Chapter 5 – Impedance Matching and Tuning

One of the most important and fundamental two-port networks that microwave engineers design is a lossless matching network (otherwise known as an impedance transformer).

**HO: Matching Networks**

**Q:** In microwave circuits, a source and load are connected by a transmission line. Can we implement matching networks in transmission line circuits?

**A:** *HO: Matching Networks and Transmission Lines*

**Q:** These matching networks seem too good to be true—can we really design and construct them to provide a perfect match?

**A:** We can easily provide a near perfect match at precisely one frequency.

But, since lossless matching and transmission lines are made of entirely reactive elements (not to mention the reactive components of source and load impedance), we find that changing the frequency will typically “unmatch” our circuit!
Thus, a difficult challenge for any microwave design engineer is to design a **wideband** matching network—a matching network that provides an “adequate” match over a wide range of frequencies.

Generally speaking, matching network design requires a **trade-off** between these for desirable attributes:

1. *Bandwidth*
2. *Complexity*
3. *Implementation*
4. *Adjustability*

**5.1 – Matching with Lumped Elements**

*Reading Assignment: pp. 222-228*

Now let’s begin to examine how matching networks are built!

We begin with the **simplest** solution: An **L-network**, consisting of a **single capacitor** and a **single inductor**.

**Q:** Just two elements! That seems simple enough. Do we **always** use these L-networks when constructing lossless matching networks?
A: Nope. L-networks have two major drawbacks:

1. They are narrow-band.

2. Capacitors and inductors are difficult to make at microwave frequencies!

Now, let's see how these L-networks actually work:

HO: L-NETWORK ANALYSIS
Matching Networks

Consider again the problem where a passive load is attached to an active source:

\[ Z_L = R_L + jX_L \]

The load will absorb power—power that is delivered to it by the source.

\[ P_L = \frac{1}{2} V_g^2 \frac{R_L}{|Z_g + Z_L|^2} \]

Recall that the power delivered to the load will be maximized (for a given \( V_g \) and \( Z_g \)) if the load impedance is equal to the complex conjugate of the source impedance (\( Z_L = Z_g^* \)). We call this maximum power the available power \( P_{\text{avl}} \) of the source—it is, after all, the largest amount of power that the source can ever deliver!
\[ P_{L}^{\text{max}} = P_{avl} = \frac{1}{2} |V_g|^2 \frac{R_g}{|Z_g + Z_r|^2} = \frac{1}{2} |V_g|^2 \frac{R_g}{2R_g} = \frac{|V_g|^2}{8R_g} \]

* Note the available power of the source is dependent on source parameters only (i.e., \(V_g\) and \(R_g\)). This makes sense! Do you see why?

* Thus, we can say that to “take full advantage” of all the available power of the source, we must to make the load impedance the complex conjugate of the source impedance.

* Otherwise, the power delivered to the load will be less than power made available by the source! In other “words”:

\[ P_L \leq P_{avl} \]

Q: But, you said that the load impedance typically models the input impedance of some useful device. We don’t typically get to “select” or adjust this impedance—it is what it is. Must we then simply accept the fact that the delivered power will be less than the available power?
A: NO! We can in fact modify our circuit such that all available source power is delivered to the load—without in any way altering the impedance value of that load!

To accomplish this, we must insert a **matching network** between the source and the load:

The sole purpose of this matching network is to “transform” the load impedance into an input impedance that is conjugate matched to the source! I.E.:

\[ Z_{in} = Z_g^* \]

\[ Z_L = R_L + jX_L \]
Because of this, all available source power is delivered to the input of the matching network (i.e., delivered to $Z_{in}$):

$$P_{in} = P_{avl}$$

**Q:** Wait just one second! The matching network ensures that all available power is delivered to the input of the matching network, but that does not mean (necessarily) that this power will be delivered to the load $Z_L$. The power delivered to the load could still be much less than the available power!

**A:** True! To ensure that the available power delivered to the input of the matching network is entirely delivered to the load, we must construct our matching network such that it cannot absorb any power—the matching network must be lossless!

We must construct our matching network entirely with reactive elements!

Examples of reactive elements include inductors, capacitors, transformers, as well as lengths of lossless transmission lines.
Thus, constructing a proper lossless matching network will lead to the **happy** condition where:

\[ P_L = P_{in} = P_{avl} \]

* Note that the design and construction of this lossless network will depend on **both** the value of source impedance \( Z_g \) and load impedance \( Z_L \).

* However, the matching network does **not physically alter** the values of either of these two quantities—the source and load are left physically unchanged!

Now, let’s consider the matching network from a **different perspective**. Instead of defining it in terms of its input impedance when attached the load, let’s describe it in terms of its output impedance when attached to the source:
This “new” source (i.e., the original source with the matching network attached) can be expressed in terms of its Thevenin’s equivalent circuit:

\[ Z_{out} = R_{out} + jX_{out} \]

Note that in general that \( V_s \neq V_g \) and \( Z_{out} \neq Z_g \)—the matching network “transforms” both the values of both the impedance and the voltage source.

**Q:** Arrrgg! Doesn’t that mean that the available power of this “transformed” source will be different from the original?

**A:** Nope. If the matching network is lossless, the available power of this equivalent source is identical to the available power of the original source—the lossless matching network does not alter the available power!
Now, for a **properly** designed, **lossless** matching network, it turns out that (as you might have expected!) the output impedance $Z_{out}$ is equal to the **complex conjugate** of the load impedance. I.E.:

$$Z_{out} = Z_L^*$$

The source and load are again matched!

Thus, we can look at the matching network in two equivalent ways:
1. As a network attached to a load, one that “transforms” its impedance to $Z_{in}$—a value matched to the source impedance $Z_g$:

\[ Z_{in} = Z_g^* \]

2. Or, as network attached to a source, one that “transforms” its impedance to $Z_{out}$—a value matched to the load impedance $Z_L$:

\[ Z_{out} = Z_L^* \]

Either way, the source and load impedance are conjugate matched—all the available power is delivered to the load!
Matching Networks and Transmission Lines

Recall that a primary purpose of a transmission line is to allow the transfer of power from a source to a load.

Q: So, say we directly connect an arbitrary source to an arbitrary load via a length of transmission line. Will the power delivered to the load be equal to the available power of the source?

A: Not likely! Remember we determined earlier that the efficacy of power transfer depends on:

1. the source impedance $Z_s$.
2. load impedance $Z_L$.

3. the transmission line characteristic impedance $Z_0$.

4. the transmission line length $\ell$.

Recall that maximum power transfer occurred only when these four parameters resulted in the input impedance of the transmission line being equal to the complex conjugate of the source impedance (i.e., $Z_{in}^* = Z_g$).

It is of course unlikely that the very specific conditions of a conjugate match will occur if we simply connect a length of transmission line between an arbitrary source and load, and thus the power delivered to the load will generally be less than the available power of the source.

**Q:** Is there any way to use a matching network to fix this problem? Can the power delivered to the load be increased to equal the available power of the source if there is a transmission line connecting them?

**A:** There sure is! We can likewise construct a matching network for the case where the source and load are connected by a transmission line.
For example, we can construct a network to transform the input impedance of the transmission line into the complex conjugate of the source impedance:

Q: But, do we have to place the matching network between the source and the transmission line?

A: Nope! We could also place a (different) matching network between the transmission line and the load.
In either case, we find that at any and all points along this matched circuit, the output impedance of the equivalent source (i.e., looking left) will be equal to the complex conjugate of the input impedance (i.e., looking right).

\[ Z_{out} = Z_{in}^* \]

**Q:** So which method should we chose? Do engineers typically place the matching network between the source and the transmission line, or place it between the transmission line and the load?

**A:** Actually, the typical solution is to do both!
We find that often there is a matching network between the a source and the transmission line, and between the line and the load.

The first network matches the source to the transmission line—in other words, it transforms the output impedance of the equivalent source to a value numerically equal to characteristic impedance $Z_0$.

The second network matches the load to the transmission line—in other words it transforms the load impedance to a value numerically equal to characteristic impedance $Z_0$:

\[ Z_{in} = Z_0 \]

**Q:** Yikes! Why would we want to build two separate matching networks, instead of just one?

**A:** By using two separate matching networks, we can decouple the design problem. Recall again that the design of a single matching network solution would depend on four separate parameters:

1. the source impedance $Z_s$.
2. load impedance $Z_L$.
3. the transmission line characteristic impedance $Z_0$.
4. the transmission line length $\ell$. 
Alternatively, the design of the network matching the source and transmission line depends on only:

1. the load impedance $Z_L$.
2. the transmission line characteristic impedance $Z_0$.

Whereas, the design of the network matching the load and transmission line depends on only:

1. the source impedance $Z_S$.
2. the transmission line characteristic impedance $Z_0$.

Note that neither design depends on the transmission line length $\ell$!

**Q:** How is that possible?

**A:** Remember the case where $Z_L = Z_0 = Z_L$. For that special case, we found that a conjugate match was the result—regardless of the transmission line length.
Thus, by matching the source to line impedance $Z_0$ and likewise matching the load to the line impedance, a conjugate match is assured—but the length of the transmission line does not matter!

In fact, the typically problem for microwave engineers is to match a load (e.g., device input impedance) to a standard transmission line impedance (typically $Z_0 = 50\,\Omega$); or to independently match a source (e.g., device output impedance) to a standard line impedance.

A conjugate match is thus obtained by connecting the two with a transmission line of any length!
L-Network Analysis

Consider the first matching L-network, which we shall denote as matching network (A):

Note that this matching network consists of just two lumped elements, which must be purely reactive—in other words, a capacitor and an inductor!

To make $\Gamma_{in} = 0$, the input impedance of the network must be:

$$Z_{in} = Z_0$$

Note that using basic circuit analysis we find that this input impedance is:

$$Z_{in} = jX + \left(\frac{1}{jB}\right)Z_L = jX + \frac{Z_L}{1 + jBZ_L}$$
Note that a **matched** network, with $Z_{in} = Z_0$, means that:

$$\text{Re}\{Z_{in}\} = Z_0 \quad \text{AND} \quad \text{Im}\{Z_{in}\} = 0$$

Note that there are **two** equations.

This works out well, since we have **two** unknowns ($B$ and $X$)!

Essentially, the L-network matching network can be viewed as consisting of **two distinct parts**, each attempting to satisfy a specific requirement.

**Part 1: Selecting** $Y = jB$

Since the shunt element $Y$ and $Z_L$ are in **parallel**, we can combine them into one element that we shall call $Y_1$:

$$Y_1 = Y + \frac{1}{Z_L} = jB + Y_L$$

The impedance of this element is therefore:

$$Z_1 = \frac{1}{Y_1} = \frac{1}{jB + Y_L} = \frac{Z_L}{Z_L + jBZ_L}$$
To achieve a perfect match, we must set the value of susceptance $B$ such that:

$$\Re \{Z_1\} = \Re \left\{ \frac{Z_L}{Z_L + jBZ_L} \right\} = Z_0$$

Thus, if $B$ is properly selected:

$$Z_1 = Z_0 + jX_1$$

Hopefully, the second part of the matching is now very obvious to you!
Part 2: Selecting $Z = jX$

Note that the impedance $Z_1 = Z_L \| \frac{1}{j\beta}$ has the ideal real value of $Z_0$. However, it likewise possesses an annoying imaginary part of:

$$X_1 = \text{Im}\{Z_1\} = \text{Im}\left(\frac{Z_L}{Z_L + j\beta Z_L}\right)$$

However, this imaginary component is easily removed by setting the series element $Z = jX$ to its equal but opposite value! I.E.,

$$X = -X_1 = -\text{Im}\left(\frac{Z_L}{Z_L + j\beta Z_L}\right)$$
Thus, we find that:

\[ Z_{in} = Z + Z_1 \]
\[ = -jX_1 + Z_0 + jX_1 \]
\[ = Z_0 \]

We have created a **perfect match**!

Going through this complex algebra, we can solve for the required values \( X \) and \( B \) to satisfy these two equations—to create a matched network!

\[
B = \frac{X_L \pm \sqrt{R_L/Z_0} \sqrt{R_L^2 + X_L^2 - Z_0 R_L}}{R_L^2 + X_L^2}
\]

and,

\[
X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{B R_L}
\]

where \( Z_L = R_L + jX_L \).
Note:

1) Because of the ±, there are two solutions for \( B \) (and thus \( X \)).

2) For \( j\beta \) to be purely imaginary (i.e., reactive), \( B \) must be real. From the term:

\[
\sqrt{R_L^2 + X_L^2 - Z_0 R_L}
\]

in the expression for \( B \), we note that \( R_L \) must be greater than \( Z_0 \) (\( R_L > Z_0 \)) to insure that \( B \) and thus \( X \) is real.

In other words, this matching network can only be used when \( R_L > Z_0 \). Notice that this condition means that the normalized load \( z'_L \) lies inside the \( r = 1 \) circle on the Smith Chart!
Now let's consider the **second** of the two L-networks, which we shall call network (B). Note it also is formed with just two lumped elements.

![Diagram of L-network](image)

$Z_0, \beta \quad \Gamma(0) = \Gamma_{in} \quad Z(0) = Z_{in} \quad Y = jB \quad Z = jX \quad Z_L$

To make $\Gamma_{in} = 0$, the **input admittance** of the network must be:

$$Y_{in} = Y_0$$

Note from circuit theory that the input **admittance** for this network is:

$$Y_{in} = jB + \frac{1}{jX + Z_L}$$

Therefore a **matched** network, with $Y_{in} = Y_0$, is described as:

$$\text{Re}\{Y_{in}\} = Y_0 \quad \text{AND} \quad \text{Im}\{Y_{in}\} = 0$$
For this design, we set the value of $Z = jX$ such that the admittance $Y_1$:

$$Y_1 = \frac{1}{Z + Z_L} = \frac{1}{jX + Z_L}$$

has a real part equal to $Y_0$:

$$Y_0 = \text{Re} \{Y\}_1 = \text{Re} \left\{ \frac{1}{jX + Z_L} \right\}$$

Now, it is evident that a perfect match will occur if the shunt element $Y = jB$ is set to "cancel" the reactive component of $Y_1$:

$$B = -\text{Im} \{Y\}_1 = -\text{Im} \left\{ \frac{1}{jX + Z_L} \right\}$$
So that we find:

\[ Y_m = Y + Y_1 = -j\beta_1 + (Y_0 + j\beta_1) = Y_0 \]

A perfect match!

With these two equations, we can directly solve for the required values \( X \) and \( B \) for a matched network:

\[ X = \pm \sqrt{R_L (Z_0 - R_L)} - X_L \]

and,

\[ B = \pm \sqrt{(Z_0 - R_L) / R_L} \]

where \( Z_L = R_L + jX_L \).
Note:

1) Because of the ±, there are two solutions for $B$ (and thus $X$).

2) For $jB$ and $jX$ to be purely imaginary (i.e., reactive), $B$ and $X$ must be real. We note from the term:

$$ (Z_0 - R_L) $$

that $R_L$ must be less than $Z_0$ ($R_L < Z_0$) to insure that $B$ and thus $X$ are real.

In other words, this matching network can only be used when $R_L < Z_0$. Notice that this condition means that the normalized load $z_L'$ lies outside the $r = 1$ circle on the Smith Chart!
Once the values of $X$ and $B$ are found, we can determine the required values of inductance $L$ and/or capacitance $C$, for the signal frequency $\omega_0$!

Recall that:

$$X = \begin{cases} 
\omega_0 L & \text{if } X > 0 \\
-\frac{1}{\omega_0 C} & \text{if } X < 0 
\end{cases}$$

and that:

$$B = \begin{cases} 
\omega_0 C & \text{if } B > 0 \\
-\frac{1}{\omega_0 L} & \text{if } B < 0 
\end{cases}$$

Make sure that you see and know why these equations are true.

As a result, we see that the reactance or susceptance of the elements of our L-network will have the proper values for matching at precisely one and only one frequency!

$\rightarrow$ And this frequency better be the signal frequency $\omega_0$!
If the signal frequency changes from this design frequency, the reactance and susceptance of the matching network inductors and capacitors will likewise change. The circuit will no longer be matched.

→ This matching network has a narrow bandwidth!

One other problem; it becomes very difficult to build quality lumped elements with useful values past 1 or 2 GHz. Thus, L-Network solutions are generally applicable only in the RF region (i.e., $< 2$GHz).

An L-Network Design Example