

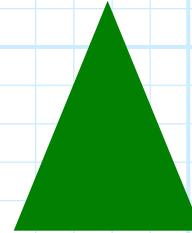
Initially, the symmetric "objects" under consideration by Galois were **polynomial functions**, but group theory can likewise be applied to evaluate the symmetry of **structures**.

For example, consider an ordinary **equilateral triangle**; we find that it is a highly **symmetric** object!

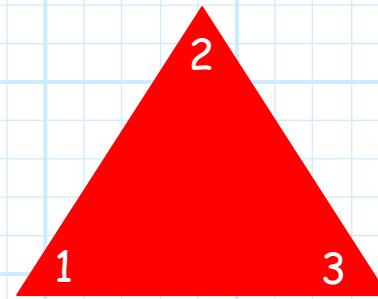


Q: Obviously this is true. We don't need a mathematician to tell us that!

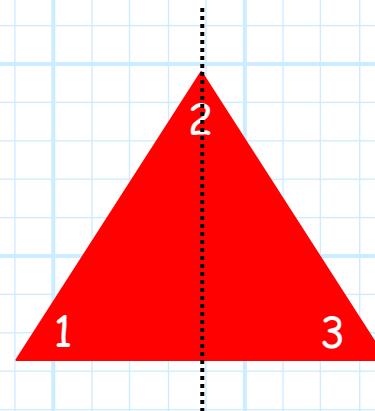
A: Yes, but how symmetric is it? How does the symmetry of an equilateral triangle compare to that of an isosceles triangle, a rectangle, or a square?



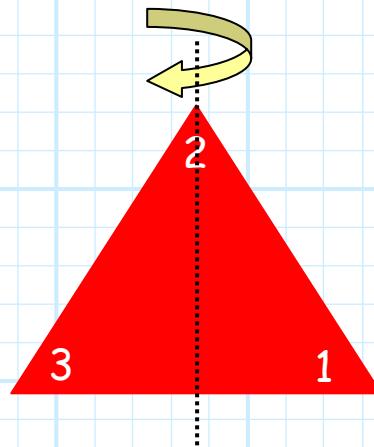
To determine its level of symmetry, let's first label each corner as corner 1, corner 2, and corner 3.



First, we note that the triangle exhibits a plane of reflection symmetry:



Thus, if we "reflect" the triangle across this plane we get:



Note that although corners 1 and 3 have changed places, the triangle itself remains **unchanged**—that is, it has the same **shape**, same **size**, and same **orientation** after reflecting across the symmetric plane!

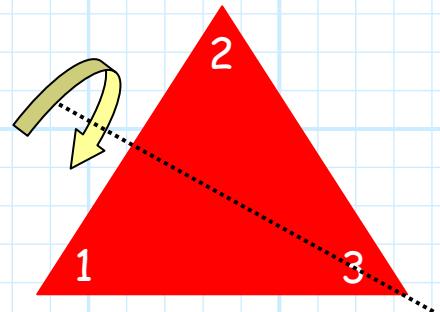
Mathematicians say that these two triangles are **congruent**.

Note that we can write this reflection operation as a **permutation** (an exchange of position) of the corners, defined as:

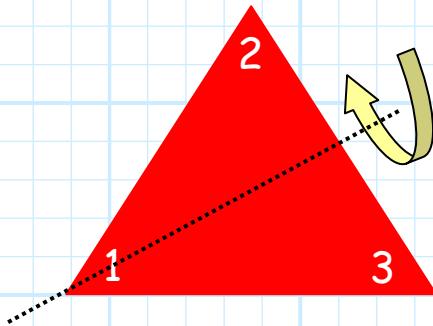
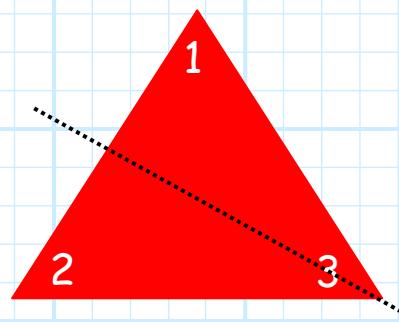
$$\begin{aligned}1 &\rightarrow 3 \\2 &\rightarrow 2 \\3 &\rightarrow 1\end{aligned}$$

Q: But wait! Isn't there is more than just one plane of reflection symmetry?

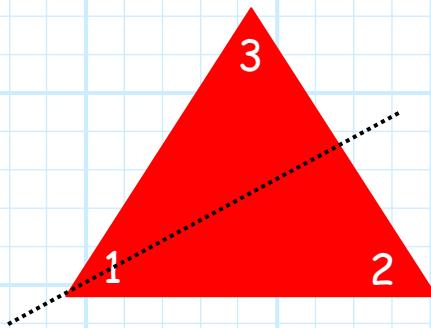
A: Definitely! There are two more:



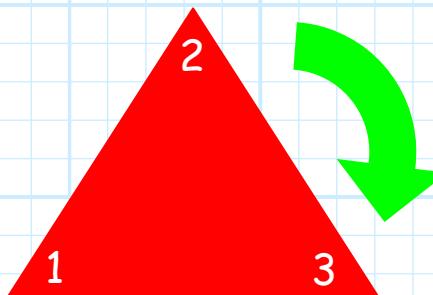
$$\begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 1 \\ 3 \rightarrow 3 \end{array}$$



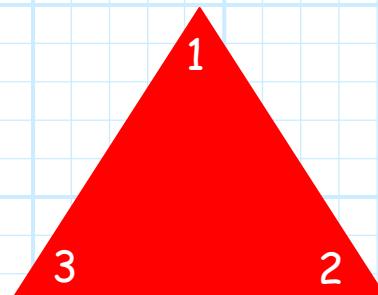
$$\begin{array}{l} 1 \rightarrow 1 \\ 2 \rightarrow 3 \\ 3 \rightarrow 2 \end{array}$$



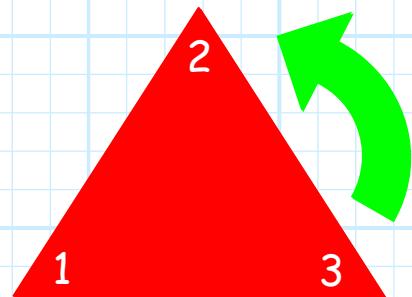
In addition, an equilateral triangle exhibits **rotation symmetry!** Rotating the triangle 120° clockwise also results in a congruent triangle:



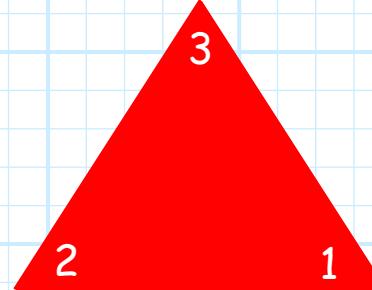
$$\begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 3 \\ 3 \rightarrow 1 \end{array}$$



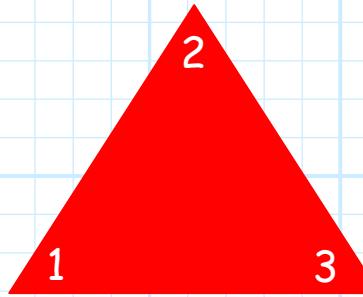
Likewise, rotating the triangle 120° counter-clockwise results in a congruent triangle:



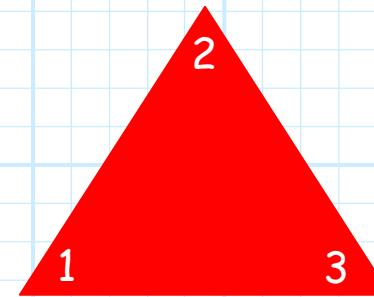
$$\begin{aligned}1 &\rightarrow 3 \\2 &\rightarrow 1 \\3 &\rightarrow 2\end{aligned}$$



Additionally, there is one more operation that will result in a congruent triangle—do nothing!



$$\begin{aligned}1 &\rightarrow 1 \\2 &\rightarrow 2 \\3 &\rightarrow 3\end{aligned}$$



This seemingly trivial operation is known as the identity operation, and is an element of every symmetry group.

These 6 operations form the dihedral symmetry group D_3 which has order six (i.e., it consists of six operations). An object that remains congruent when operated on by any and all of these six operations is said to have D_3 symmetry.



An equilateral triangle has D_3 symmetry!

By applying a similar analysis to a isosceles triangle, rectangle, and square, we find that:



An isosceles trapezoid has D_1 symmetry, a dihedral group of order 2.



A rectangle has D_2 symmetry, a dihedral group of order 4.



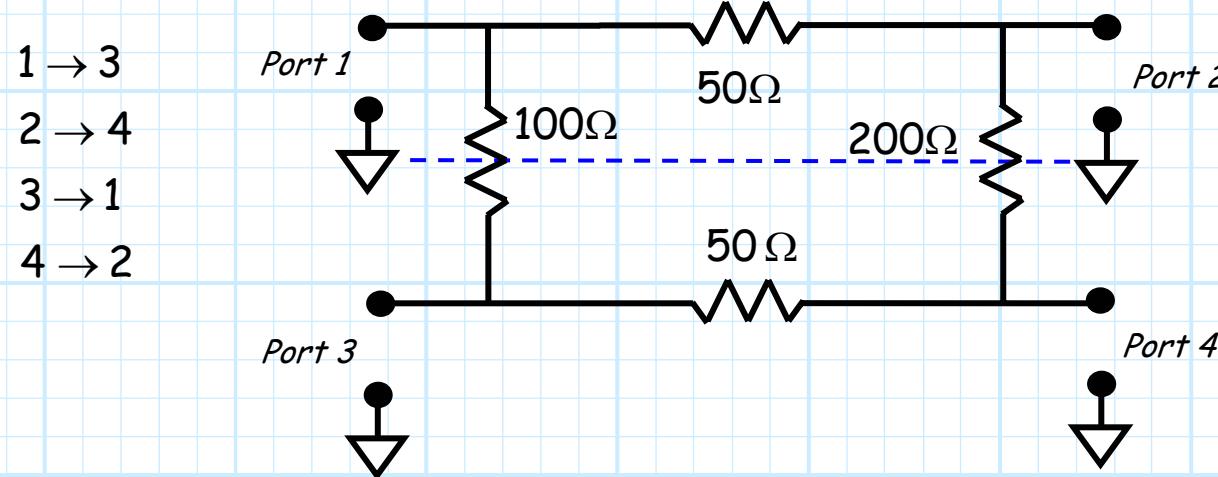
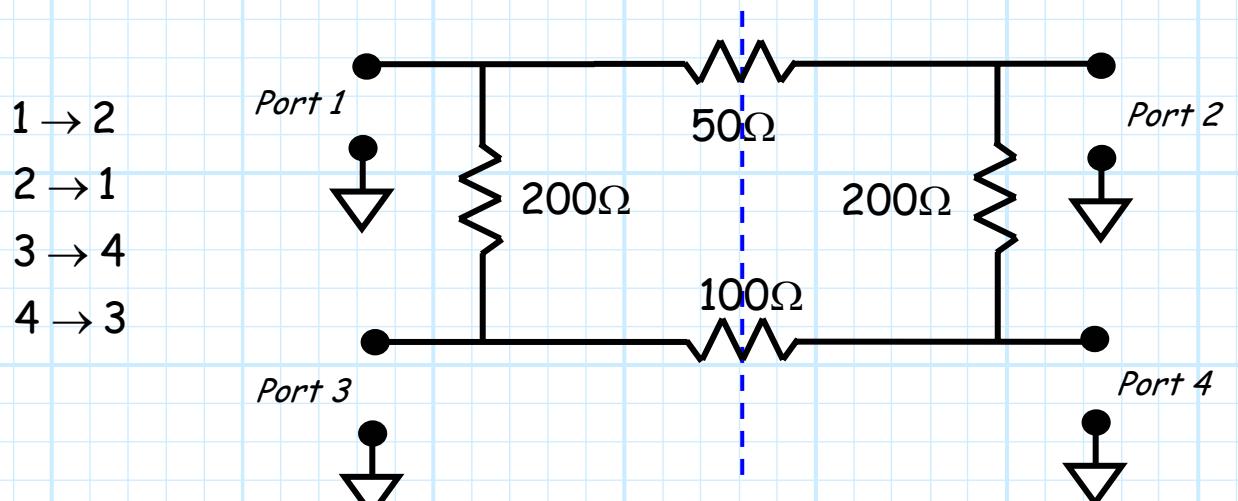
A square has D_4 symmetry, a dihedral group of order 8.

Thus, a square is the **most** symmetric object of the four we have discussed; the isosceles trapezoid is the **least**.

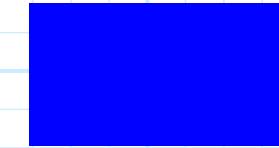
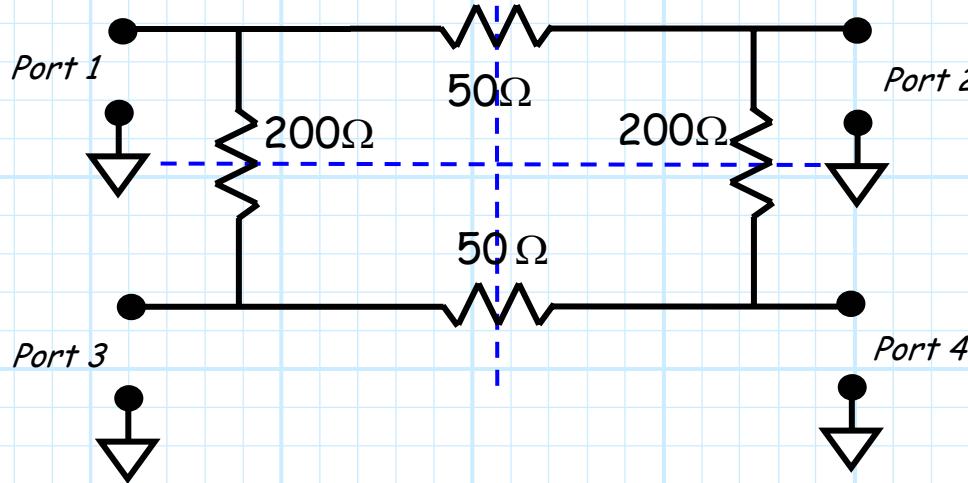
Q: Well that's all just fascinating—but just what the heck does this have to do with microwave circuits!?

A: Plenty! Useful circuits often display high levels of symmetry.

For example, consider these D_1 symmetric multi-port circuits:



Or this circuit with D_2 symmetry:



which is congruent under these permutations:

$$1 \rightarrow 3$$

$$2 \rightarrow 4$$

$$3 \rightarrow 1$$

$$4 \rightarrow 2$$

$$1 \rightarrow 2$$

$$2 \rightarrow 1$$

$$3 \rightarrow 4$$

$$4 \rightarrow 3$$

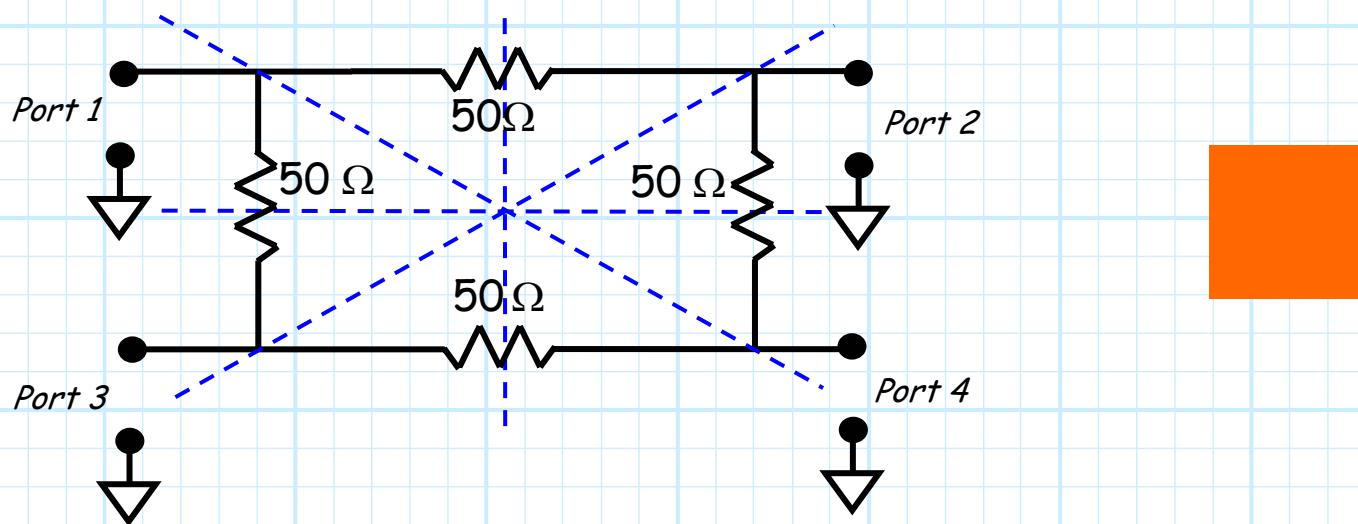
$$1 \rightarrow 4$$

$$2 \rightarrow 3$$

$$3 \rightarrow 2$$

$$4 \rightarrow 1$$

Or this circuit with D_4 symmetry:



which is **congruent** under these permutations:

$$1 \rightarrow 3$$

$$2 \rightarrow 4$$

$$3 \rightarrow 1$$

$$4 \rightarrow 2$$

$$1 \rightarrow 2$$

$$2 \rightarrow 1$$

$$3 \rightarrow 4$$

$$4 \rightarrow 3$$

$$1 \rightarrow 4$$

$$2 \rightarrow 3$$

$$3 \rightarrow 2$$

$$4 \rightarrow 1$$

$$1 \rightarrow 4$$

$$2 \rightarrow 2$$

$$3 \rightarrow 3$$

$$4 \rightarrow 1$$

$$1 \rightarrow 1$$

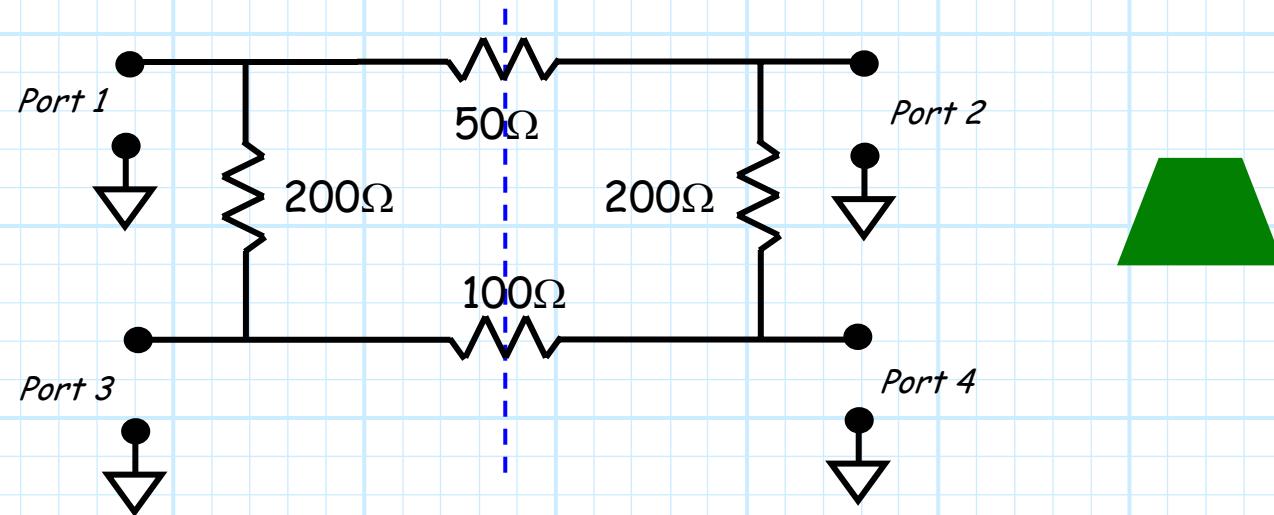
$$2 \rightarrow 3$$

$$3 \rightarrow 2$$

$$4 \rightarrow 4$$

The **importance** of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

For example, consider again this symmetric circuit:



This four-port network has a single plane of **reflection symmetry** (i.e., D_1 symmetry), and thus is congruent under the permutation:

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \\ 3 &\rightarrow 4 \\ 4 &\rightarrow 3 \end{aligned}$$

So, since (for example) $1 \rightarrow 2$, we find that for this circuit:

$$S_{11} = S_{22} \quad Z_{11} = Z_{22} \quad Y_{11} = Y_{22}$$

must be true!

Or, since $1 \rightarrow 2$ and $3 \rightarrow 4$ we find:

$$S_{13} = S_{24} \quad Z_{13} = Z_{24} \quad Y_{13} = Y_{24}$$

$$S_{31} = S_{42} \quad Z_{31} = Z_{42} \quad Y_{31} = Y_{42}$$

Continuing for all elements of the permutation, we find that for this symmetric circuit, the scattering matrix must have this form:

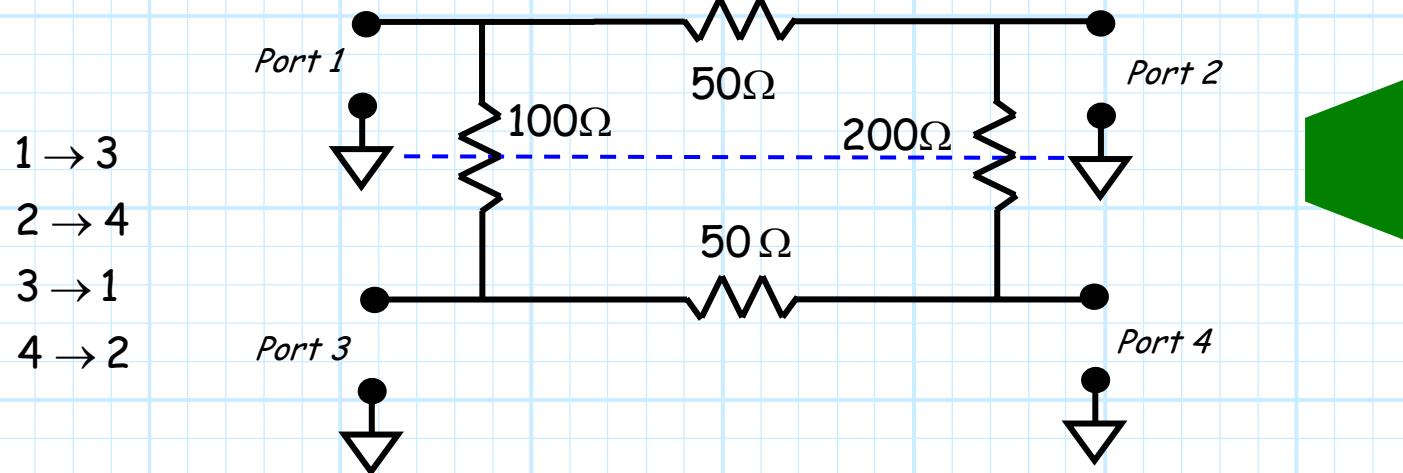
$$S = \begin{bmatrix} S_{11} & S_{21} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{14} & S_{13} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

and the **impedance** and **admittance** matrices would likewise have this same form.

Note there are just **8** independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that $S_{31} = S_{13}$ and $S_{41} = S_{14}$, and the matrix reduces further to one with just **6** independent elements:

$$S = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

Or, for circuits with this D_1 symmetry:

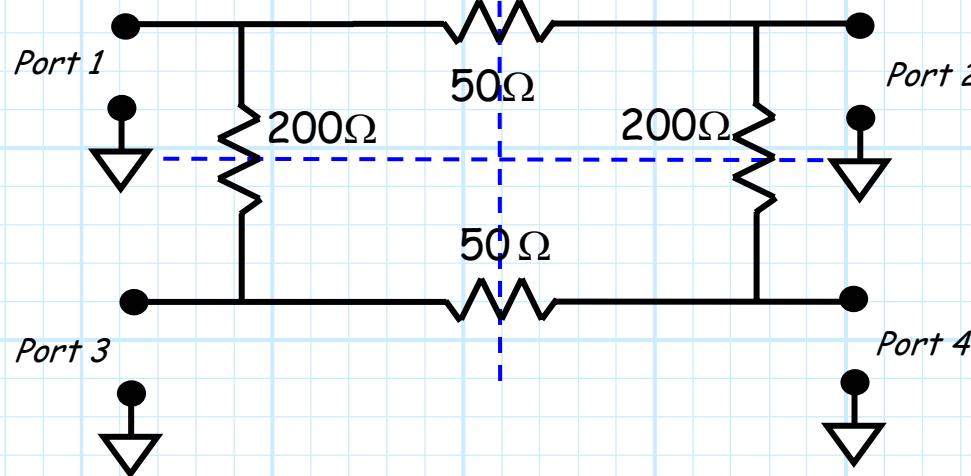


$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{22} \end{bmatrix}$$

Q: Interesting. But why do we care?

A: This will greatly simplify the analysis of this symmetric circuit, as we need to determine only six matrix elements!

For a circuit with D_2 symmetry:

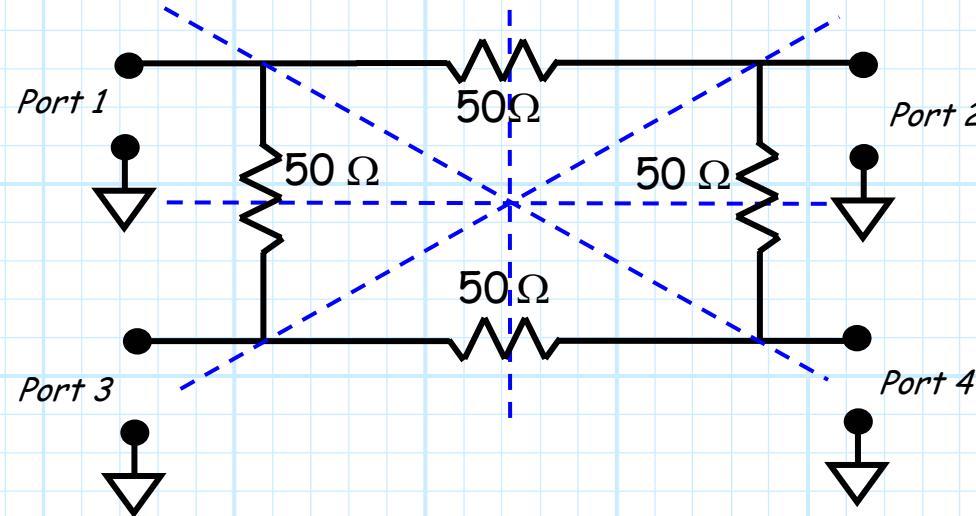


we find that the impedance (or scattering, or admittance) matrix has the form:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{21} & Z_{31} & Z_{41} \\ Z_{21} & Z_{11} & Z_{41} & Z_{31} \\ Z_{31} & Z_{41} & Z_{11} & Z_{21} \\ Z_{41} & Z_{31} & Z_{21} & Z_{11} \end{bmatrix}$$

Note here that there are just **four** independent values!

For a circuit with D_4 symmetry:



we find that the admittance (or scattering, or impedance) matrix has the form:

$$\mathcal{Y} = \begin{bmatrix} Y_{11} & Y_{21} & Y_{21} & Y_{41} \\ Y_{21} & Y_{11} & Y_{41} & Y_{21} \\ Y_{21} & Y_{41} & Y_{11} & Y_{21} \\ Y_{41} & Y_{21} & Y_{21} & Y_{11} \end{bmatrix}$$

Note here that there are just **three** independent values!

One more interesting thing (yet another one!); recall that we earlier found that a matched, lossless, reciprocal **4-port** device must have a scattering matrix with one of two forms:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix} \quad \text{The "symmetric" solution}$$

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix} \quad \text{The "anti-symmetric" solution}$$

Compare these to the matrix forms above. The "symmetric solution" has the **same form** as the scattering matrix of a circuit with D_2 symmetry!

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Q: Does this mean that a matched, lossless, reciprocal four-port device with the "symmetric" scattering matrix **must** exhibit D_2 symmetry?

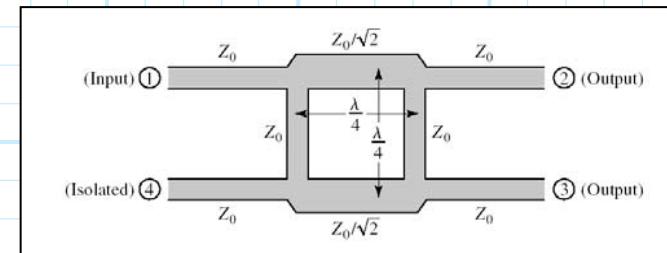
A: That's exactly what it means!

Not only can we determine from the form of the scattering matrix **whether** a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the **general structure** of a possible solutions (e.g. the circuit must have D_2 symmetry).

Likewise, the "anti-symmetric" matched, lossless, reciprocal four-port network **must** exhibit D_1 symmetry!

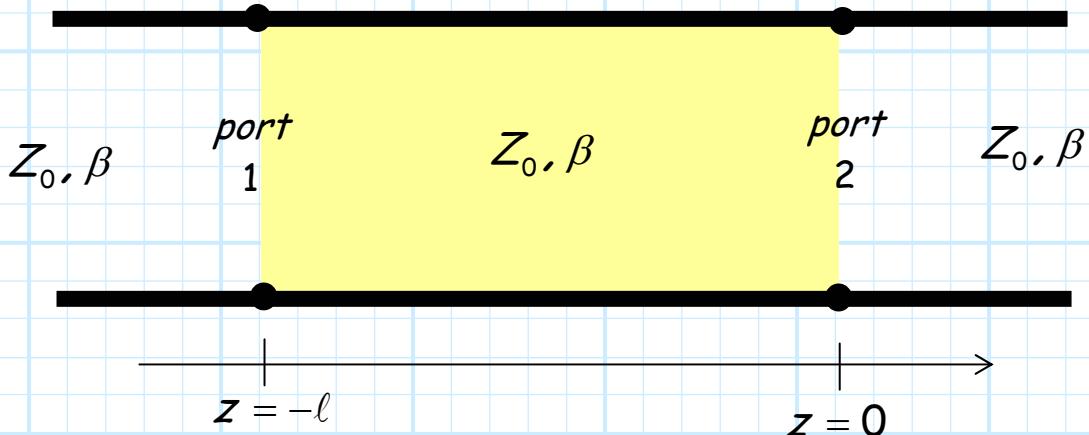
$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

We'll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually **are** later in the course!



Example: Using Symmetry to Determine a Scattering Matrix

Say we wish to determine the scattering matrix of the simple two-port device shown below:



We note that attaching transmission lines of characteristic impedance Z_0 to each port of our "circuit" forms a **continuous** transmission line of characteristic impedance Z_0 .

Thus, the voltage all along this transmission line thus has the form:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

We begin by defining the location of port 1 as $z_{1P} = -\ell$, and the port location of port 2 as $z_{2P} = 0$:

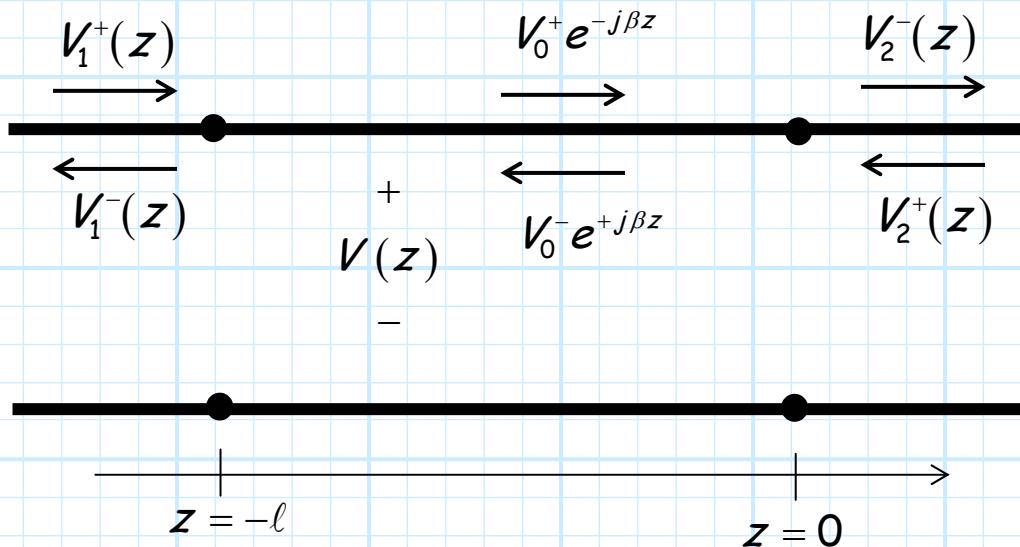
We can thus conclude:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \quad (z \leq -\ell)$$

$$V_1^-(z) = V_0^- e^{+j\beta z} \quad (z \leq -\ell)$$

$$V_2^+(z) = V_0^- e^{+j\beta z} \quad (z \geq 0)$$

$$V_2^-(z) = V_0^+ e^{-j\beta z} \quad (z \geq 0)$$



Say the transmission line on port 2 is terminated in a **matched load**. We know that the $-z$ wave must be **zero** ($V_0^- = 0$), and so the voltage along the transmission line becomes simply the $+z$ wave voltage:

$$V(z) = V_0^+ e^{-j\beta z}$$

and so:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \quad V_1^-(z) = 0 \quad (z \leq -\ell)$$

$$V_2^+(z) = 0 \quad V_2^-(z) = V_0^+ e^{-j\beta z} \quad (z \geq 0)$$

Now, because port 2 is terminated in a matched load, we can determine the scattering parameters S_{11} and S_{21} :

$$S_{11} = \frac{V_1^-(z = z_{1P})}{V_1^+(z = z_{1P})} \Big|_{V_2^+ = 0} = \frac{V_1^-(z = -\ell)}{V_1^+(z = -\ell)} \Big|_{V_2^+ = 0} = \frac{0}{V_0^+ e^{-j\beta(-\ell)}} = 0$$

$$S_{21} = \frac{V_2^-(z = z_{2P})}{V_1^+(z = z_{1P})} \Big|_{V_2^+ = 0} = \frac{V_2^-(z = 0)}{V_1^+(z = -\ell)} \Big|_{V_2^+ = 0} = \frac{V_0^+ e^{-j\beta(0)}}{V_0^+ e^{-j\beta(-\ell)}} = \frac{1}{e^{+j\beta\ell}} = e^{-j\beta\ell}$$

From the symmetry of the structure, we can conclude:

$$S_{22} = S_{11} = 0$$

And from both reciprocity and symmetry:

$$S_{12} = S_{21} = e^{-j\beta\ell}$$



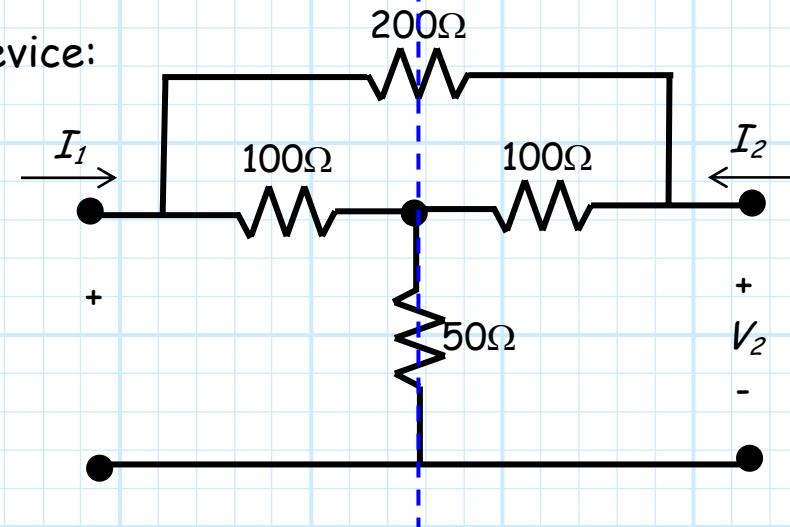
Thus:

$$\underline{\underline{S}} = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}$$

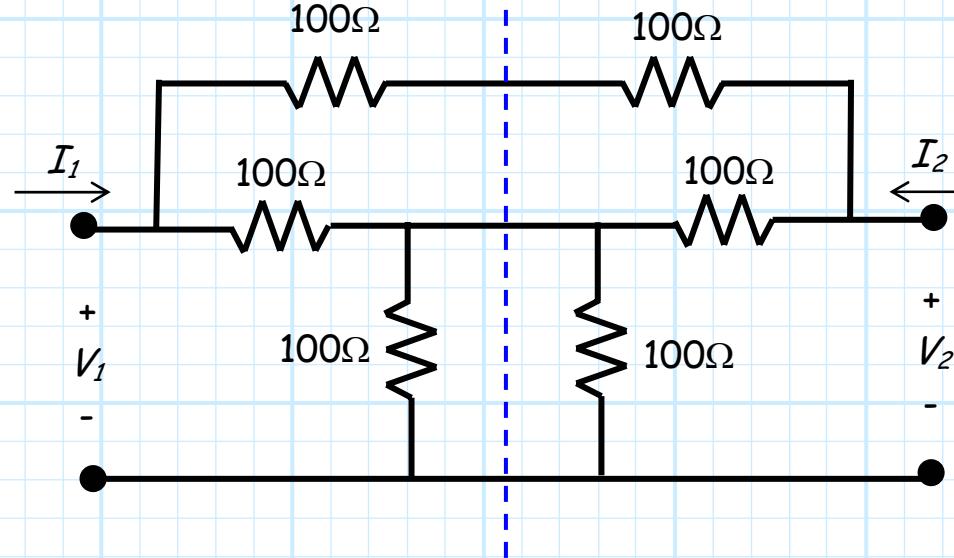
Symmetric Circuit Analysis

Consider the following D₁ symmetric two-port device:

Q: Yikes! The plane of reflection symmetry slices through two resistors. What can we do about that?



A: Resistors are easily split into two equal pieces: the 200Ω resistor into two 100Ω resistors in **series**, and the 50Ω resistor as two 100 Ω resistors in **parallel**.



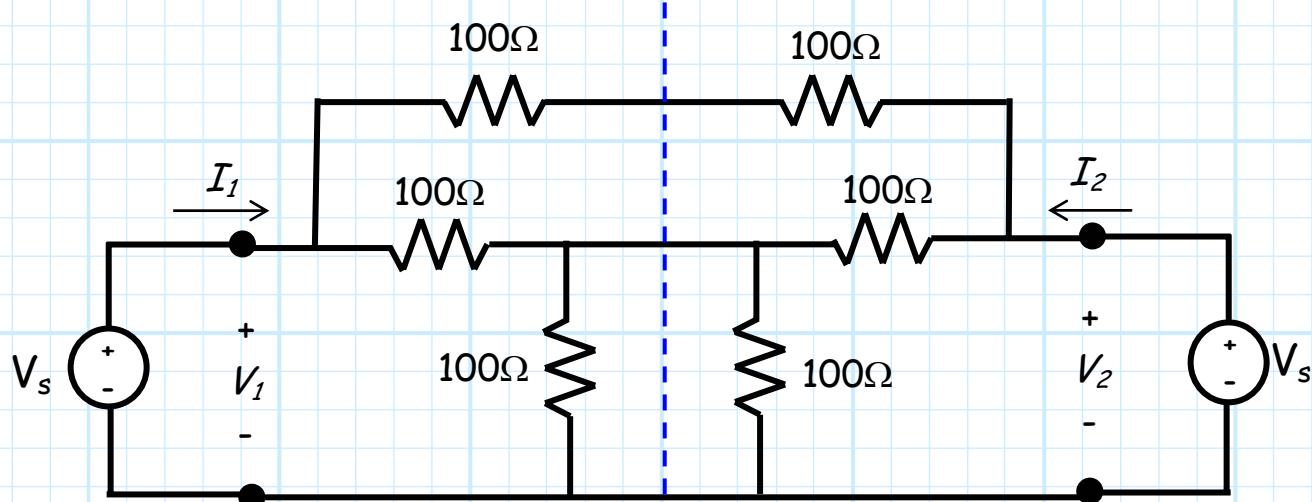
Recall that the **symmetry** of this 2-port device leads to **simplified** network matrices:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{11} \end{bmatrix} \quad \mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{11} \end{bmatrix} \quad \mathcal{Y} = \begin{bmatrix} Y_{11} & Y_{21} \\ Y_{21} & Y_{11} \end{bmatrix}$$

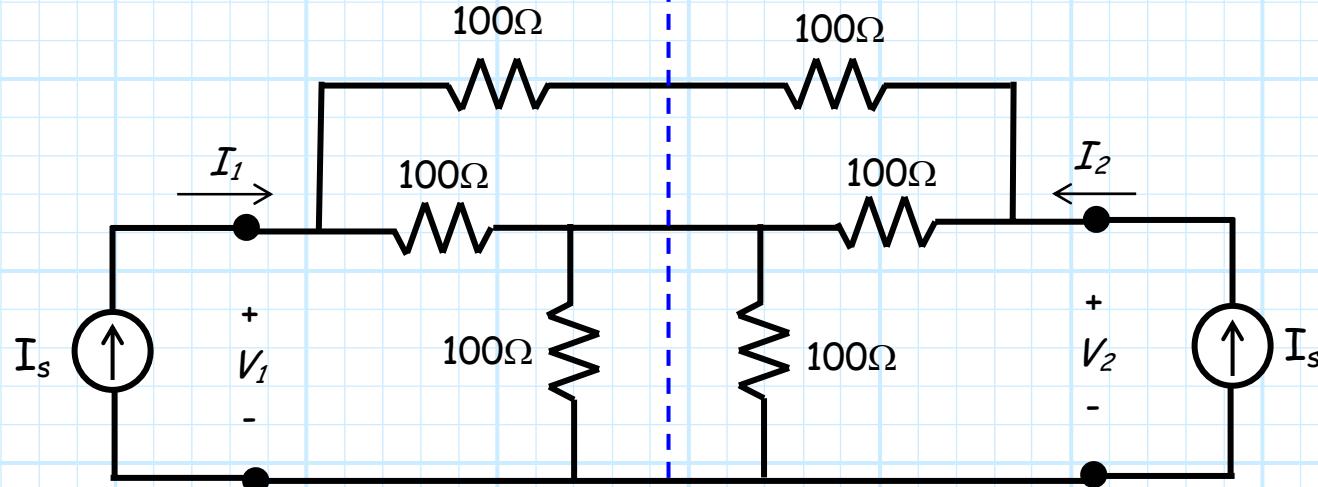
Q: Yes, but can circuit symmetry likewise simplify the procedure of **determining** these elements? In other words, can symmetry be used to **simplify** circuit analysis?

A: You bet!

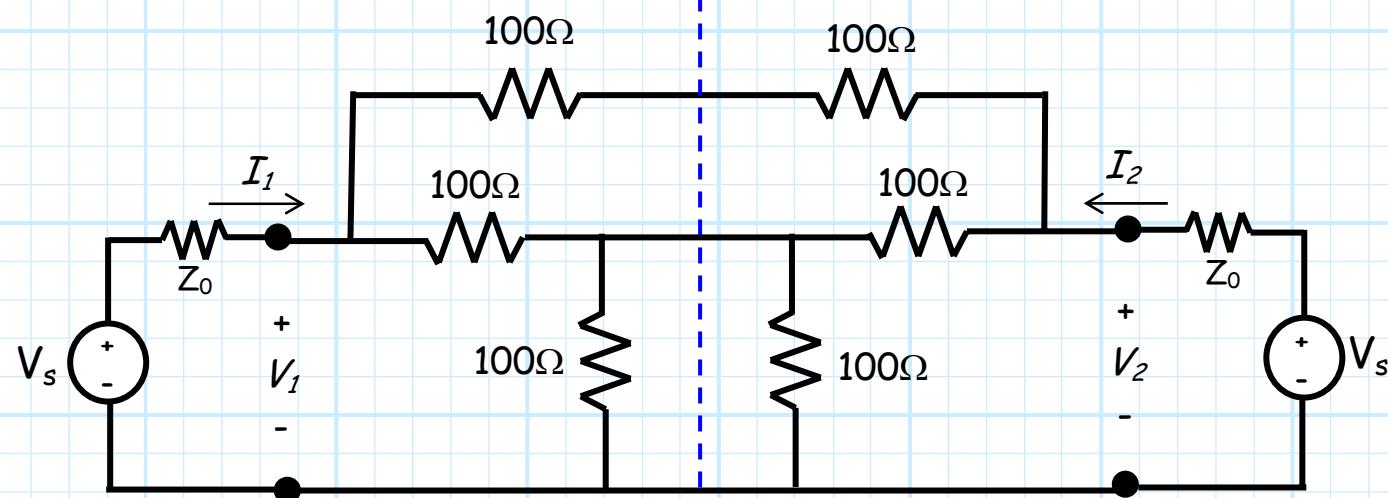
First, consider the case where we **attach sources** to circuit in a way that **preserves** the circuit **symmetry**:



Or,

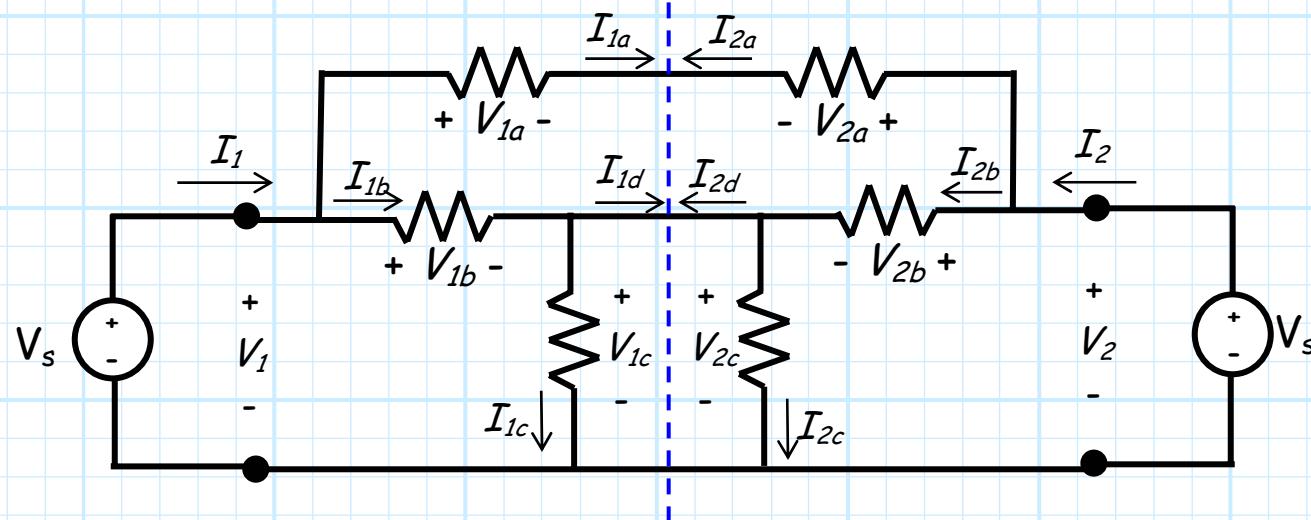


Or,



But remember! In order for **symmetry to be preserved**, the source values on both sides (i.e., I_s, V_s, Z_0) must be **identical**!

Now, consider the **voltages** and **currents** within this circuit under this symmetric configuration:



Since this circuit possesses **bilateral** (reflection) symmetry ($1 \rightarrow 2, 2 \rightarrow 1$), symmetric currents and voltages must be equal:

$$V_1 = V_2$$

$$V_{1a} = V_{2a}$$

$$V_{1b} = V_{2b}$$

$$V_{1c} = V_{2c}$$

$$I_1 = I_2$$

$$I_{1a} = I_{2a}$$

$$I_{1b} = I_{2b}$$

$$I_{1c} = I_{2c}$$

$$I_{1d} = I_{2d}$$

Q: Wait! This can't possibly be correct! Look at currents I_{1a} and I_{2a} , as well as currents I_{1d} and I_{2d} . From KCL, this must be true:

$$I_{1a} = -I_{2a}$$

$$I_{1d} = -I_{2d}$$

Yet you say that *this* must be true:

$$I_{1a} = I_{2a}$$

$$I_{1d} = I_{2d}$$

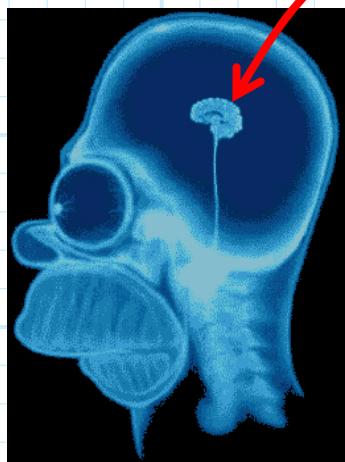
There is an **obvious contradiction** here! There is no way that both sets of equations can simultaneously be correct, is there?

A: Actually there is! There is **one** solution that will satisfy **both** sets of equations:

$$I_{1a} = I_{2a} = 0$$

$$I_{1d} = I_{2d} = 0$$

The currents are **zero**!

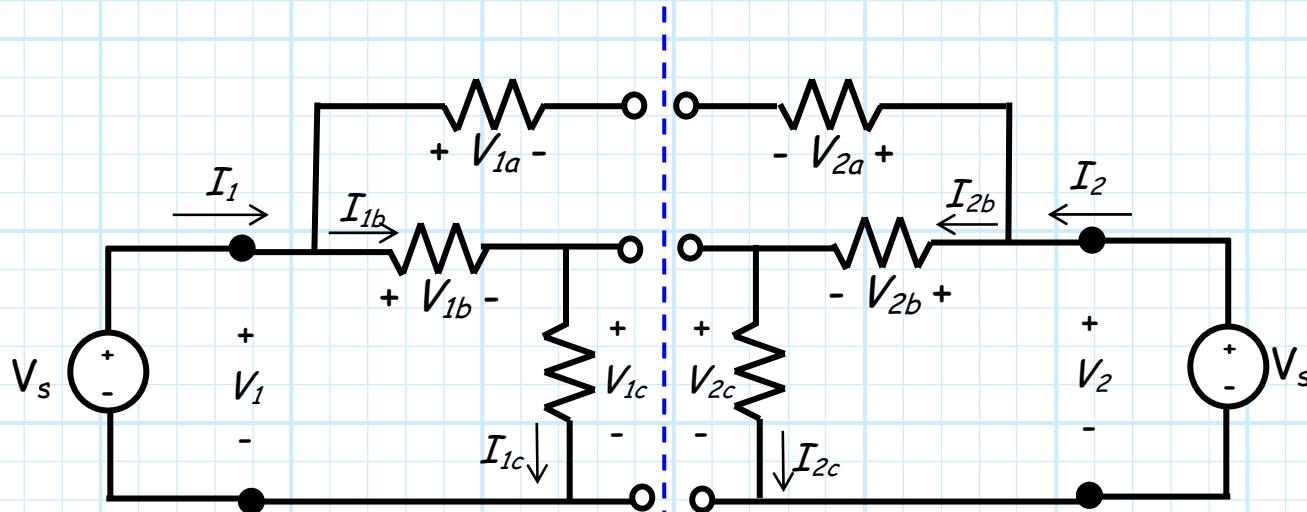


If you think about it, this makes **perfect sense**! The result says that **no current** will flow from one side of the symmetric circuit into the other.

If current **did** flow across the symmetry plane, then the circuit symmetry would be **destroyed**—one side would effectively become the “**source side**”, and the other the “**load side**” (i.e., the source side delivers current to the load side).

Thus, **no current** will flow **across** the reflection symmetry plane of a **symmetric circuit**—the symmetry plane thus acts as a **open circuit**!

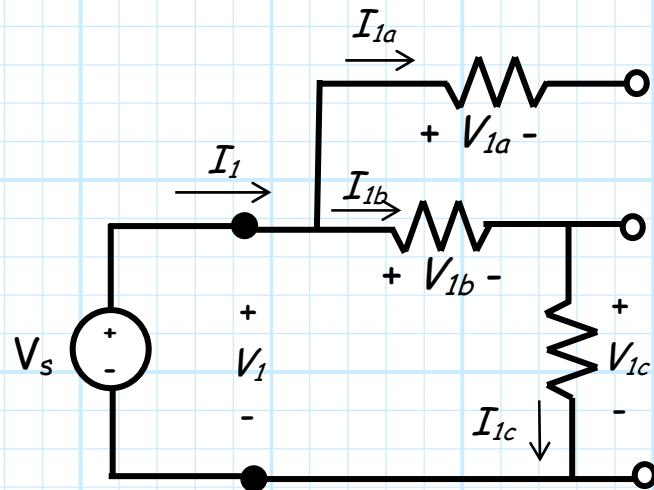
The plane of symmetry thus becomes a **virtual open!**



Q: So what?

Virtual Open
 $I=0$

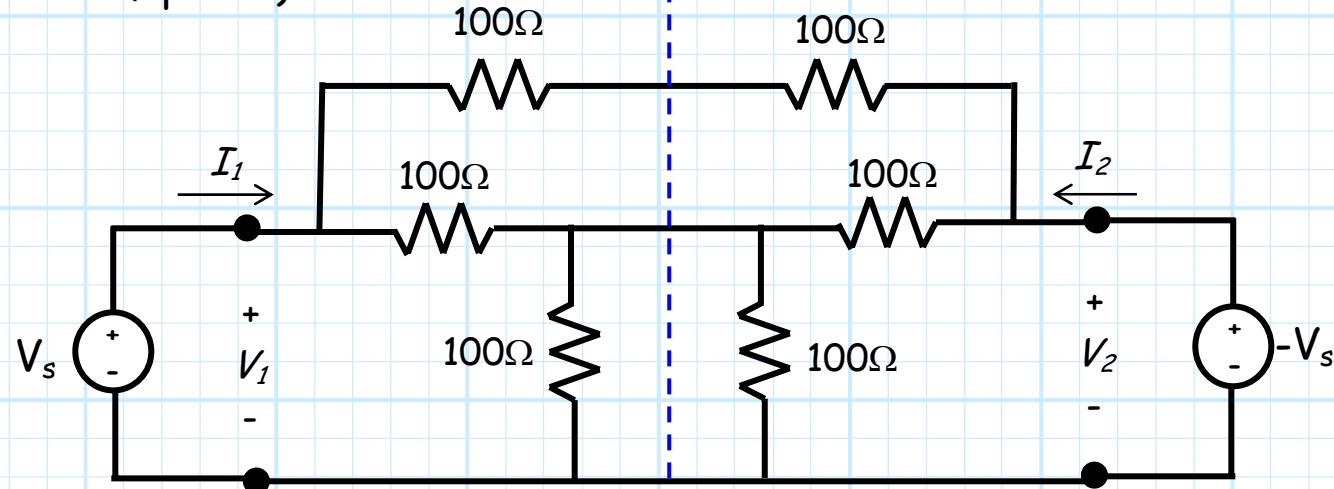
A: So what! This means that our circuit can be split apart into two separate but identical circuits. Solve one half-circuit, and you have solved the other!



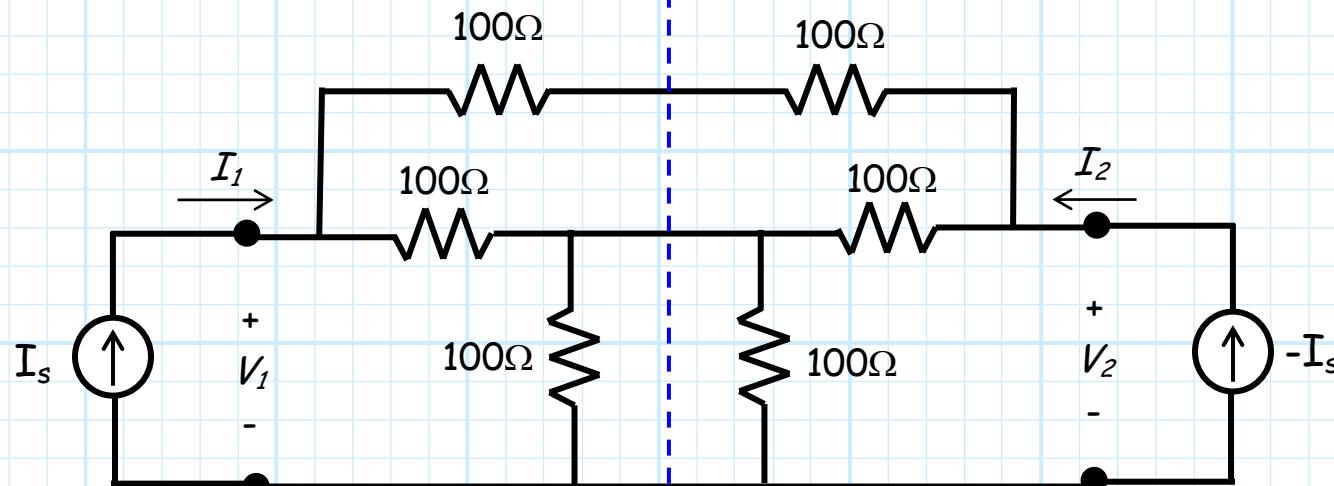
$$\begin{aligned}V_1 &= V_2 = V_s \\V_{1a} &= V_{2a} = 0 \\V_{1b} &= V_{2b} = V_s/2 \\V_{1c} &= V_{2c} = V_s/2\end{aligned}$$

$$\begin{aligned}I_1 &= I_2 = V_s/200 \\I_{1a} &= I_{2a} = 0 \\I_{1b} &= I_{2b} = V_s/200 \\I_{1c} &= I_{2c} = V_s/200 \\I_{1d} &= I_{2d} = 0\end{aligned}$$

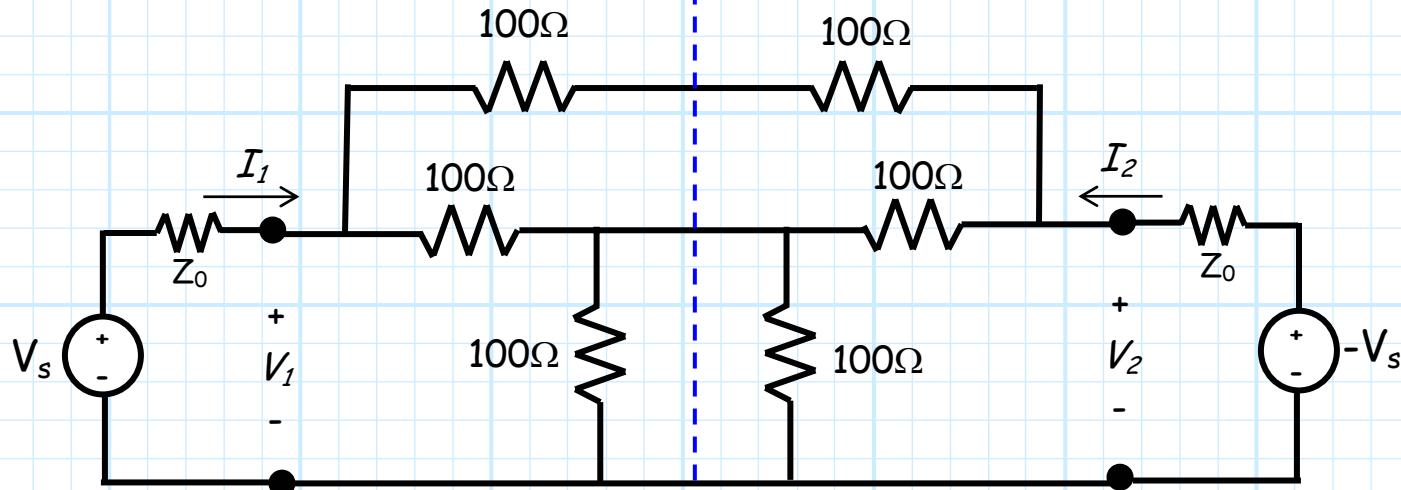
Now, consider another type of symmetry, where the sources are **equal but opposite** (i.e., **180 degrees** out of phase).



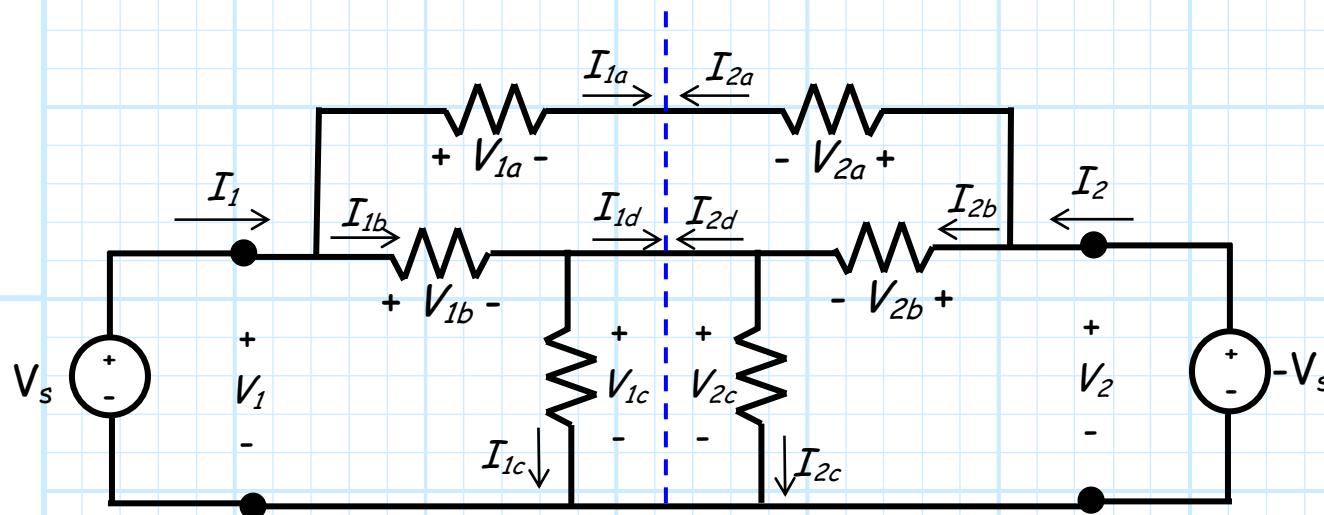
Or,



Or,

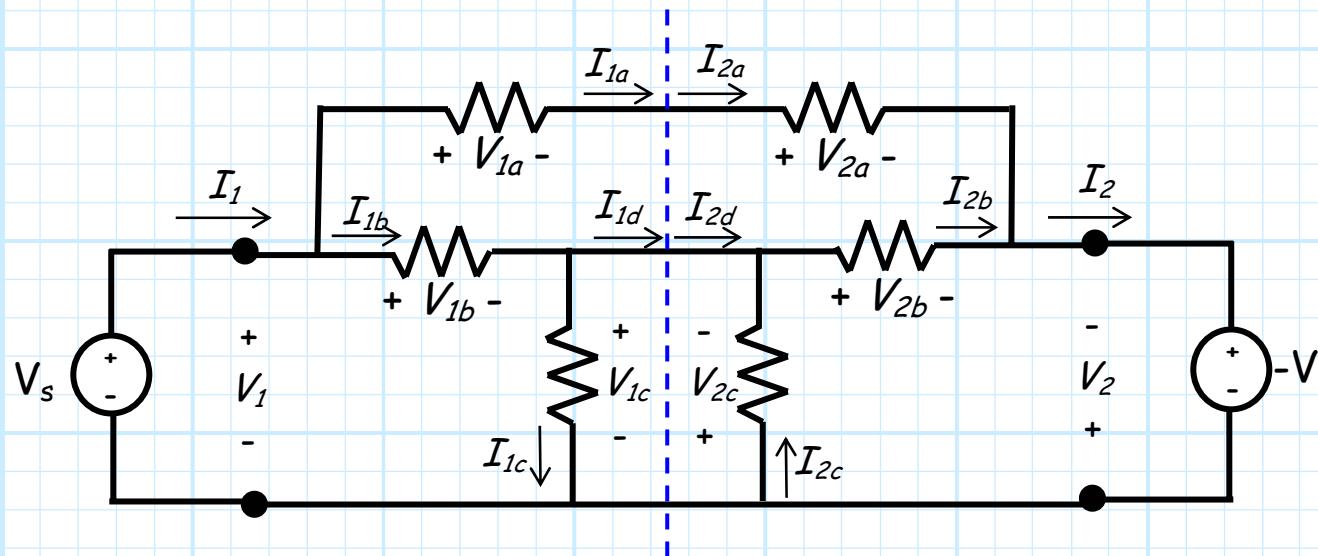


This situation still preserves the **symmetry** of the circuit—somewhat. The voltages and currents in the circuit will now possess **odd symmetry**—they will be **equal but opposite** (180 degrees out of phase) at symmetric points across the symmetry plane.



$$\begin{aligned}
 I_1 &= -I_2 \\
 I_{1a} &= -I_{2a} \\
 I_{1b} &= -I_{2b} \\
 I_{1c} &= -I_{2c} \\
 I_{1d} &= -I_{2d} \\
 V_1 &= -V_2 \\
 V_{1a} &= -V_{2a} \\
 V_{1b} &= -V_{2b} \\
 V_{1c} &= -V_{2c}
 \end{aligned}$$

Perhaps it would be easier to **redefine** the circuit variables as:



$$\begin{aligned}
 I_1 &= I_2 \\
 I_{1a} &= I_{2a} \\
 I_{1b} &= I_{2b} \\
 I_{1c} &= I_{2c} \\
 I_{1d} &= I_{2d}
 \end{aligned}$$

Q: But wait! Again I see a problem. By KVL it is evident that:

$$V_{1c} = -V_{2c}$$

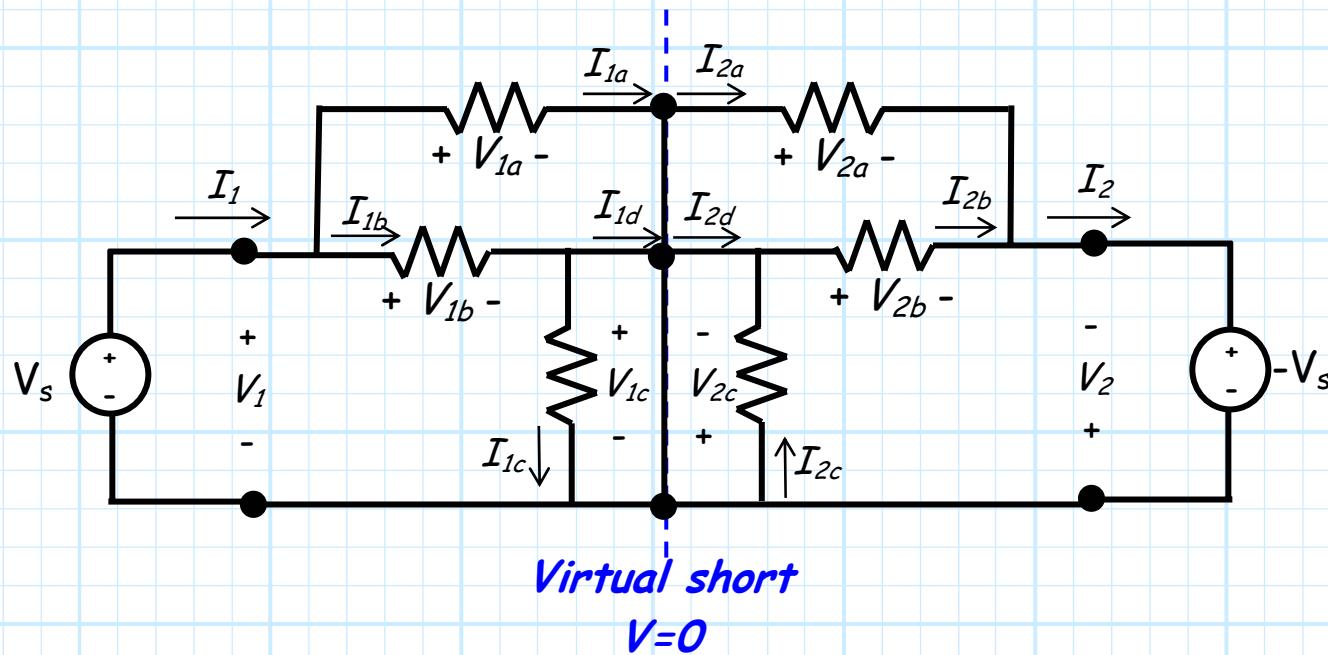
Yet you say that $V_{1c} = V_{2c}$ must be true!

A: Again, the solution to both equations is zero!

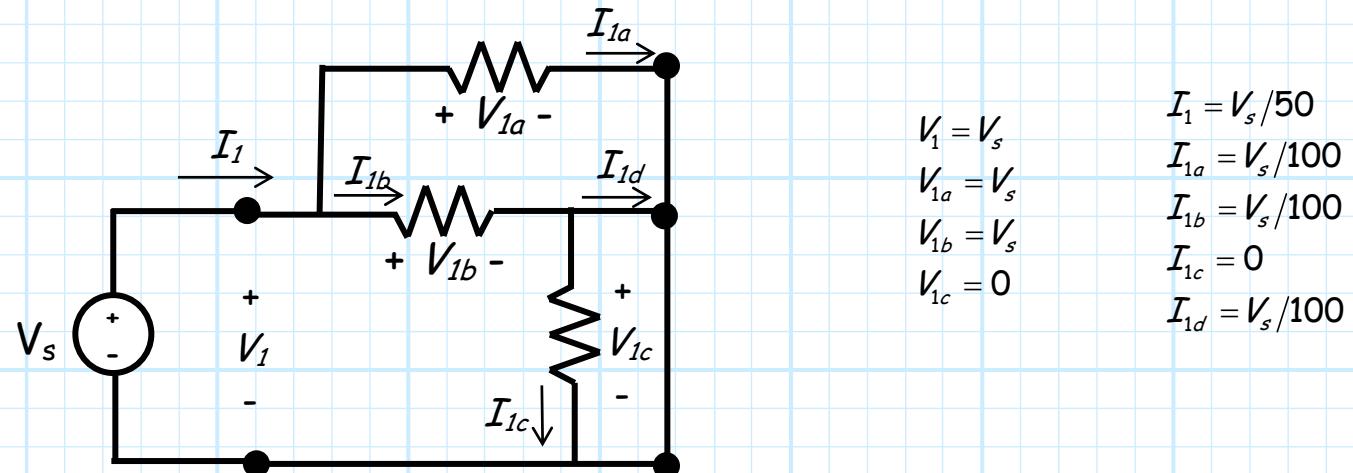
$$V_{1c} = V_{2c} = 0$$

For the case of **odd symmetry**, the symmetric plane must be a plane of **constant potential** (i.e., constant voltage)—just like a **short circuit**!

Thus, for odd symmetry, the symmetric plane forms a **virtual short**.



This greatly simplifies things, as we can again break the circuit into two independent and (effectively) identical circuits!



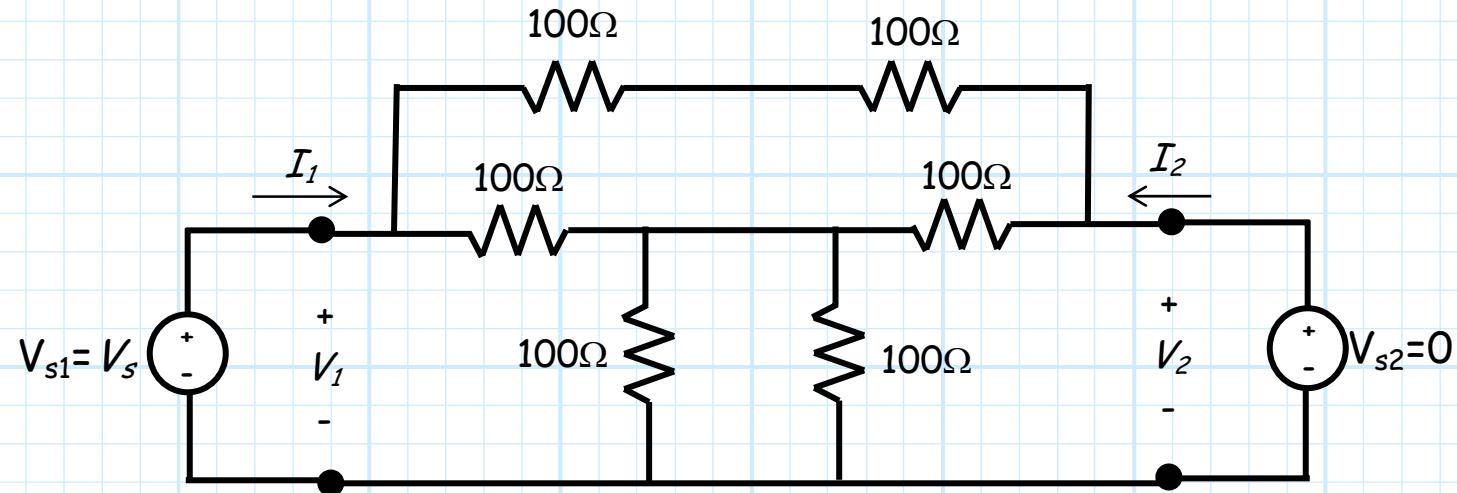
Odd/Even Mode Analysis

Q: Although symmetric circuits appear to be plentiful in microwave engineering, it seems unlikely that we would often encounter symmetric sources. Do virtual shorts and opens typically ever occur?

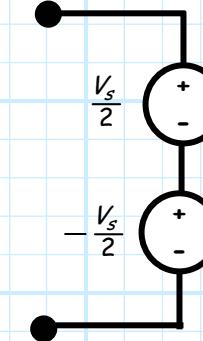
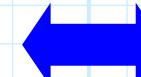
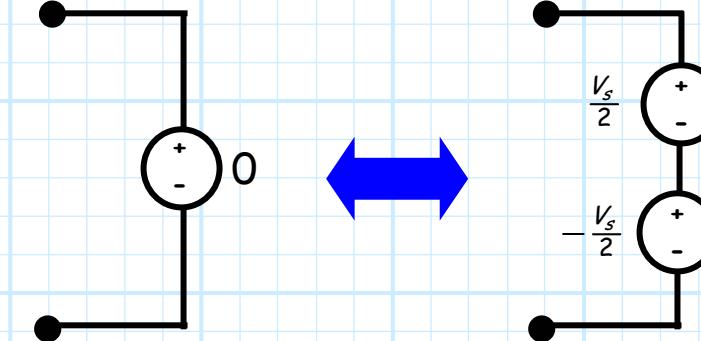
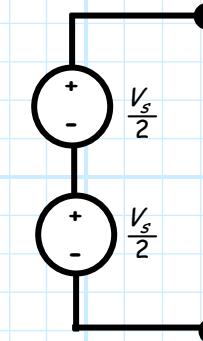
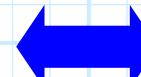
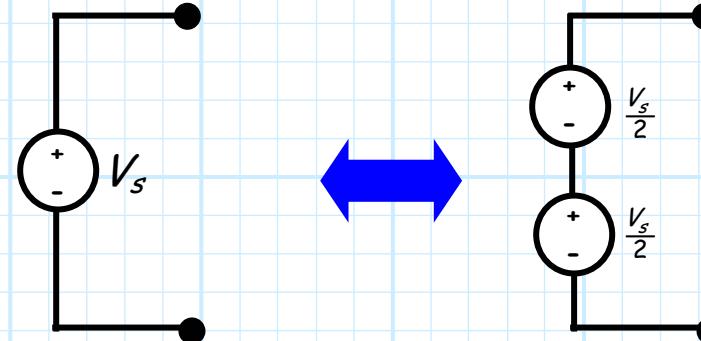
A: One word—superposition!

If the elements of our circuit are independent and linear, we can apply superposition to analyze symmetric circuits when non-symmetric sources are attached.

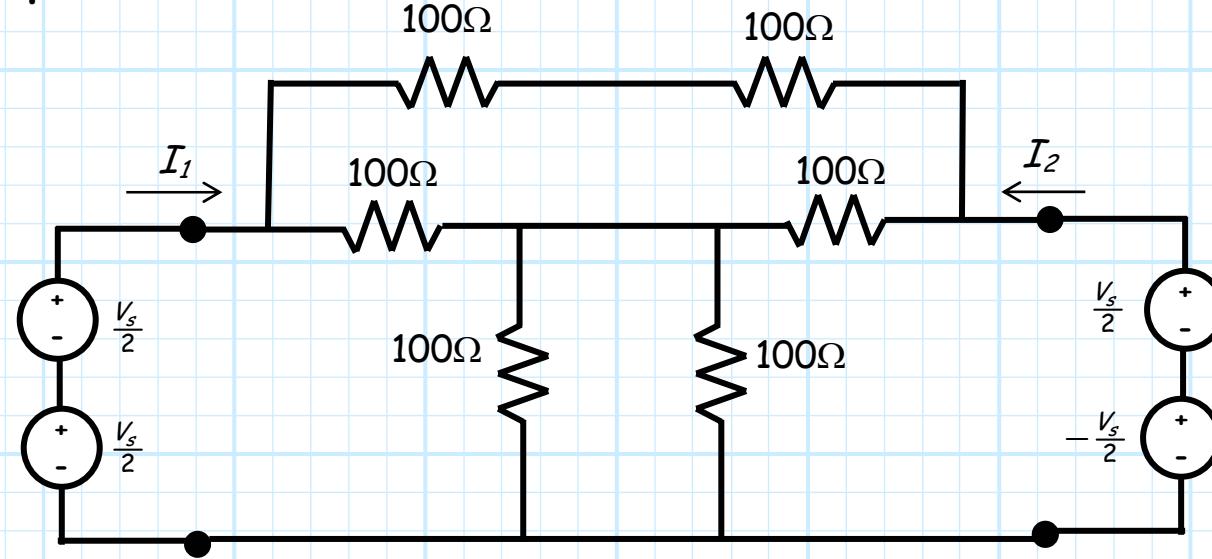
For example, say we wish to determine the admittance matrix of this circuit. We would place a voltage source at port 1, and a short circuit at port 2—a set of asymmetric sources if there ever was one!



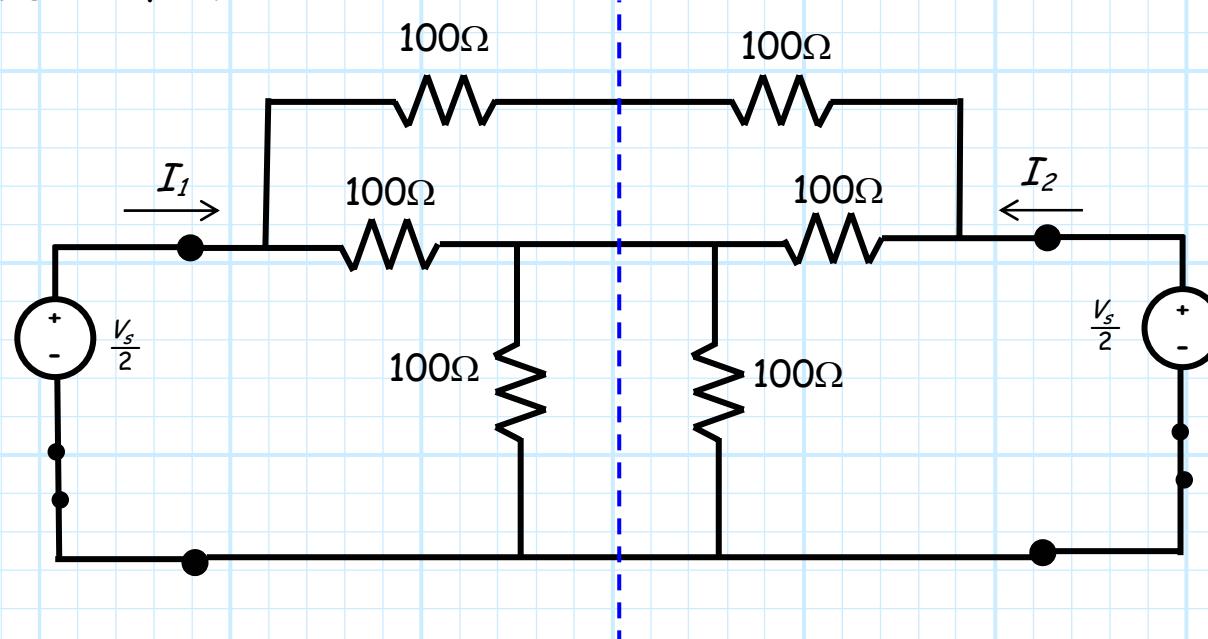
Here's the really **neat** part. We find that the source on port 1 can be modeled as **two equal voltage sources in series**, whereas the source at port 2 can be modeled as **two equal but opposite sources in series**.



Therefore an **equivalent** circuit is:



Now, the **above** circuit (due to the sources) is obviously **asymmetric**—no virtual ground, nor virtual short is present. But, let's say we **turn off** (i.e., set to $V=0$) the **bottom** source on **each side** of the circuit:

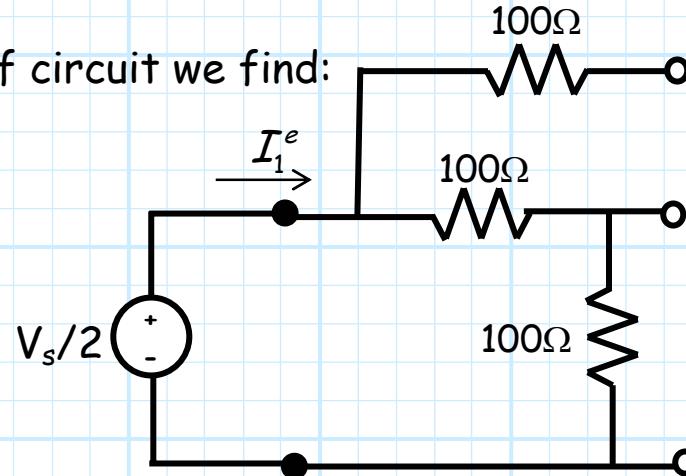


Our symmetry has been restored! The symmetry plane is a virtual open.

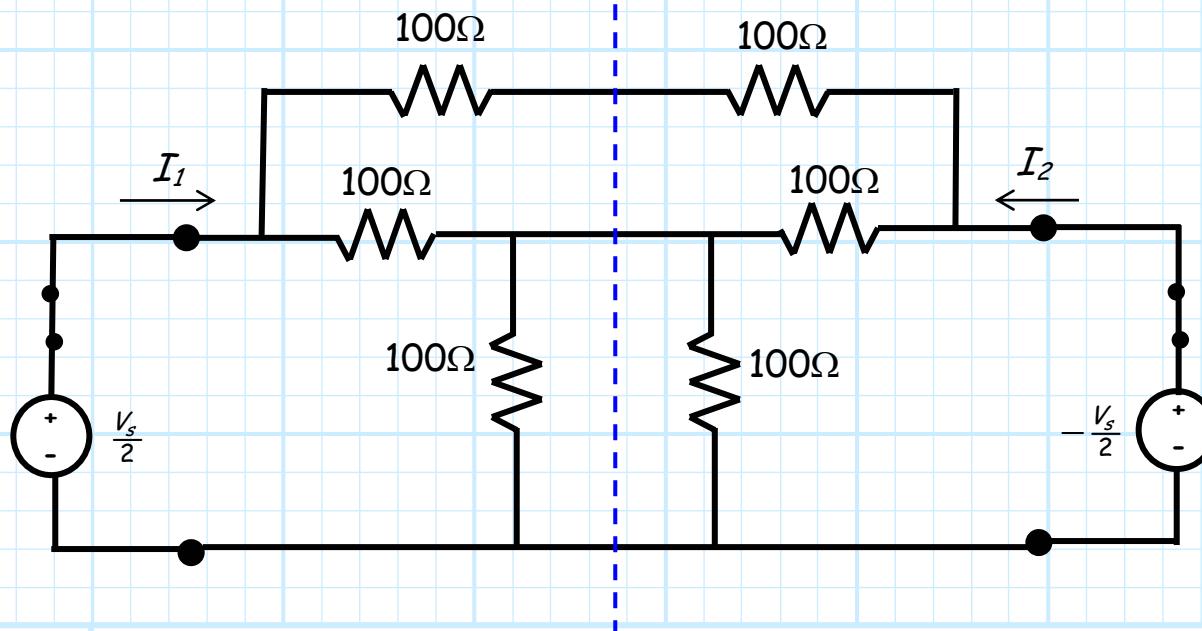
This circuit is referred to as its **even mode**, and analysis of it is known as the **even mode analysis**. The solutions are known as the even mode **currents** and **voltages**!

Evaluating the resulting **even mode** half circuit we find:

$$I_1^e = \frac{V_s}{2} \cdot \frac{1}{200} = \frac{V_s}{400} = I_2^e$$



Now, let's turn the bottom sources back on—but turn off the top two!

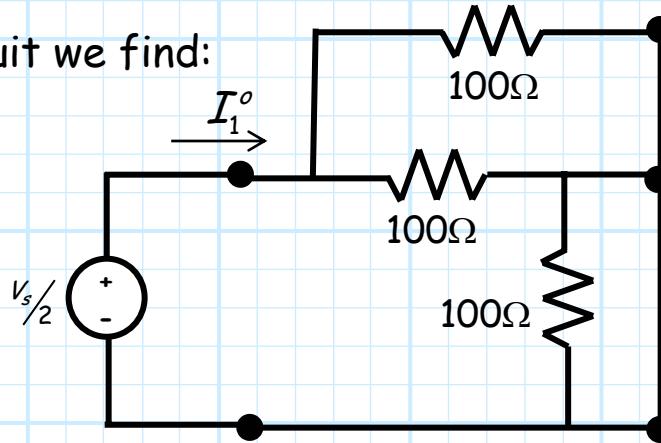


We now have a circuit with **odd symmetry**—the symmetry plane is a **virtual short**!

This circuit is referred to as its **odd mode**, and analysis of it is known as the **odd mode analysis**. The solutions are known as the **odd mode currents and voltages**!

Evaluating the resulting **odd mode** half circuit we find:

$$I_1^o = \frac{V_s}{2} \cdot \frac{1}{50} = \frac{V_s}{100} = -I_2^o$$



Q: But what good is this "even mode" and "odd mode" analysis? After all, the source on port 1 is $V_{s1}=V_s$, and the source on port 2 is $V_{s2}=0$. What are the currents I_1 and I_2 for these sources?

A: Recall that these sources are the **sum** of the even and odd mode sources:

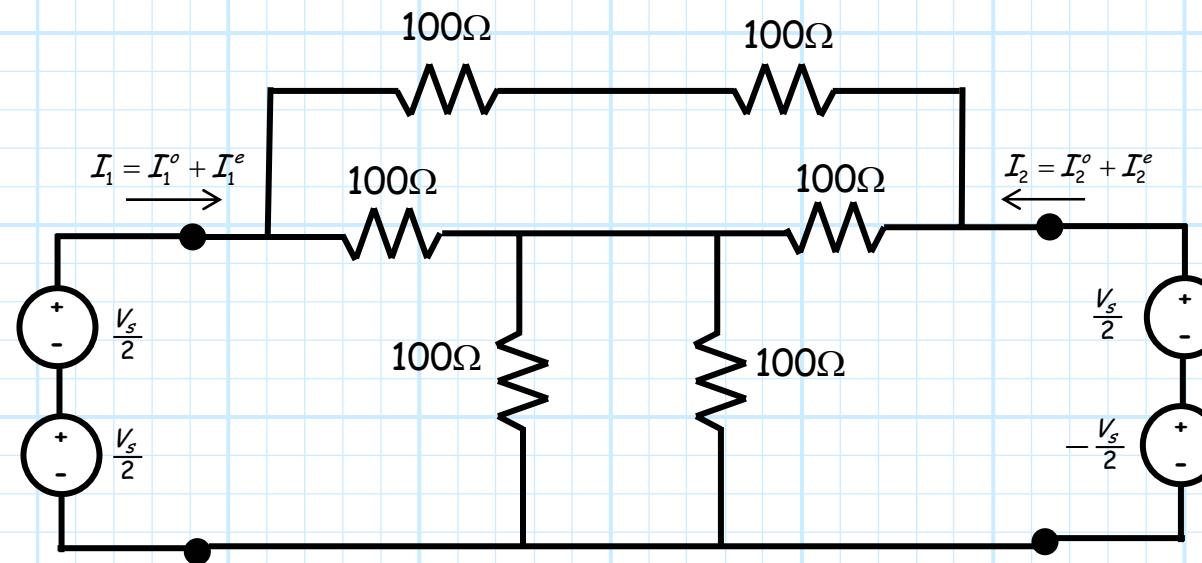
$$V_{s1} = V_s = \frac{V_s}{2} + \frac{V_s}{2}$$

$$V_{s2} = 0 = \frac{V_s}{2} - \frac{V_s}{2}$$

and thus—since all the devices in the circuit are **linear**—we know from superposition that the currents I_1 and I_2 are simply the **sum** of the **odd** and **even** mode currents!

$$I_1 = I_1^e + I_1^o$$

$$I_2 = I_2^e + I_2^o$$



Thus, adding the odd and even mode analysis results together:

$$I_1 = I_1^e + I_1^o$$

$$\begin{aligned} &= \frac{V_s}{400} + \frac{V_s}{100} \\ &= \frac{V_s}{80} \end{aligned}$$

$$I_2 = I_2^e + I_2^o$$

$$\begin{aligned} &= \frac{V_s}{400} - \frac{V_s}{100} \\ &= -\frac{3V_s}{400} \end{aligned}$$

And then the **admittance parameters** for this two port network is:

$$Y_{11} = \left. \frac{I_1}{V_{s1}} \right|_{V_{s2}=0} = \frac{V_s}{80} \frac{1}{V_s} = \frac{1}{80}$$

$$Y_{21} = \left. \frac{I_2}{V_{s1}} \right|_{V_{s2}=0} = -\frac{3V_s}{400} \frac{1}{V_s} = -\frac{3}{400}$$

And from the **symmetry** of the device we know:

$$Y_{22} = Y_{11} = \frac{1}{80}$$

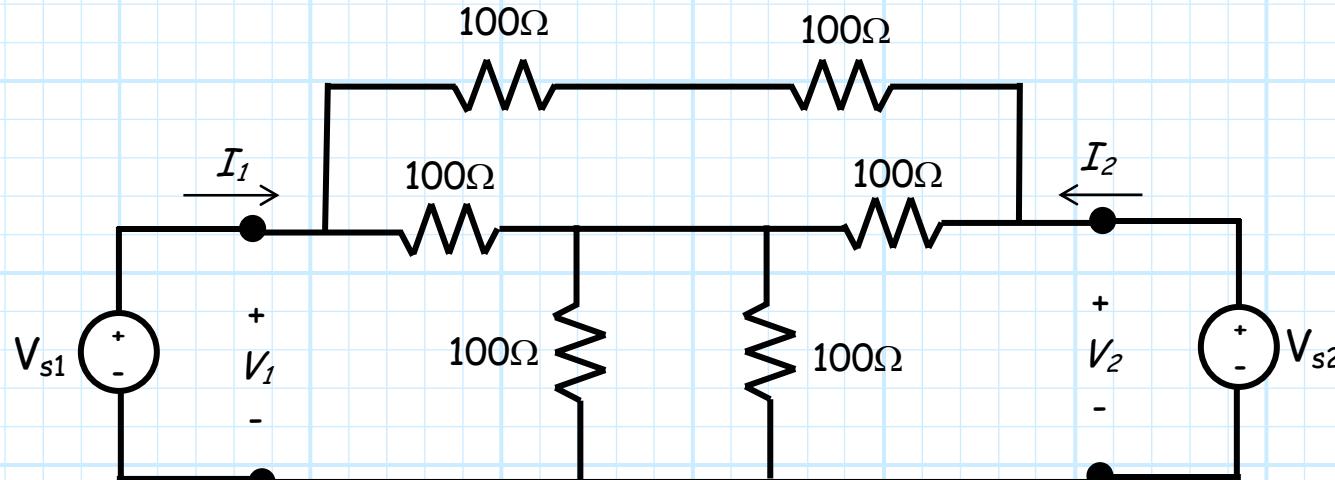
$$Y_{12} = Y_{21} = \frac{-3}{400}$$

Thus, the full admittance matrix is:

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{80} & \frac{-3}{400} \\ \frac{-3}{400} & \frac{1}{80} \end{bmatrix}$$

Q: What happens if both sources are non-zero? Can we use symmetry then?

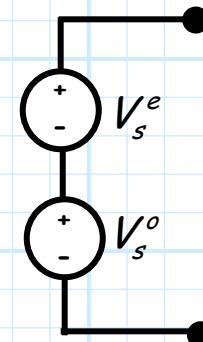
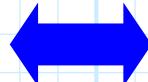
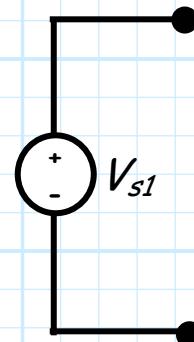
A: Absolutely! Consider the problem below, where **neither** source is equal to zero:



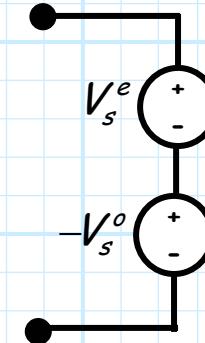
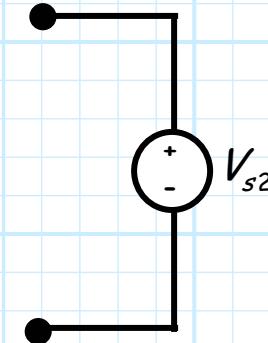
In this case we can define an even mode and an odd mode source as:

$$V_s^e = \frac{V_{s1} + V_{s2}}{2}$$

$$V_s^o = \frac{V_{s1} - V_{s2}}{2}$$

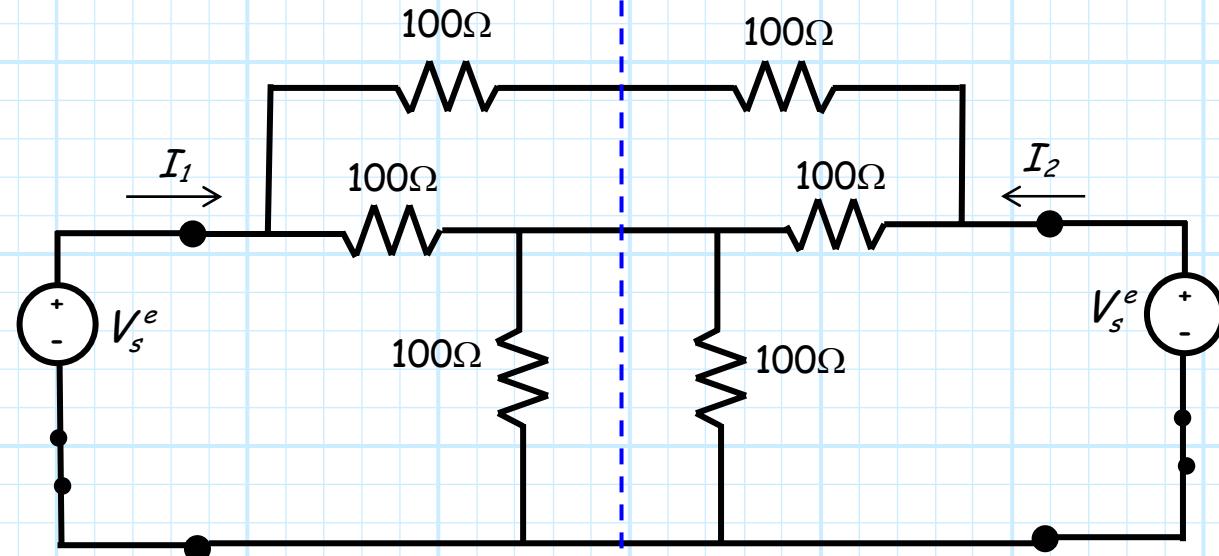


$$V_{s1} = V_s^e + V_s^o$$

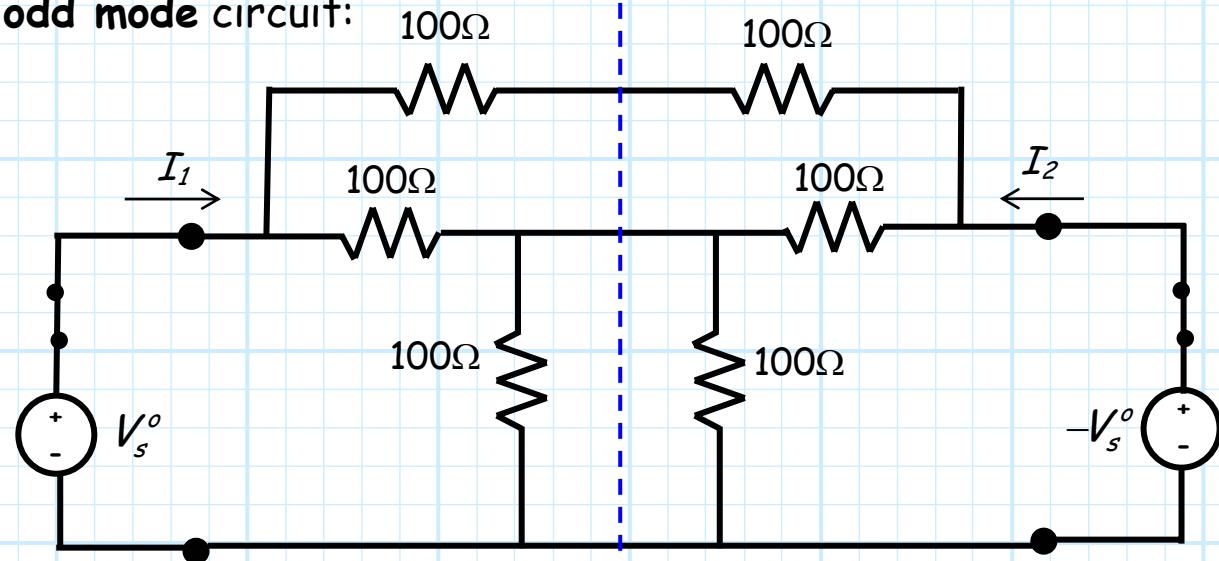


$$V_{s2} = V_s^e - V_s^o$$

We then can analyze the **even mode** circuit:



And then the **odd mode** circuit:



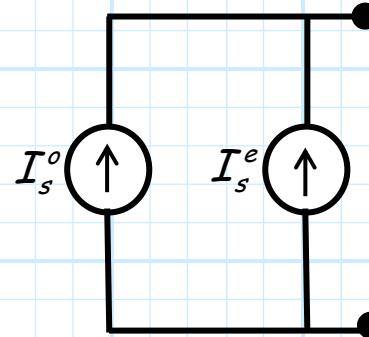
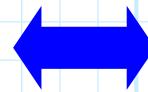
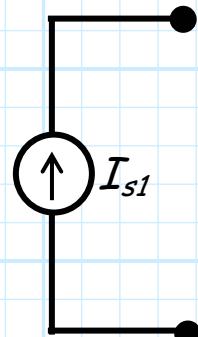
And then combine these results in a **linear superposition**!

Q: What about current sources? Can I likewise consider them to be a sum of an odd mode source and an even mode source?

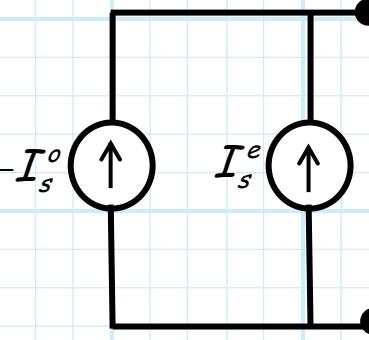
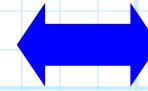
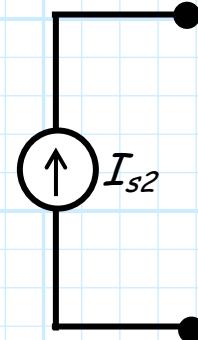
A: Yes, but be **very** careful! The current of two source will add if they are placed in parallel—not in series! Therefore:

$$I_s^e = \frac{I_{s1} + I_{s2}}{2}$$

$$I_s^o = \frac{I_{s1} - I_{s2}}{2}$$



$$I_{s1} = I_s^e + I_s^o$$



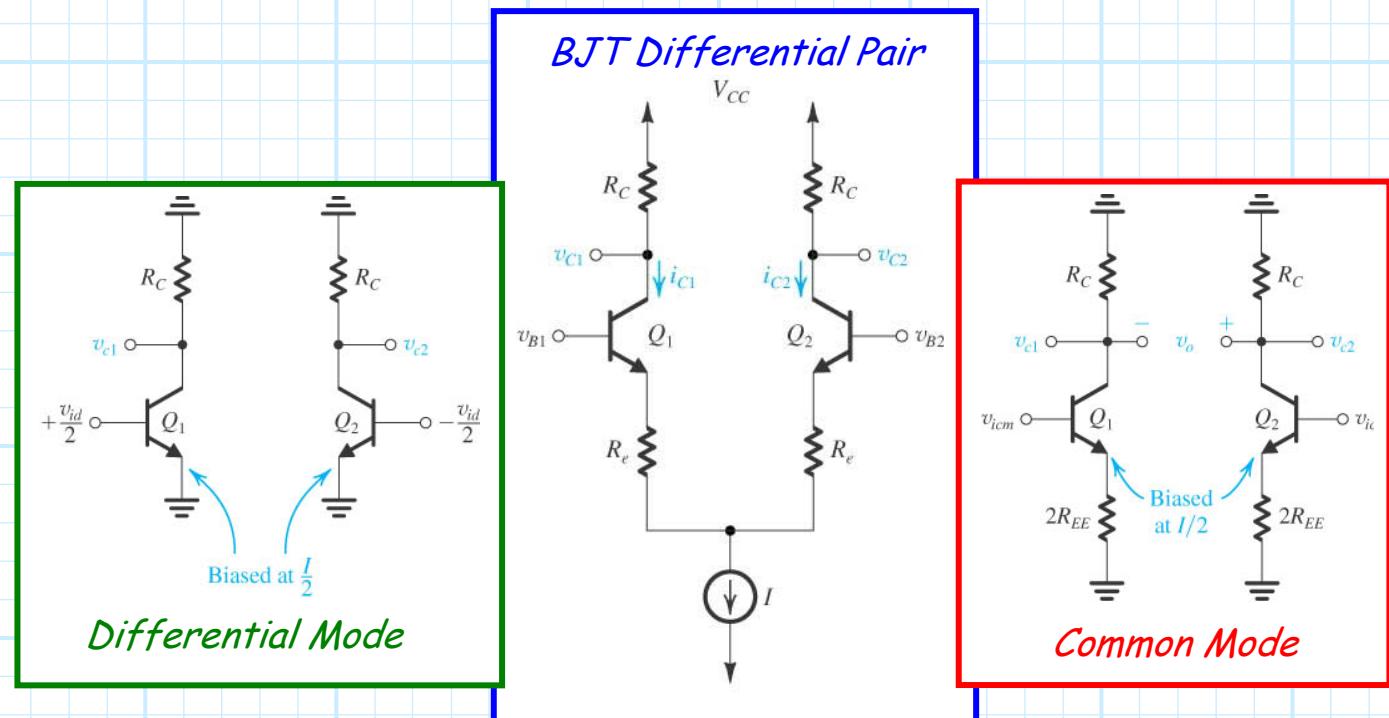
$$I_{s2} = I_s^e - I_s^o$$

One final word (I promise!) about circuit symmetry and even/odd mode analysis: precisely the same concept exists in **electronic circuit design!**

Specifically, the **differential** (odd) and **common** (even) mode analysis of bilaterally symmetric electronic circuits, such as **differential amplifiers!**

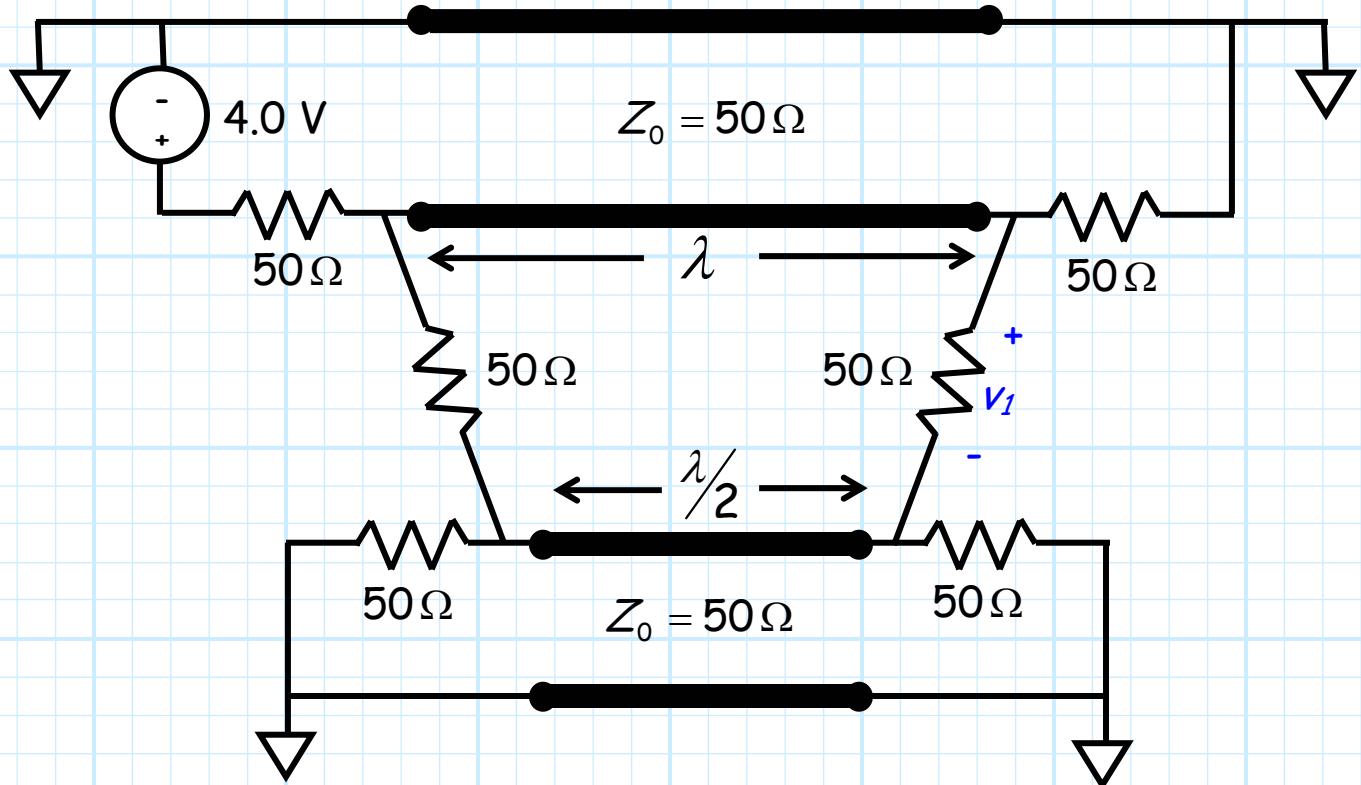


Hi! You might remember differential and common mode analysis from such classes as "EECS 412- Electronics II", or handouts such as "Differential Mode Small-Signal Analysis of BJT Differential Pairs"



Example: Odd-Even Mode Circuit Analysis

Carefully (**very** carefully) consider the **symmetric** circuit below.

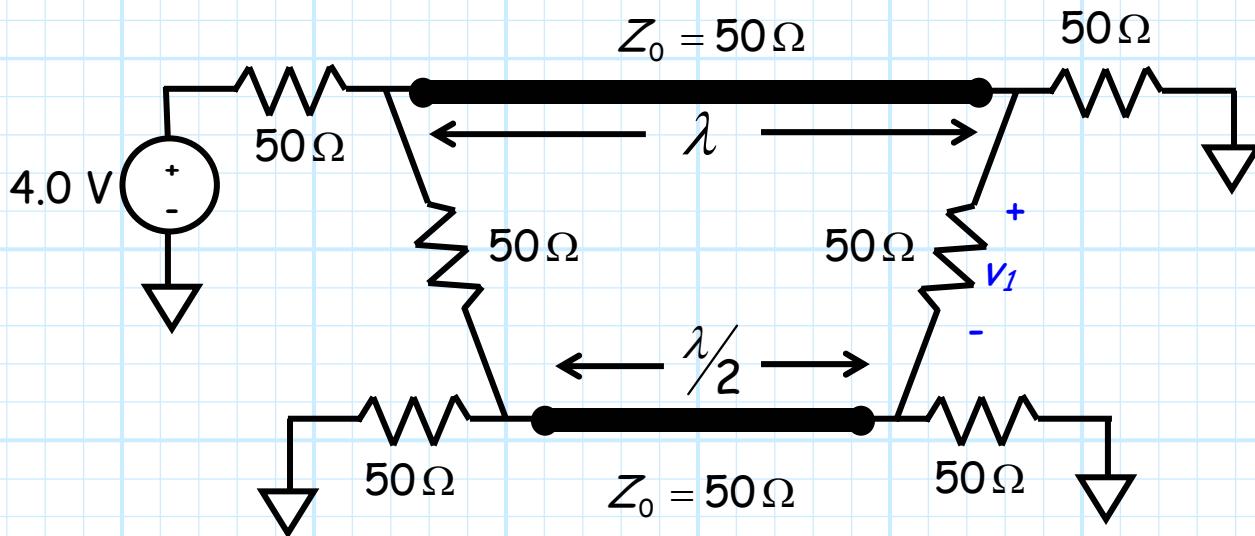


The two transmission lines each have a characteristic impedance of .

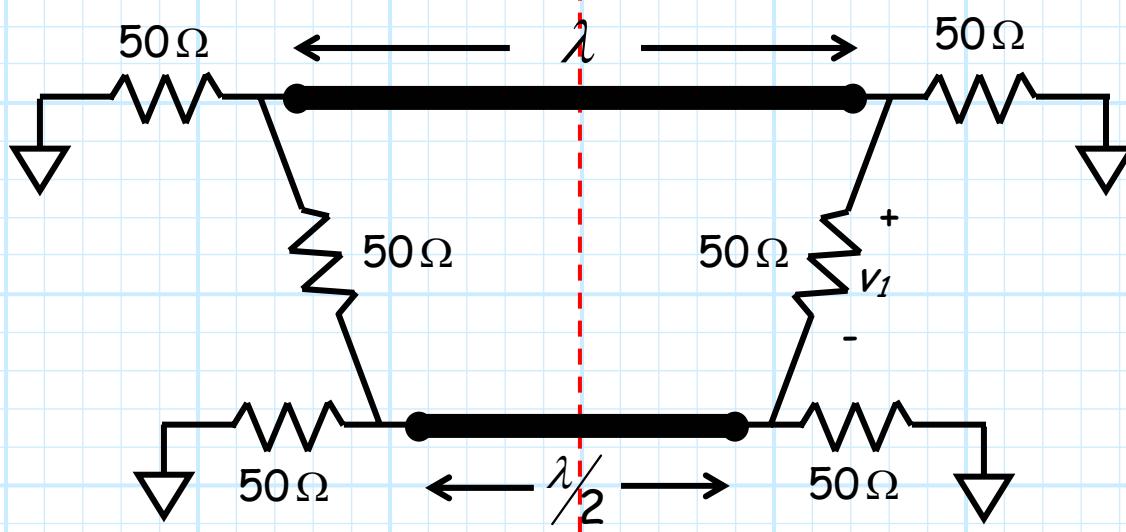
Use odd-even mode analysis to determine the value of voltage v_1 .

Solution

To simplify the circuit schematic, we first remove the bottom (i.e., ground) conductor of each transmission line:

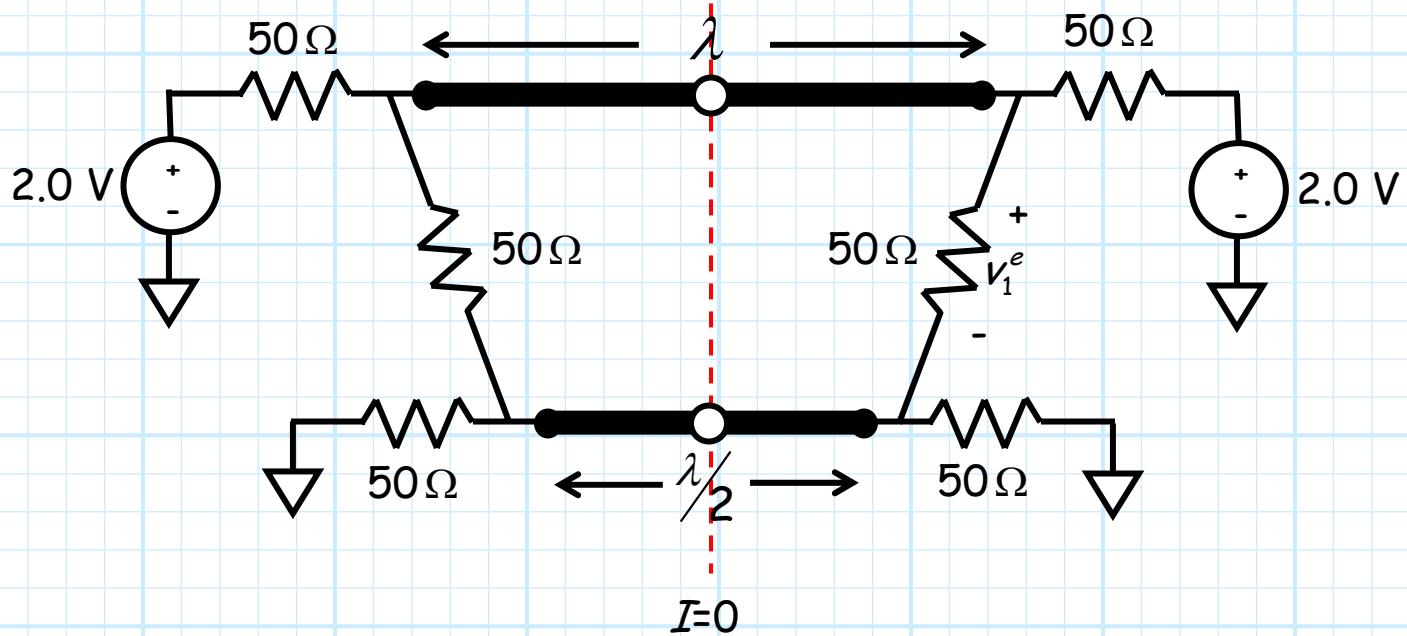


Note that the circuit has one plane of **bilateral symmetry**:

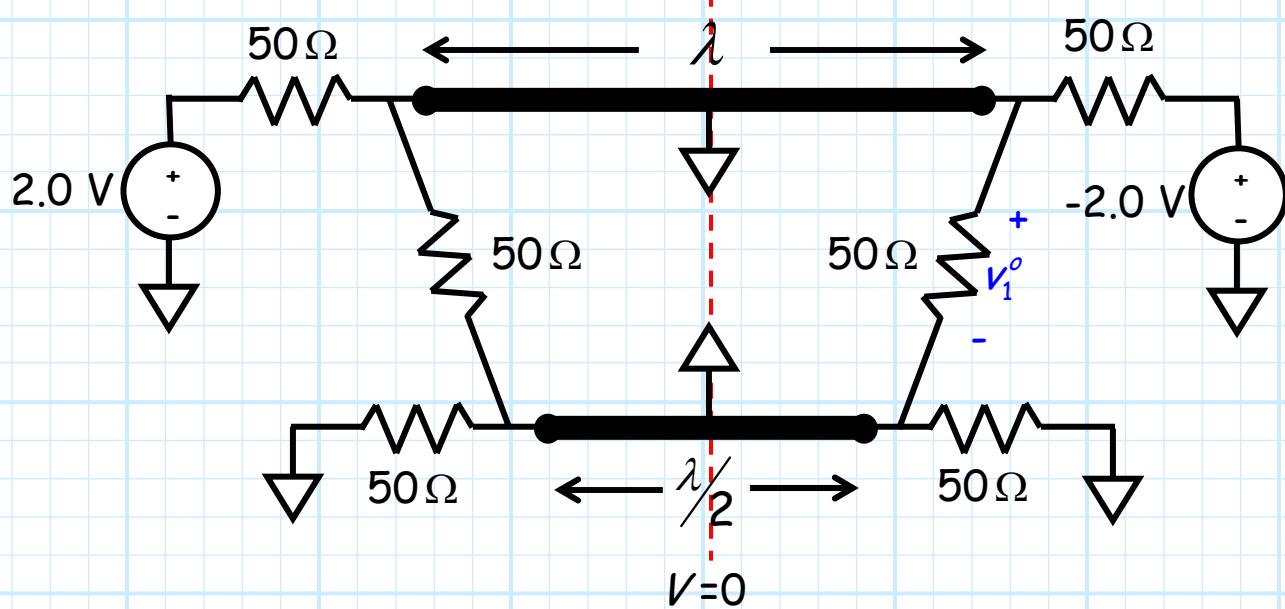


Thus, we can analyze the circuit using **even/odd mode analysis** (Yeah!).

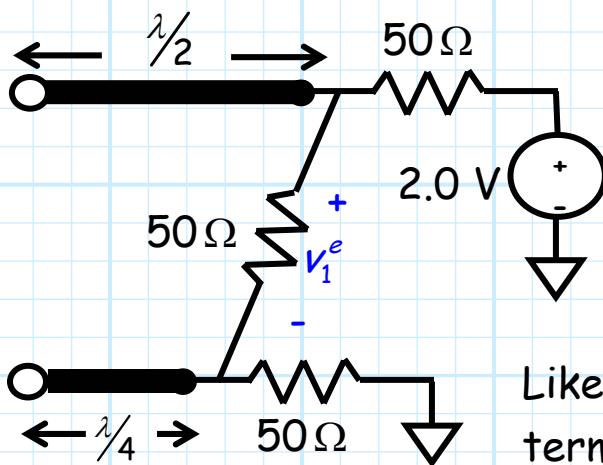
The **even mode** circuit is:



Whereas the **odd mode** circuit is:



We split the modes into half-circuits from which we can determine voltages v_1^e and v_1^o :

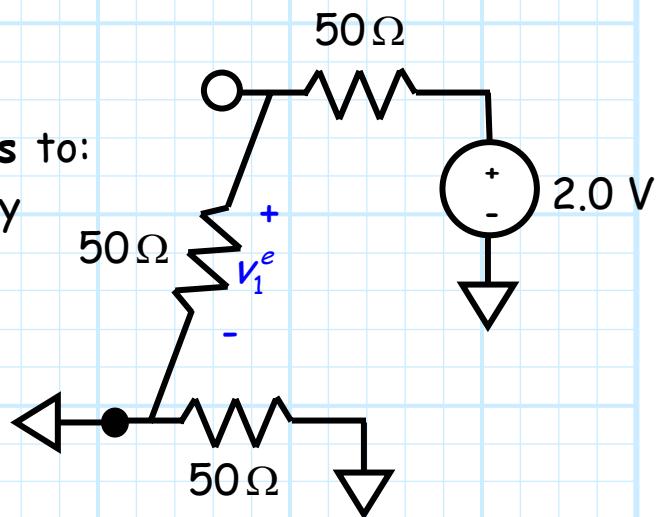


Recall that a $\ell = \frac{\lambda}{2}$ transmission line terminated in an open circuit has an input impedance of $Z_{in} = \infty$ — an **open** circuit!

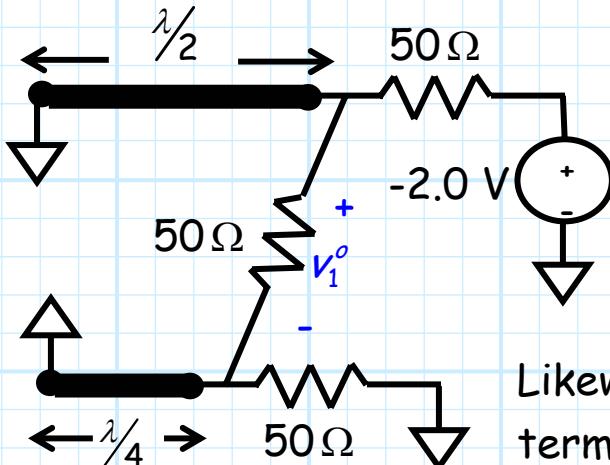
Likewise, a transmission line $\ell = \frac{\lambda}{4}$ terminated in an open circuit has an input impedance of $Z_{in} = 0$ — a **short** circuit!

Therefore, this half-circuit **simplifies** to:
And therefore the voltage v_1^e is easily determined via voltage division:

$$v_1^e = 2 \left(\frac{50}{50 + 50} \right) = 1.0\text{ V}$$



Now, examine the right half-circuit of the **odd mode**:

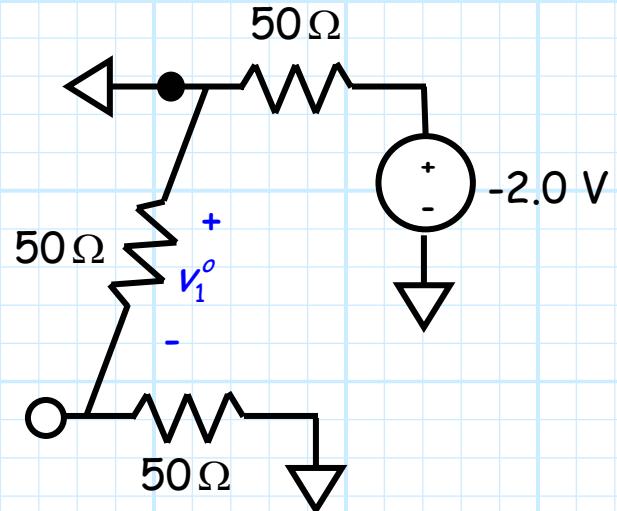


Recall that a $\ell = \frac{\lambda}{2}$ transmission line terminated in a short circuit has an input impedance of $Z_{in} = 0$ — a **short** circuit!

Likewise, a transmission line $\ell = \frac{\lambda}{4}$ terminated in a short circuit has an input impedance of $Z_{in} = \infty$ — an **open** circuit!

This half-circuit simplifies to →

It is apparent from the circuit
that the voltage $v_1^o = 0$!



Thus, the superposition of the odd and even modes leads to the result:

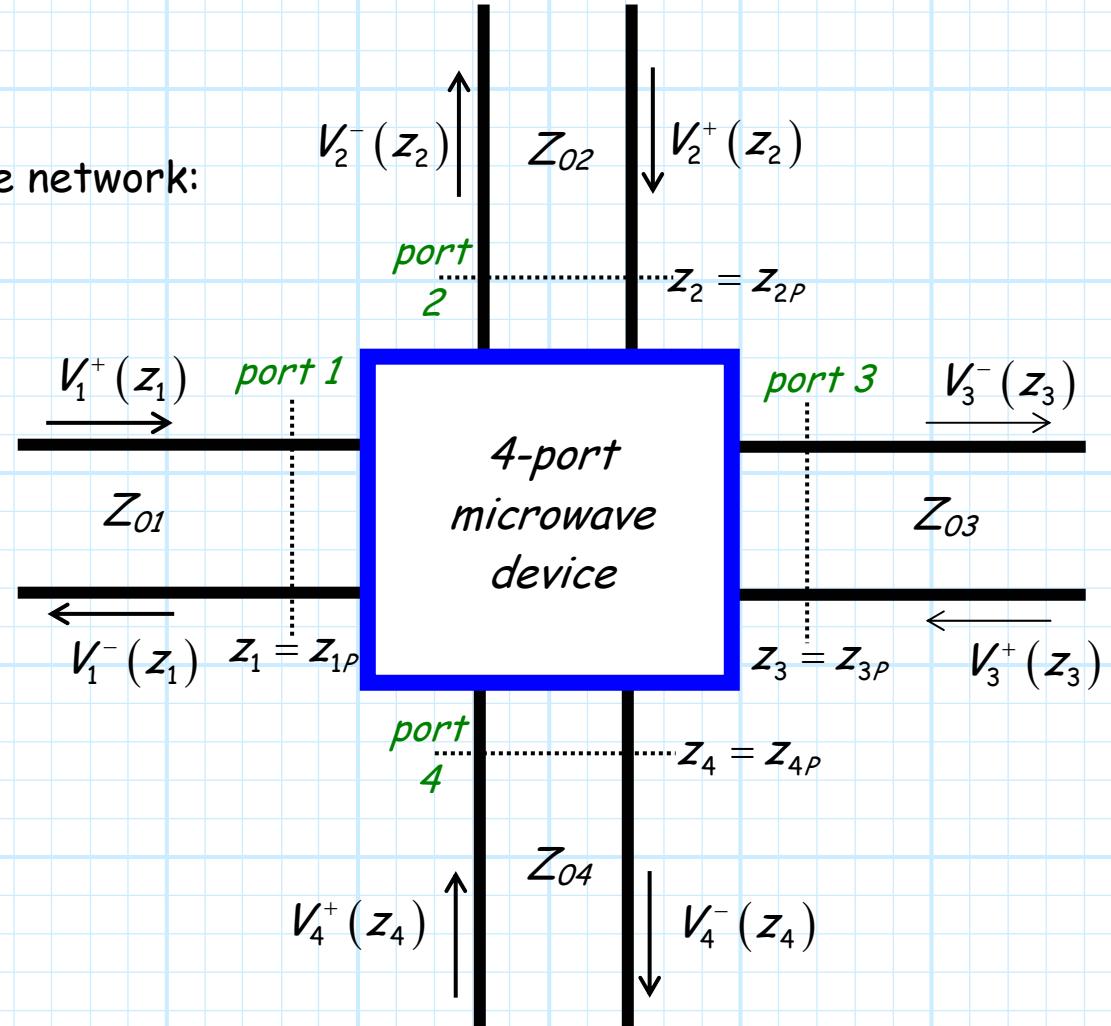
$$\underline{v_1} = \underline{v_1^e} + \underline{v_1^o} = \underline{1.0} + \underline{0} = \underline{\underline{1.0\text{V}}}$$

Generalized Scattering Parameters

Consider now this microwave network:

Q: Boring! We studied this before; this will lead to the definition of scattering parameters, right?

A: Not exactly. For this network, the characteristic impedance of each transmission line is different (i.e., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$)!



Q: Yikes! You said scattering parameters are **dependent** on transmission line characteristic impedance Z_0 . If these values are **different** for each port, which Z_0 do we use?

A: For this general case, we must use **generalized scattering parameters**! First, we define a slightly new form of complex wave amplitudes:

$$a_n = \frac{V_{0n}^+}{\sqrt{Z_{0n}}} \quad b_n = \frac{V_{0n}^-}{\sqrt{Z_{0n}}}$$

So for example:

$$a_1 = \frac{V_{01}^+}{\sqrt{Z_{01}}} \quad b_3 = \frac{V_{03}^-}{\sqrt{Z_{03}}}$$

The key things to note are:

a

A variable a (e.g., a_1, a_2, \dots) denotes the complex amplitude of an **incident** (i.e., plus) wave.

b

A variable b (e.g., b_1, b_2, \dots) denotes the complex amplitude of an **existing** (i.e., minus) wave.

We now get to **rewrite** all our transmission line knowledge in terms of these generalized complex amplitudes!



First, our two propagating wave amplitudes (i.e., plus and minus) are **compactly** written as:

$$V_{0n}^+ = a_n \sqrt{Z_{0n}}$$

$$V_{0n}^- = b_n \sqrt{Z_{0n}}$$

And so:

$$V_n^+(z_n) = a_n \sqrt{Z_{0n}} e^{-j\beta z_n}$$

$$V_n^-(z_n) = b_n \sqrt{Z_{0n}} e^{+j\beta z_n}$$

$$\Gamma(z_n) = \frac{b_n}{a_n} e^{+j2\beta z_n}$$

Likewise, the total voltage, current, and impedance are:

$$V_n(z_n) = \sqrt{Z_{0n}} (a_n e^{-j\beta z_n} + b_n e^{+j\beta z_n})$$

$$I_n(z_n) = \frac{a_n e^{-j\beta z_n} - b_n e^{+j\beta z_n}}{\sqrt{Z_{0n}}}$$

$$Z(z_n) = \frac{a_n e^{-j\beta z_n} + b_n e^{+j\beta z_n}}{a_n e^{-j\beta z_n} - b_n e^{+j\beta z_n}}$$

Assuming that our port planes are defined with $z_{n,p} = 0$, we can determine the total voltage, current, and impedance at port n as:

$$V_n \doteq V_n(z_n = 0) = \sqrt{Z_{0n}}(a_n + b_n) \quad I_n \doteq I_n(z_n = 0) = \frac{a_n - b_n}{\sqrt{Z_{0n}}}$$

$$Z_n \doteq Z(z_n = 0) = \frac{a_n + b_n}{a_n - b_n}$$

Likewise, the power associated with each wave is:

$$P_n^+ = \frac{|V_{0n}^+|^2}{2Z_{0n}} = \frac{|a_n|^2}{2} \quad P_n^- = \frac{|V_{0n}^-|^2}{2Z_{0n}} = \frac{|b_n|^2}{2}$$

As such, the power delivered to port n (i.e., the power absorbed by port n) is:

$$P_n = P_n^+ - P_n^- = \frac{|a_n|^2 - |b_n|^2}{2}$$

This result is also verified:

$$\begin{aligned}
 P_n &= \frac{1}{2} \operatorname{Re} \{ V_n I_n^* \} \\
 &= \frac{1}{2} \operatorname{Re} \{ (a_n + b_n)(a_n^* - b_n^*) \} \\
 &= \frac{1}{2} \operatorname{Re} \{ a_n a_n^* + b_n a_n^* - a_n b_n^* - b_n b_n^* \} \\
 &= \frac{1}{2} \operatorname{Re} \{ |a_n|^2 + b_n a_n^* - (b_n a_n^*)^* - |b_n|^2 \} \\
 &= \frac{1}{2} \operatorname{Re} \{ |a_n|^2 + j \operatorname{Im} \{ b_n a_n^* \} - |b_n|^2 \} \\
 &= \frac{|a_n|^2 - |b_n|^2}{2}
 \end{aligned}$$

Q: So what's the big deal? This is yet another way to express transmission line activity. Do we really need to know this, or is this simply a strategy for making the next exam even harder?

$$Z_1 = \frac{a_1 + b_1}{a_1 - b_1}$$



You may have noticed that this notation (a_n, b_n) provides descriptions that are a bit "cleaner" and more symmetric between current and voltage.

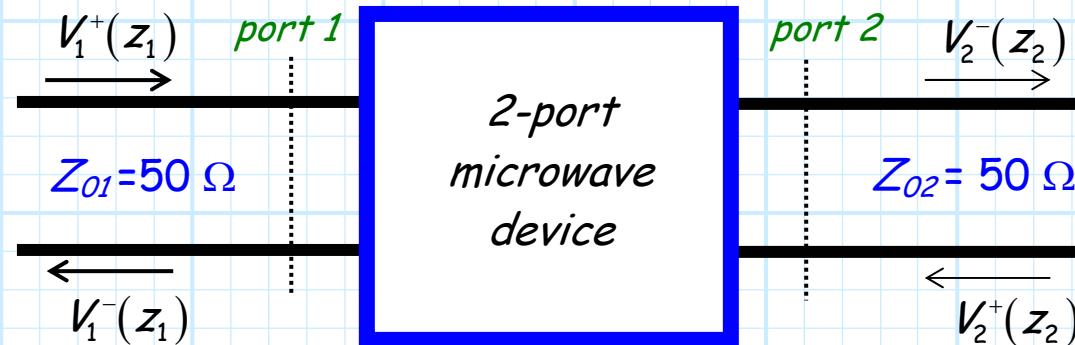
However, the main reason for this notation is for evaluating the scattering parameters of a device with dissimilar transmission line impedance (e.g., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$).

For these cases we must use **generalized scattering parameters**:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} \quad (\text{when } V_k^+(z_k) = 0 \text{ for all } k \neq n)$$

Note that if the transmission lines at each port are identical ($Z_{0m} = Z_{0n}$), the scattering parameter definition "reverts back" to the original (i.e., $S_{mn} = V_{0m}^- / V_{0n}^+$ if $Z_{0m} = Z_{0n}$). E.G.:

$$S_{21} = \frac{V_{02}^-}{V_{01}^+} \quad \text{when } V_{02}^+ = 0$$



But, if the transmission lines at each port are **dissimilar** ($Z_{0m} \neq Z_{0n}$), our original scattering parameter definition is **not correct** (i.e., $S_{mn} \neq V_{0m}^-/V_{0n}^+$ if $Z_{0m} \neq Z_{0n}$)! E.G.:

$$S_{21} \neq \frac{V_{02}^-}{V_{01}^+} \quad \text{when } V_{02}^+ = 0$$



$$S_{21} = \frac{V_{02}^-}{V_{01}^+} \frac{\sqrt{50}}{\sqrt{75}} \quad \text{when } V_{02}^+ = 0$$

Note that the generalized scattering parameters can be more **compactly** written in terms of our **new** wave amplitude notation:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} = \frac{b_m}{a_n} \quad (\text{when } a_k = 0 \text{ for all } k \neq n)$$

Remember, this is the **generalized** form of scattering parameter—it **always** provides the correct answer, regardless of the values of Z_{0m} or Z_{0n} !

Q: But why can't we define the scattering parameter as $S_{mn} = V_{0m}^- / V_{0n}^+$, regardless of Z_{0m} or Z_{0n} ? Who says we must define it with those awful $\sqrt{Z_{0n}}$ values in there?

A: Good question! Recall that a lossless device is will **always** have a **unitary** scattering matrix. As a result, the scattering parameters of a lossless device will **always** satisfy, for example:

$$1 = \sum_{m=1}^M |S_{mn}|^2$$

This is true **only** if the scattering parameters are **generalized**!

The scattering parameters of a lossless device will form a unitary matrix **only** if defined as $S_{mn} = b_m / a_n$. If we use $S_{mn} = V_{0m}^- / V_{0n}^+$, the matrix will be unitary **only** if the connecting transmission lines have the **same** characteristic impedance.

Q: Do we really care if the matrix of a lossless device is unitary or not?

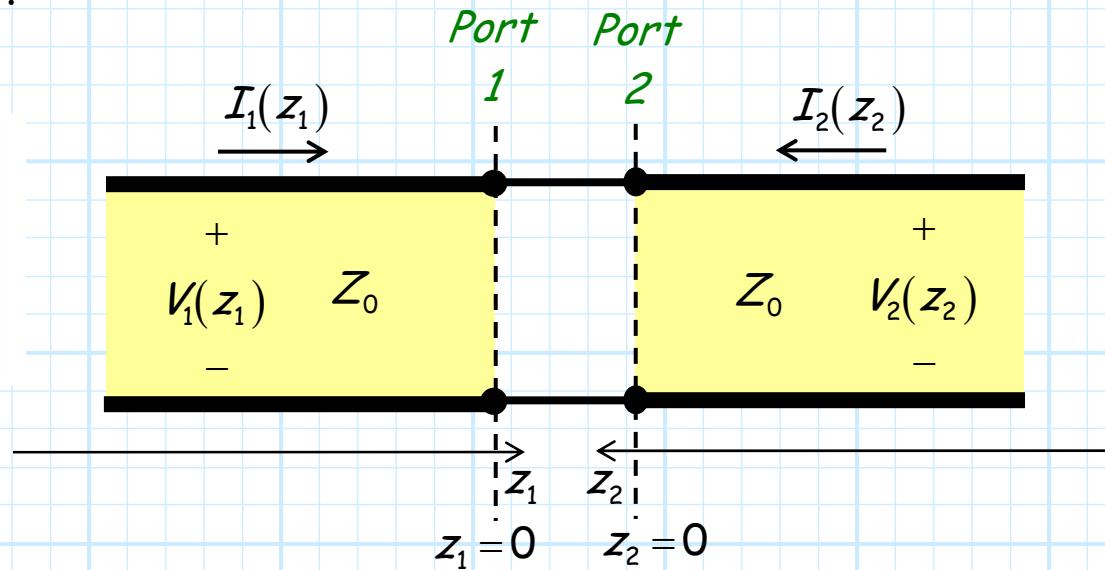
A: Absolutely we do! The:

$$\text{lossless device} \Leftrightarrow \text{unitary scattering matrix}$$

relationship is a very powerful one. It allows us to identify lossless devices, and it allows us to determine if specific lossless devices are even possible!

Example: The Scattering Matrix of a Connector

First, let's consider the scattering matrix of a **perfect connector**—an electrically **very small** two-port device that allows us to connect the ends of different transmission lines together.



If the connector is ideal, then it will exhibit **no** series inductance **nor** shunt capacitance, and thus from KVL and KCL:

$$V_1(z_1=0) = V_2(z_2=0)$$

$$I_1(z_1=0) = -I_2(z_2=0)$$

Terminating port 2 in a **matched load**, and then analyzing the resulting circuit, we find that (not surprisingly!):

$$V_{01}^- = 0 \quad \text{and} \quad V_{02}^- = V_{01}^+$$

From this we conclude that (since $V_{02}^+ = 0$):

$$S_{11} = \frac{V_{01}^-}{V_{01}^+} = \frac{0}{V_{01}^+} = 0.0$$

$$S_{21} = \frac{V_{02}^-}{V_{01}^+} = \frac{V_{01}^+}{V_{01}^+} = 1.0$$

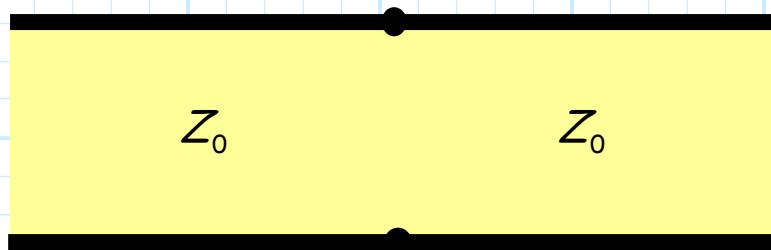
This two-port device has D_2 symmetry (a plane of bilateral symmetry), meaning:

$$S_{22} = S_{11} = 0.0 \quad \text{and} \quad S_{21} = S_{12} = 1.0$$

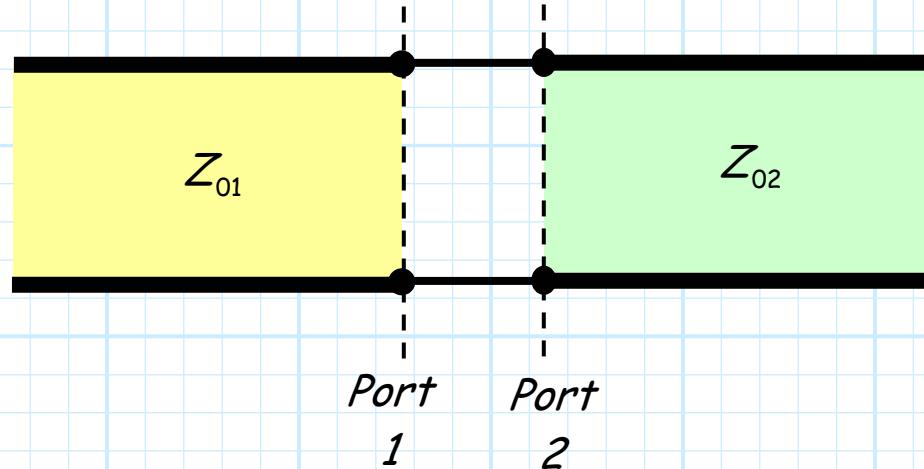
The scattering matrix for such this ideal connector is therefore:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

As a result, the perfect connector allows two transmission lines of identical characteristic impedance to be connected together into one "seamless" transmission line.



Now, however, consider the case where the transmission lines connected together have **dissimilar** characteristic impedances (i.e., $Z_0 \neq Z_1$):



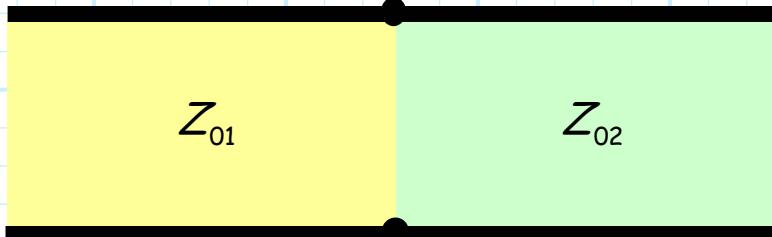
Q: Won't the scattering matrix of this ideal connector remain the same? After all, the device itself has not changed!

A: The impedance, admittance, and transmission matrix will remained unchanged—these matrix quantities do not depend on the characteristics of the transmission lines connected to the device.

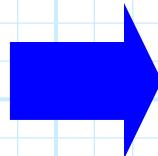
But remember, the scattering matrix depends on **both** the device and the characteristic impedance of the transmission lines attached to it.

→ After all, the **incident** and **exiting** waves are traveling on these transmission lines!

The ideal connector in this case establishes a "seamless" **interface** between two **dissimilar** transmission lines.



Remember, this is the same structure that we evaluated in an earlier handout!



In that analysis we found that—when $V_{02}^+ = 0$:

$$\frac{V_{01}^-}{V_{01}^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad \text{and} \quad \frac{V_{02}^-}{V_{01}^+} = \frac{2Z_{02}}{Z_{02} + Z_{01}}$$

And so the (generalized) scattering parameters S_{11} and S_{21} are:

$$S_{11} = \frac{V_{01}^-}{V_{01}^+} \frac{\sqrt{Z_{01}}}{\sqrt{Z_{01}}} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad \text{and} \quad S_{21} = \frac{V_{02}^-}{V_{01}^+} \frac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}} = \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{02} + Z_{01}}$$

As a result we can conclude that the scattering matrix of the ideal connector (when connecting dissimilar transmission lines) is:

$$S = \begin{bmatrix} \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} & \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{02} + Z_{01}} \\ \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{01} + Z_{02}} & \frac{Z_{01} - Z_{02}}{Z_{01} + Z_{02}} \end{bmatrix}$$

Example: The Transmission Coefficient T

Consider this circuit:

I.E., a transmission line with characteristic impedance Z_t transitions to a different transmission line at location $z=0$. This second transmission line has different characteristic impedance Z_2 ($Z_1 \neq Z_2$). This second line is terminated with a load $Z_L = Z_2$ (i.e., the second line is matched).

Q: What is the voltage and current along each of these two transmission lines? More specifically, what are V_{01} , V_{01}^- , V_{02} and V_{02}^- ??

A: Since a source has not been specified, we can only determine V_{01} , V_{01}^- and V_{02} in terms of complex constant V_{01}^+ . To accomplish this, we must apply a boundary condition at $z=0$!