5.4 - The Quarter-Wave Transformer

Reading Assignment: pp. 73-76, 240-243

By now you’ve noticed that a quarter-wave length of transmission line ($\ell = \lambda/4$, $2\beta\ell = \pi$) appears often in microwave engineering problems.

Another application of the $\ell = \lambda/4$ transmission line is as an impedance matching network.

**HO: THE QUARTER-WAVE TRANSFORMER**

**HO: THE SIGNAL-FLOW GRAPH OF A QUARTER-WAVE TRANSFORMER**

**Q:** Why does the quarter-wave matching network work—after all, the quarter-wave line is mismatched at both ends?

**A:** **HO: MULTIPLE REFLECTION VIEWPOINT**
The Quarter-Wave Transformer

Say the end of a transmission line with characteristic impedance $Z_0$ is terminated with a resistive (i.e., real) load.

Unless $R_L = Z_0$, the resistor is mismatched to the line, and thus some of the incident power will be reflected.

We can of course correct this situation by placing a matching network between the line and the load:

In addition to the designs we have just studied (e.g., L-networks, stub tuners), one of the simplest matching network designs is the quarter-wave transformer.
The quarter-wave transformer is simply a transmission line with characteristic impedance $Z_1$ and length $\ell = \lambda/4$ (i.e., a quarter-wave line).

The $\lambda/4$ line is the matching network!

**Q:** But what about the characteristic impedance $Z_1$; what should its value be??

**A:** Remember, the quarter wavelength case is one of the special cases that we studied. We know that the input impedance of the quarter wavelength line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

Thus, if we wish for $Z_{in}$ to be numerically equal to $Z_0$, we find:

$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$
Solving for $Z_1$, we find its **required** value to be:

\[
\frac{(Z_1)^2}{R_L} = Z_0
\]

\[
(Z_1)^2 = Z_0 R_L
\]

\[
Z_1 = \sqrt{Z_0 R_L}
\]

In other words, the characteristic impedance of the quarter wave line is the **geometric average** of $Z_0$ and $R_L$!

Therefore, a $\lambda/4$ line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will match a transmission line with characteristic impedance $Z_0$ to a resistive load $R_L$.

Thus, all **power** is delivered to load $R_L$!

Alas, the quarter-wave transformer (like all our designs) has a few problems!
**Problem #1**

The matching **bandwidth** is **narrow**!

In other words, we obtain a **perfect** match at precisely the frequency where the length of the matching transmission line is a **quarter**-wavelength.

⇒ But remember, this length can be a quarter-wavelength at just **one** frequency!

Remember, **wavelength** is related to **frequency** as:

\[ \lambda = \frac{v_p}{f} = \frac{1}{f \sqrt{LC}} \]

where \( v_p \) is the **propagation velocity** of the wave.

For **example**, assuming that \( v_p = c \) (\( c = \) the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm (\( \lambda = 0.3 \text{ m} \)), while one wavelength at 3 GHz is 10 cm (\( \lambda = 0.1 \text{ m} \)). As a result, a transmission line length \( \ell = 7.5 \text{ cm} \) is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a **perfect** match (\( \Gamma_{in} = 0 \)) at **one** and **only one** signal frequency!
As the signal frequency (i.e., wavelength) changes, the electrical length of the matching transmission line changes. It will no longer be a quarter wavelength, and thus we no longer will have a perfect match.

We find that the closer $R_L (R_{in})$ is to characteristic impedance $Z_0$, the wider the bandwidth of the quarter wavelength transformer.

![Figure 5.12 (p. 243)](image)

**Figure 5.12 (p. 243)** Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

We will find that the bandwidth can be increased by adding multiple $\lambda/4$ sections!
Problem #2

Recall the matching solution was limited to loads that were purely real! I.E.:

\[ Z_L = R_L + j0 \]

Of course, this is a BIG problem, as most loads will have a reactive component!

Fortunately, we have a relatively easy solution to this problem, as we can always add some length \( \ell \) of transmission line to the load to make the impedance completely real:

However, remember that the input impedance will be purely real at only one frequency!

We can then build a quarter-wave transformer to match the line \( Z_0 \) to resistance \( R_{in} \):
Again, since the transmission lines are lossless, all of the incident power is delivered to the load $Z_l$.
The Signal Flow Graph of a Quarter-Wave Transformer

A quarter wave transformer can be thought of as a cascaded series of two two-port devices, terminated with a load $R_L$:

Q: Two two-port devices? It appears to me that a quarter-wave transformer is not that complex. What are the two two-port devices?

A: The first is a “connector”. Note a connector is the interface between one transmission line (characteristic impedance $Z_0$) to a second transmission line (characteristic impedance $Z_1$).
Recall that we earlier determined the scattering matrix of this two-port device:

$$S_x = \begin{bmatrix}
\frac{Z_1 - Z_0}{Z_1 + Z_0} & \frac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} \\
\frac{Z_1 + Z_0}{Z_1 + Z_0} & \frac{Z_0 - Z_1}{Z_0 + Z_1}
\end{bmatrix}$$

This result can be more compactly stated as:

$$S = \begin{bmatrix}
\Gamma & T \\
T & -\Gamma
\end{bmatrix}$$

where \( \Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} \) and \( T = \frac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} \)

The signal flow graph of this device is therefore:
Now, the second two-port device is a quarter wavelength of transmission line.

We know that it has the scattering matrix:

\[
S_y = \begin{bmatrix}
0 & e^{-j\beta\ell} \\
e^{-j\beta\ell} & 0
\end{bmatrix}
\]

Finally, a load has a "scattering matrix" of:

\[
S = \left[ \frac{R_L - Z_1}{R_L + Z_1} \right] = \Gamma_L
\]
Of course, if we connect the ideal connector to a quarter wavelength of transmission line, and terminate the whole thing with load $R_L$, we have formed a **quarter wave matching network**!

![Quarter Wave Network Diagram]

The boundary conditions associated with these connections are likewise:

$$
\begin{align*}
a_{1y} &= b_{2x} \\
a_{2x} &= b_{1y} \\
a_{1L} &= b_{2y} \\
a_{2y} &= b_{1L}
\end{align*}
$$

We can thus put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:
And simplifying:

Now, let’s see if we can reduce this graph to determine:

\[ \Gamma_{in} = \frac{b_{1x}}{a_{1x}} \]

From the series rule:

From the splitting rule:
From the **self-loop** rule:

\[ a_{1x} \xleftarrow{} \frac{T}{1 - \Gamma \Gamma_L} \xrightarrow{} b_{1x} \]

Again with the **series** rule:

\[ a_{1x} \xleftarrow{} \Gamma \xrightarrow{\frac{T^2 \Gamma_L e^{-j2\beta t}}{1 - \Gamma \Gamma_L}} b_{1x} \]

And finally with the **parallel** rule:

\[ a_{1x} \xrightarrow{} \Gamma \xleftarrow{} b_{1x} \]

So that:

\[ \Gamma_{in} \triangleq \frac{b_{1x}}{a_{1x}} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta t}}{1 - \Gamma \Gamma_L} \]
Q: Hey wait! If the quarter-wave transformer is a matching network, shouldn’t $\Gamma_{in} = 0$?

A: Who says it isn’t! Consider now three important facts.

For a quarter wave transformer, we set $Z_1$ such that:

$$Z_1^2 = Z_0 R_L \implies Z_0 = Z_1^2 / R_L$$

Inserting this into the scattering parameter $S_{11}$ of the connector, we find:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - Z_1^2 / R_L}{Z_1 + Z_1^2 / R_L} = \frac{R_L - Z_1}{R_L + Z_1}$$

Look at this result! For the quarter-wave transformer, the connector $S_{11}$ value (i.e., $\Gamma$) is the same as the load reflection coefficient $\Gamma_L$:

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \quad \leftarrow \text{Fact 1}$$

Since the connector is lossless (unitary scattering matrix!), we can conclude (and likewise show) that:

$$1 = |S_{11}|^2 + |S_{21}|^2 = |\Gamma|^2 + |T|^2$$

Since $Z_0$, $Z_1$, and $R_L$ are all real, the values $\Gamma$ and $T$ are also real valued. As a result, $|\Gamma|^2 = \Gamma^2$ and $|T|^2 = T^2$, and we can likewise conclude:
\[ \Gamma^2 + T^2 = 1 \quad \text{← Fact 2} \]

Likewise, the Z1 transmission line has \( \ell = \frac{\lambda}{4} \), so that:

\[ 2\beta\ell = 2 \left( \frac{2\pi}{\lambda} \right) \frac{\lambda}{4} = \pi \]

where you of course recall that \( \beta = \frac{2\pi}{\lambda} \)! Thus:

\[ e^{-j2\beta\ell} = e^{-j\pi} = -1 \quad \text{← Fact 3} \]

As a result:

\[ \Gamma_{in} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta\ell}}{1 - \Gamma \Gamma_L} = \Gamma - \frac{T^2 \Gamma_L}{1 - \Gamma \Gamma_L} \]

And using the newly discovered fact that (for a correctly designed transformer) \( \Gamma_L = \Gamma \):

\[ \Gamma_{in} = \Gamma - \frac{T^2 \Gamma_L}{1 - \Gamma \Gamma_L} = \Gamma - \frac{T^2 \Gamma}{1 - \Gamma^2} \]

And also are recent discovery that \( T^2 = 1 - \Gamma^2 \):

\[ \Gamma_{in} = \Gamma - \frac{T^2 \Gamma}{1 - \Gamma^2} = \Gamma - \frac{T^2 \Gamma}{T^2} = 0 \]

A perfect match! The quarter-wave transformer does indeed work!
**Multiple Reflection Viewpoint**

The quarter-wave transformer brings up an interesting question in μ-wave engineering.

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**Q:** Why is there no reflection at \( z = -\ell \)? It appears that the line is mismatched at both \( z = 0 \) and \( z = -\ell \).

**A:** In fact there are reflections at these mismatched interfaces—an infinite number of them!

We can use our signal flow graph to determine the propagation series, once we determine all the propagation paths through the quarter-wave transformer.
Now let's try to interpret what **physically** happens when the **incident** voltage wave:

\[ V^+(z) = a \sqrt{Z_0} e^{-j \beta (z + \ell)} \]

reaches the interface at \( z = -\ell \). We find that there are **two forward paths** through the quarter-wave transformer signal flow graph.
Path 1. At $z = -\ell$, the characteristic impedance of the transmission line changes from $Z_0$ to $Z_1$. This mismatch creates a reflected wave, with complex amplitude $p_1 a$:

\[ p_1 a = \Gamma. \]

So, $p_1 = \Gamma$.

Path 2. However, a portion of the incident wave is transmitted ($T$) across the interface at $z = -\ell$, this wave travels a distance of $\beta \ell = 90^\circ$ to the load at $z = 0$, where a portion of it is reflected ($\Gamma_L$). This wave travels back $\beta \ell = 90^\circ$ to the interface at $z = -\ell$, where a portion is again transmitted ($T$) across into the $Z_0$ transmission line—another reflected wave!
So the second direct path is

\[ p_2 = T \ e^{-j90^\circ} \ \Gamma \ L \ e^{-j90^\circ} \ T = -T^2 \ \Gamma \ L \]

note that traveling \( 2\beta\ell = 180^\circ \) has produced a minus sign in the result.

**Path 3.** However, a portion of this second wave is also reflected (\( \Gamma \)) back into the \( Z_1 \) transmission line at \( z = -\ell \), where it again travels to \( \beta\ell = 90^\circ \) the load, is partially reflected (\( \Gamma \ L \)), travels \( \beta\ell = 90^\circ \) back to \( z = -\ell \), and is partially transmitted into \( Z_0 (T) \) — our third reflected wave!

where:

\[ p_3 = T \ e^{-j90^\circ} \ \Gamma \ L \ e^{-j90^\circ} \ (-\Gamma) e^{-j90^\circ} \ \Gamma \ L \ e^{-j90^\circ} \ T \\
= -T^2 \ (\Gamma \ L)^2 \ \Gamma \]
Note that path 3 is **not** a direct path!

**Path n.** We can see that this “bouncing” back and forth can go on **forever**, with each trip launching a **new** reflected wave into the $Z_0$ transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

**Q:** *But, why then is $\Gamma = 0$?*

**A:** Each reflected wave is a **coherent** wave. That is, they all oscillate at same frequency $\omega$; the reflected waves differ only in terms of their **magnitude** and **phase**.

Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—this summation of course results in our **propagation series**, a series that must converge for passive devices.

$$b = a \sum_{n=1}^{\infty} p_n$$
It can be shown that the infinite propagation series for this quarter-wavelength structure converges to the closed-form expression:

\[ \frac{b}{a} = \sum_{n=1}^{\infty} \rho_n = \frac{\Gamma - \Gamma^2 \Gamma_L - \Gamma^2 \Gamma_L}{1 - \Gamma^2} \]

Thus, the input reflection coefficient is:

\[ \Gamma_{in} = \frac{b}{a} = \frac{\Gamma - \Gamma^2 \Gamma_L - \Gamma^2 \Gamma_L}{1 - \Gamma^2} \]

Using our definitions, it can likewise be shown that the numerator of the above expression is:

\[ \Gamma - \Gamma^2 \Gamma_L - \Gamma^2 \Gamma_L = \frac{2(Z_1^2 - Z_0 R_L)}{(Z_1 + Z_0)(R_L + Z_1)} \]

It is evident that the numerator (and therefore \( \Gamma \)) will be zero if:

\[ Z_1^2 - Z_0 R_L = 0 \quad \Rightarrow \quad Z_1 = \sqrt{Z_0 R_L} \]

Just as we expected!

Physically, this results insures that all the reflected waves add coherently together to produce a zero value!

Note all of our transmission line analysis has been steady-state analysis. We assume our signals are sinusoidal, of the form \( \exp(j\omega t) \). Note this signal exists for all time \( t \)—the signal is
assumed to have been “on” forever, and assumed to continue on forever.

In other words, in steady-state analysis, all the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is zero!