5.4 – The Quarter-Wave Transformer

Reading Assignment: ***pp. 73-76, 240-243***

By now you've noticed that a **quarter-wave length** of transmission line ($\ell = \lambda/4$, $2\beta\ell = \pi$) appears **often** in microwave engineering problems.

Another application of the $\ell = \lambda/4$ transmission line is as an **impedance matching network**.

HO: THE QUARTER-WAVE TRANSFORMER

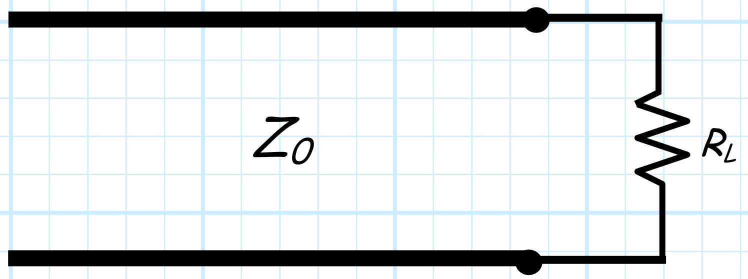
HO: THE SIGNAL-FLOW GRAPH OF A QUARTER-WAVE TRANSFORMER

Q: ***Why*** *does the quarter-wave matching network work—after all, the quarter-wave line is* ***mismatched*** *at both ends?*

A: HO: MULTIPLE REFLECTION VIEWPOINT

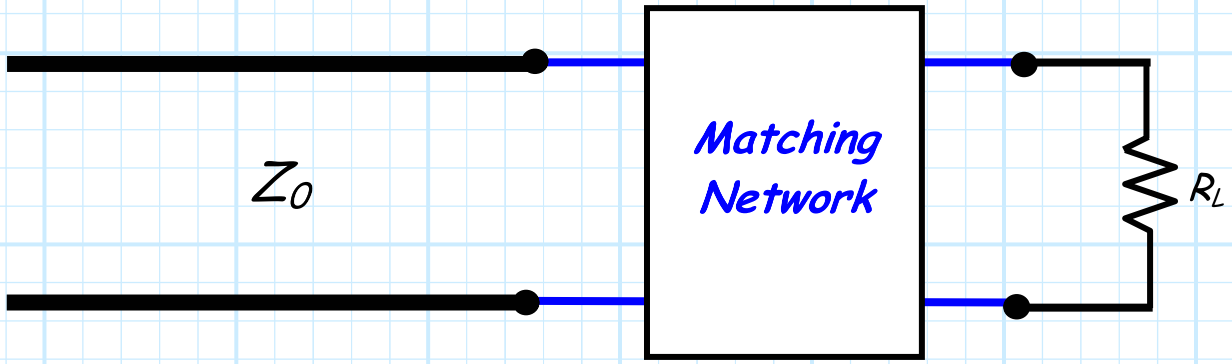
The Quarter-Wave Transformer

Say the end of a transmission line with characteristic impedance Z_0 is terminated with a **resistive** (i.e., real) load.



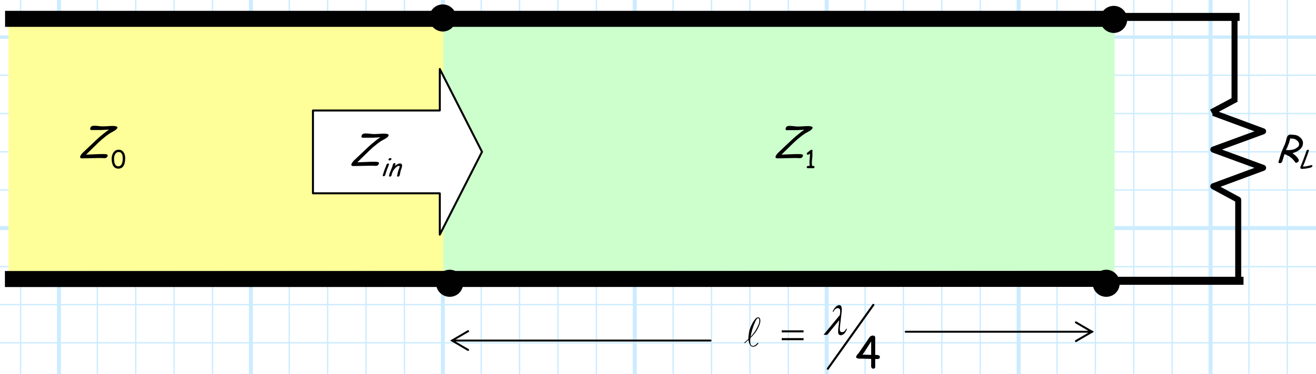
Unless $R_L = Z_0$, the resistor is **mismatched** to the line, and thus some of the incident power will be **reflected**.

We can of course correct this situation by placing a matching network between the line and the load:



In addition to the designs we have just studied (e.g., L-networks, stub tuners), one of the simplest matching network designs is the **quarter-wave transformer**.

The quarter-wave transformer is simply a transmission line with characteristic impedance Z_1 and length $\ell = \lambda/4$ (i.e., a quarter-wave line).



The $\lambda/4$ line **is** the **matching network**!

Q: *But what about the characteristic impedance Z_1; what **should** its value be??*

A: Remember, the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

Thus, if we wish for Z_{in} to be numerically equal to Z_0, we find:

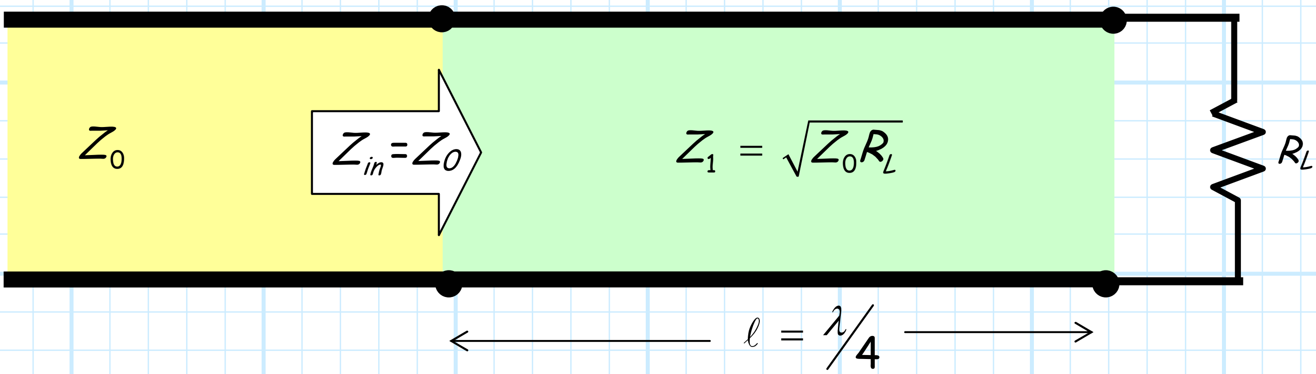
$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$

Solving for Z_1, we find its **required** value to be:

$$\left(Z_1\right)^2 / R_L = Z_0$$
$$\left(Z_1\right)^2 = Z_0 \, R_L$$
$$Z_1 = \sqrt{Z_0 \, R_L}$$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** of Z_0 and R_L!

Therefore, a $\lambda/4$ line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will **match** a transmission line with characteristic impedance Z_0 **to** a resistive load R_L.



Thus, **all power** is delivered to load R_L !

Alas, the quarter-wave transformer (like all our designs) has a few problems!

Problem #1

The matching **bandwidth** is **narrow** !

In other words, we obtain a **perfect** match at precisely the frequency where the length of the matching transmission line is a **quarter**-wavelength.

→ But remember, this length can be a quarter-wavelength at just **one** frequency!

Remember, **wavelength** is related to **frequency** as:

$$\lambda = \frac{v_p}{f} = \frac{1}{f\sqrt{LC}}$$

where v_p is the **propagation velocity** of the wave .

For **example**, assuming that $v_p = c$ (c = the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm ($\lambda = 0.3\ m$), while one wavelength at 3 GHz is 10 cm ($\lambda = 0.1\ m$). As a result, a transmission line length $\ell = 7.5\ cm$ is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a **perfect** match ($\Gamma_{in} = 0$) at **one** and **only one** signal frequency!

As the signal frequency (i.e., wavelength) changes, the **electrical** length of the matching transmission line changes. It will **no longer** be a **quarter** wavelength, and thus we **no longer** will have a **perfect** match.

We find that the **closer** R_L (R_{in}) is to characteristic impedance Z_0, the **wider** the bandwidth of the quarter wavelength transformer.

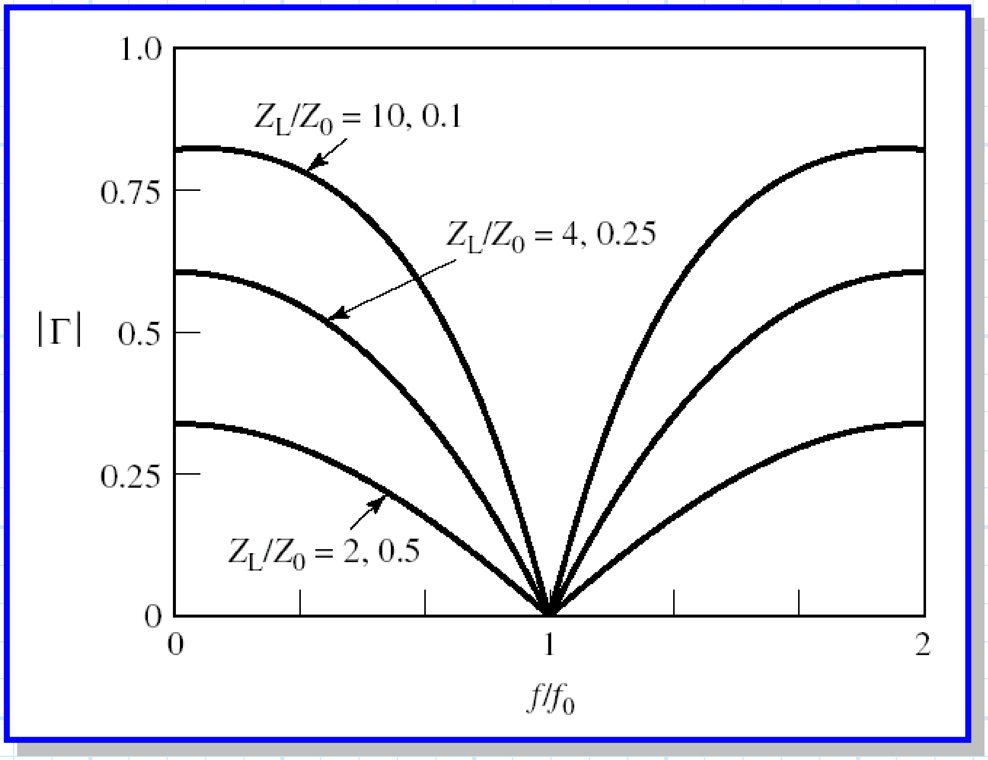


Figure 5.12 (p. 243) *Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.*

We will find that the bandwidth can be **increased** by adding **multiple** $\lambda/4$ sections!

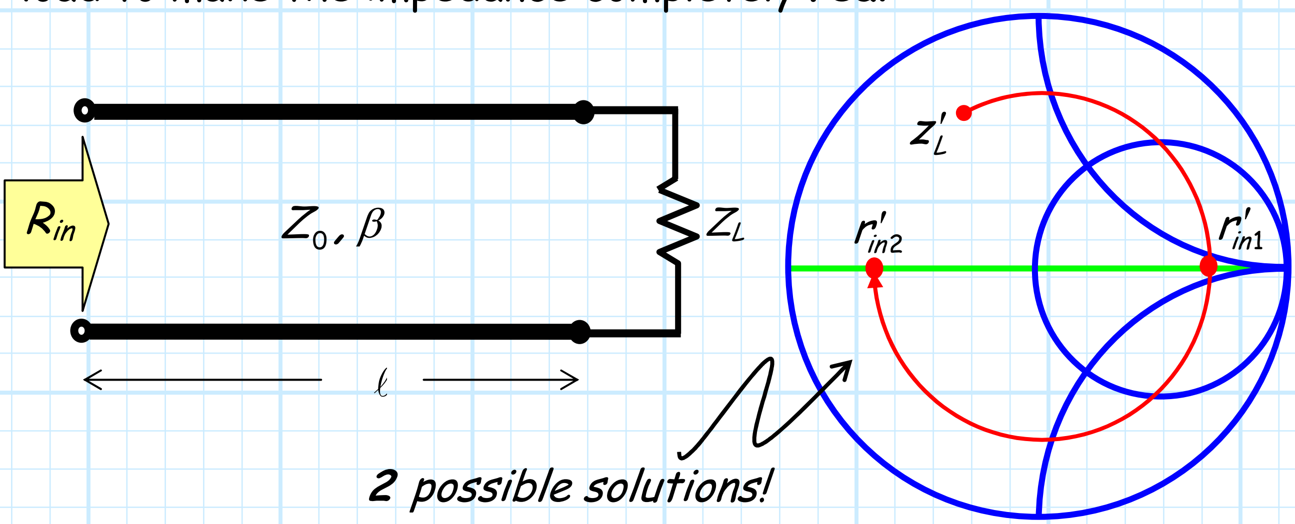
Problem #2

Recall the matching solution was limited to loads that were **purely real**! I.E.:

$$Z_L = R_L + j0$$

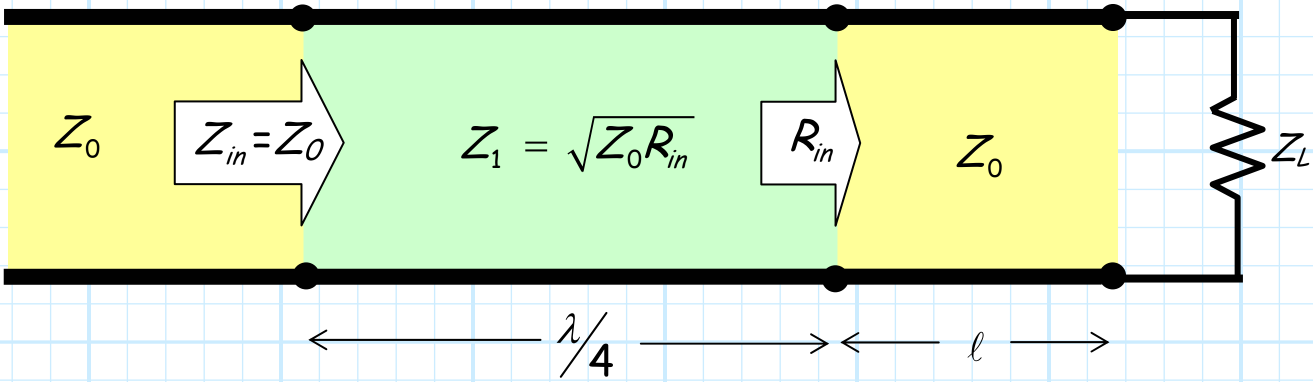
Of course, this is a BIG problem, as most loads will have a **reactive** component!

Fortunately, we have a relatively easy **solution** to this problem, as we can always add some **length** ℓ of transmission line to the load to make the impedance completely **real**:



However, remember that the input impedance will be purely real at only **one** frequency!

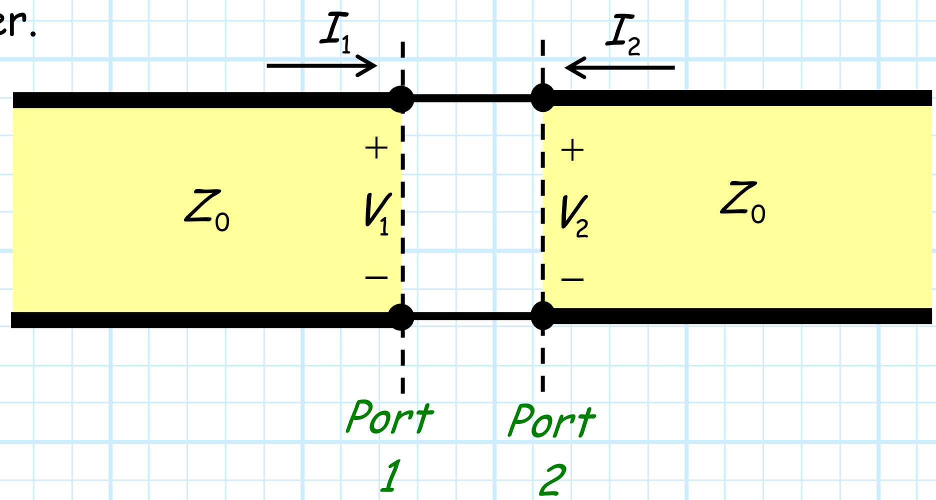
We can then build a quarter-wave transformer to **match** the line Z_0 to resistance R_{in}:



Again, since the transmission lines are lossless, **all** of the incident power is delivered to the **load** Z_L.

The Signal Flow Graph of a Quarter-Wave Transformer

First, let's consider the scattering matrix of a **perfect connector**—an electrically **very small** two-port device that allows us to connect the ends of different transmission lines together.

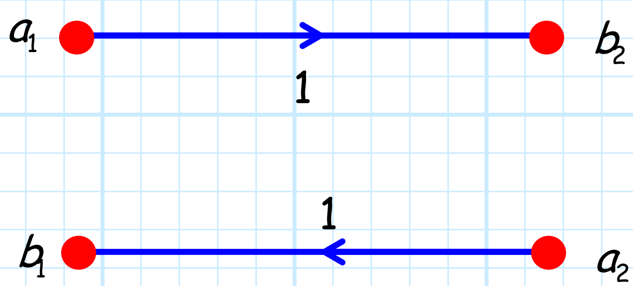


If the connector is ideal, then it will exhibit **no** series inductance **nor** shunt capacitance, and thus:

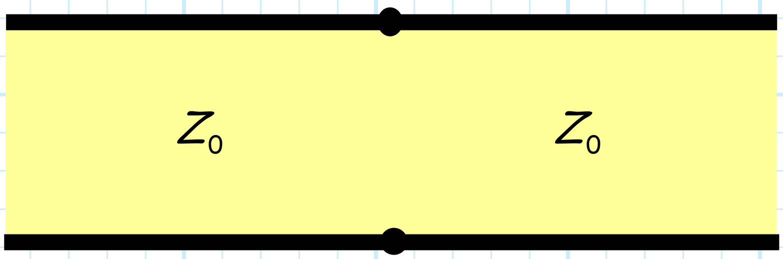
$$V_1 = V_2 \qquad\qquad I_1 = -I_2$$

The **scattering matrix** for such this ideal connector is therefore:

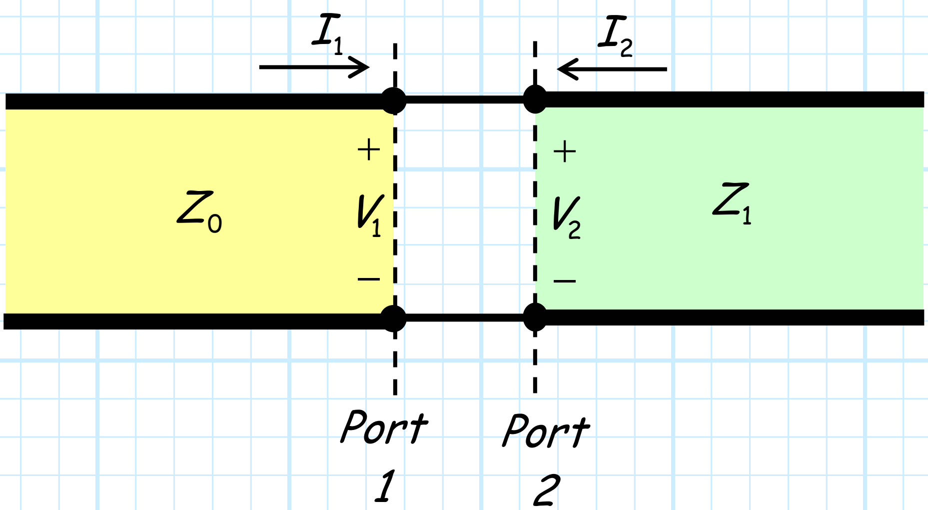
$$\mathcal{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



As a result, the perfect connector allows two transmission lines of **identical characteristic impedance** to be connected together into **one "seamless" transmission line**.



Now, however, consider the case where the transmission lines connected together have **dissimilar** characteristic impedances (i.e., $Z_0 \neq Z_1$):



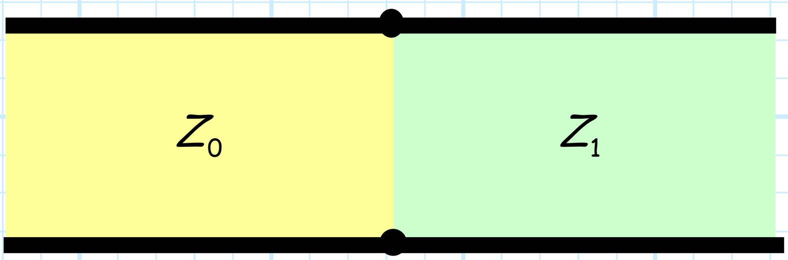
Q: *Won't the scattering matrix of this ideal connector remain the **same**? After all, the **device itself** has not changed!*

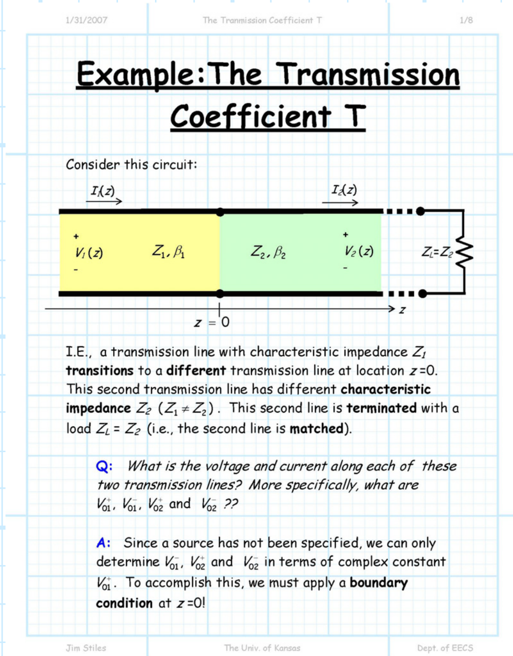
A: The impedance, admittance, and transmission matrix **will** remained unchanged—these matrix quantities **do not** depend on the characteristics of the transmission lines connected to the device.

But remember, the **scattering matrix** depends on **both** the device **and** the characteristic impedance of the transmission lines attached to it.

After all, the **incident** and **exiting** waves are traveling on these transmission lines!

The ideal connector in this case establishes a "seamless" **interface** between two **dissimilar** transmission lines.





1/31/2007 The Tranmission Coefficient T 1/8

Example:The Transmission Coefficient T

Consider this circuit:

I.E., a transmission line with characteristic impedance Z_1 **transitions** to a **different** transmission line at location z=0. This second transmission line has different **characteristic impedance** Z_2 ($Z_1 \neq Z_2$). This second line is **terminated** with a load $Z_L = Z_2$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^+, V_{01}^-, V_{02}^+ *and* V_{02}^- *??*

A: Since a source has not been specified, we can only determine V_{01}^-, V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+. To accomplish this, we must apply a **boundary condition** at z=0!

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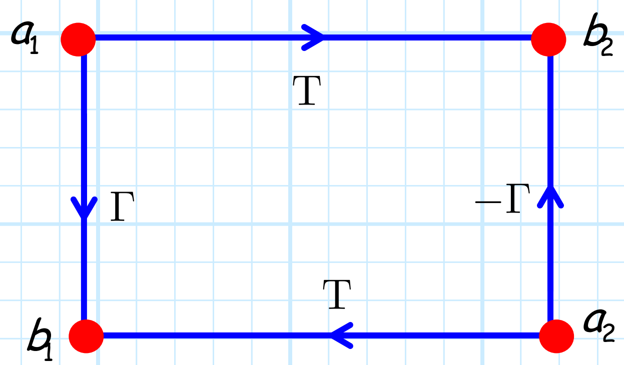
Remember, this is the **same** structure that we evaluated in an **earlier** handout!

From the results of **that** analysis we can conclude that the **scattering matrix** of the ideal connector (when connecting dissimilar transmission lines) is:

$$\mathcal{S} = \begin{bmatrix} \dfrac{Z_1 - Z_0}{Z_1 + Z_0} & \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} \\ \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} & \dfrac{Z_0 - Z_1}{Z_0 + Z_1} \end{bmatrix}$$

Or more **compactly** stated:

$$\mathcal{S} = \begin{bmatrix} \Gamma & T \\ T & -\Gamma \end{bmatrix}$$



where

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad T = \frac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1}$$

For a **quarter wave transformer**, we set Z_1 such that:

$$Z_1^2 = Z_0 \, R_L \qquad \Rightarrow \qquad Z_0 = Z_1^2 / R_L$$

Inserting this into the expressions above, we find:

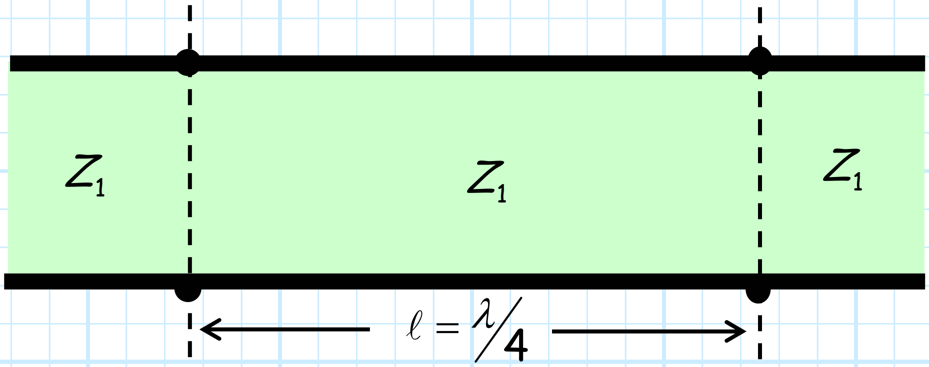
$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} \qquad T = \frac{2\sqrt{R_L Z_1}}{R_L + Z_1}$$

Since the device is **lossless,** we can conclude (and likewise show) that:

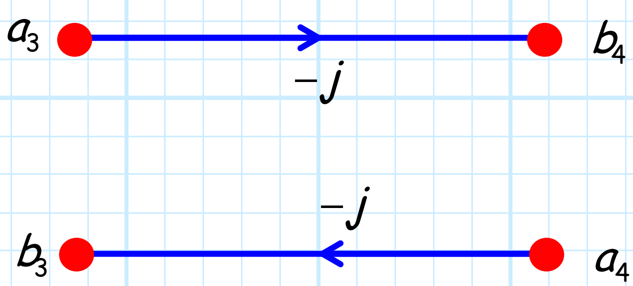
$$1 = |\Gamma|^2 + |T|^2 = \Gamma^2 + T^2$$

where this last expression is (only) true because Γ and T are **real** valued.

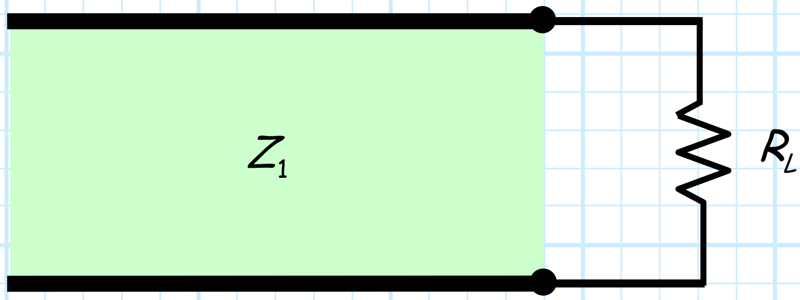
Now, a quarter wavelength of **transmission line** has the scattering matrix:



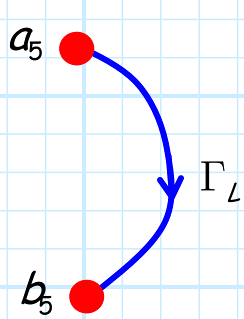
$$\mathcal{S} = \begin{bmatrix} 0 & -j \\ -j & 0 \end{bmatrix}$$



While a **load** has a "scattering matrix" of:

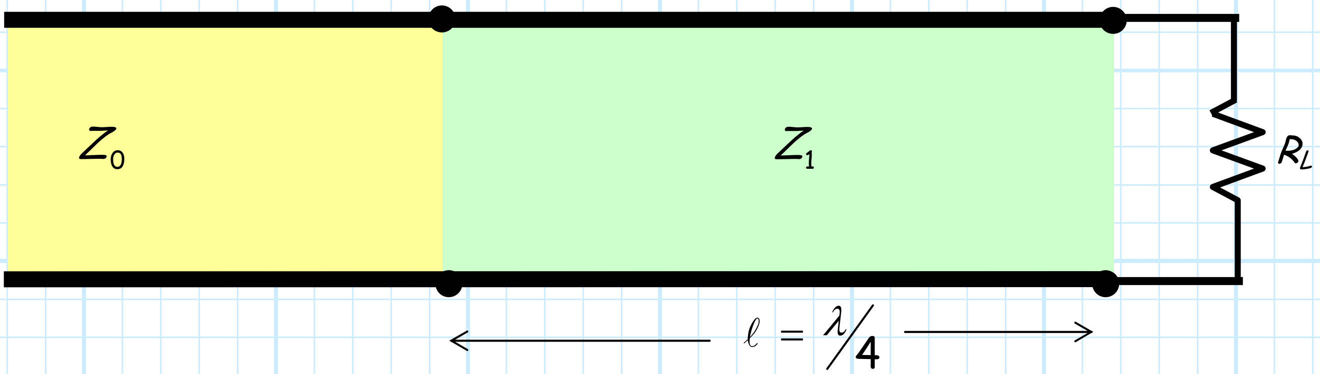


$$\mathcal{S} = \left[\frac{R_L - Z_1}{R_L + Z_1} \right] = \Gamma_L$$

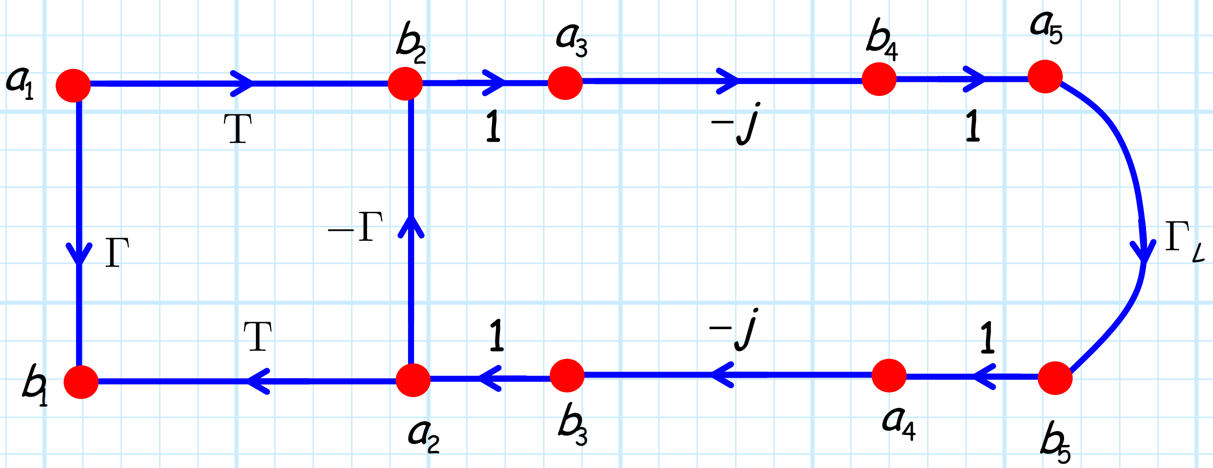


Note that $\Gamma_L = \Gamma$!!!

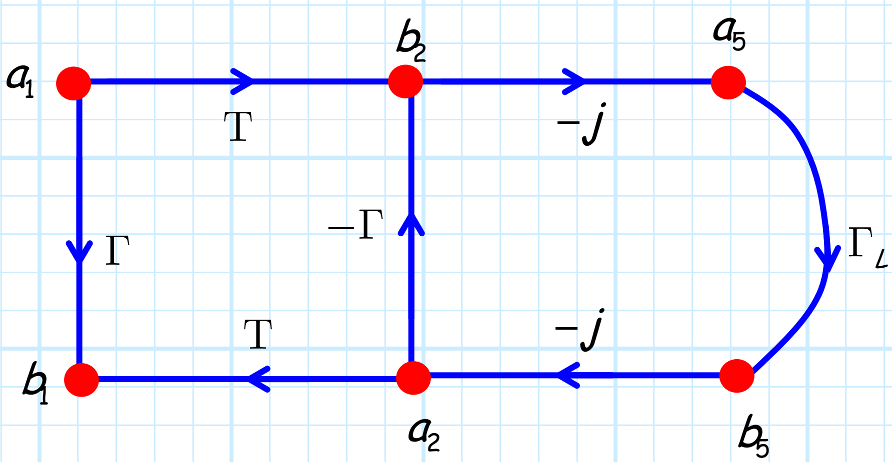
Of course, if we connect the ideal connector to a quarter wavelength of transmission line, and terminate the whole thing with load R_L, we have formed a **quarter wave matching network**!



We can likewise put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:



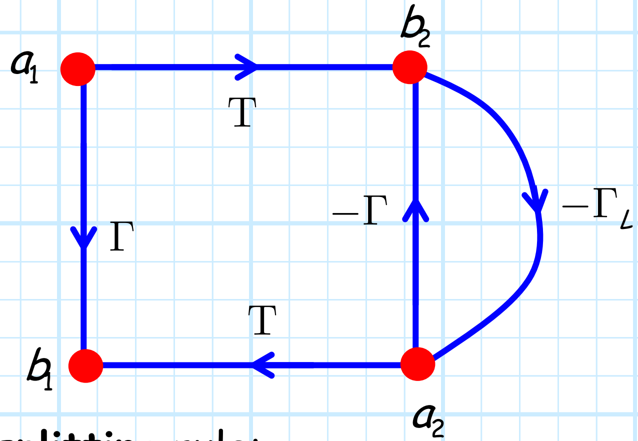
And simplifying:



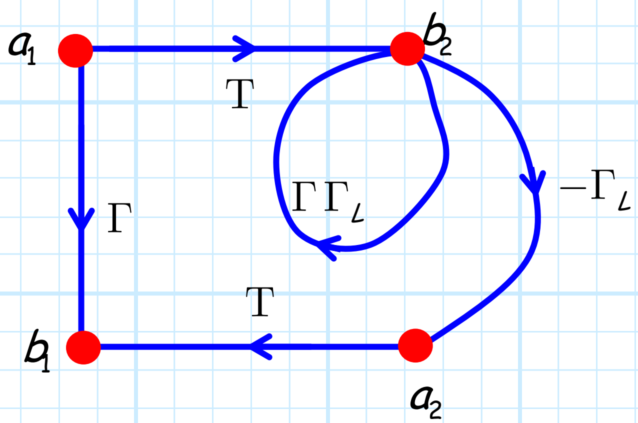
Now, let's see if we can **reduce** this graph to determine:

$$\Gamma_{in} \doteq \frac{b_1}{a_1}$$

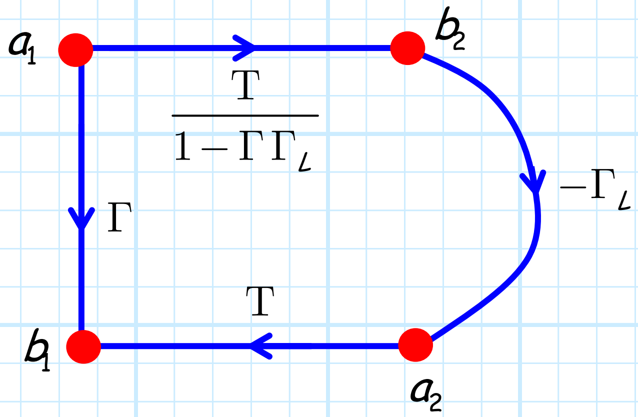
From the **series** rule:



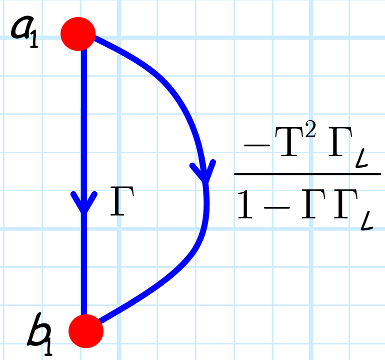
From the **splitting** rule:



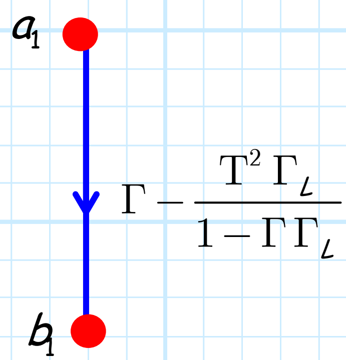
From the **self-loop** rule:



Again with the **series** rule:



And finally with the **parallel** rule:



So that:

$$\Gamma_{in} \doteq \frac{b_1}{a_1} = \Gamma - \frac{T^2\,\Gamma_L}{1-\Gamma\,\Gamma_L}$$

Q: *Hey wait! If the quarter-wave transformer is a matching network, shouldn't* $\Gamma_{in} = 0$ *??*

A: Who says it doesn't!

Recall that for the quarter-wave transformer, we found that $\Gamma_L = \Gamma$, thus:

$$\Gamma_{in} = \Gamma - \frac{T^2\,\Gamma_L}{1-\Gamma\,\Gamma_L}$$

$$= \Gamma - \frac{T^2\,\Gamma}{1-\Gamma^2}$$

And likewise since $T^2 = 1-\Gamma^2$:

$$\Gamma_{in} = \Gamma - \frac{T^2\,\Gamma}{1-\Gamma^2}$$

$$= \Gamma - \frac{T^2\,\Gamma}{T^2}$$

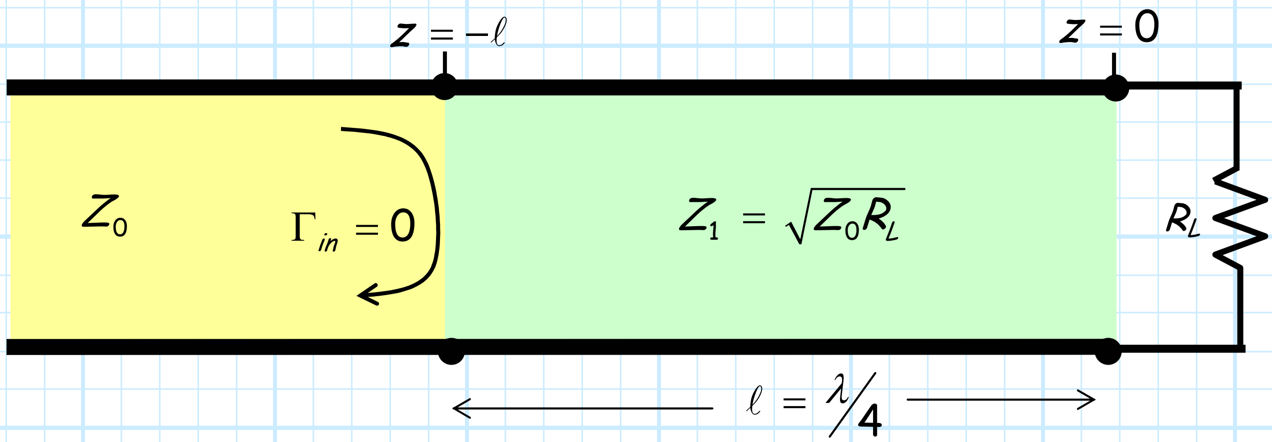
$$= \Gamma - \Gamma$$

$$= 0$$

A **perfect match**!

Multiple Reflection Viewpoint

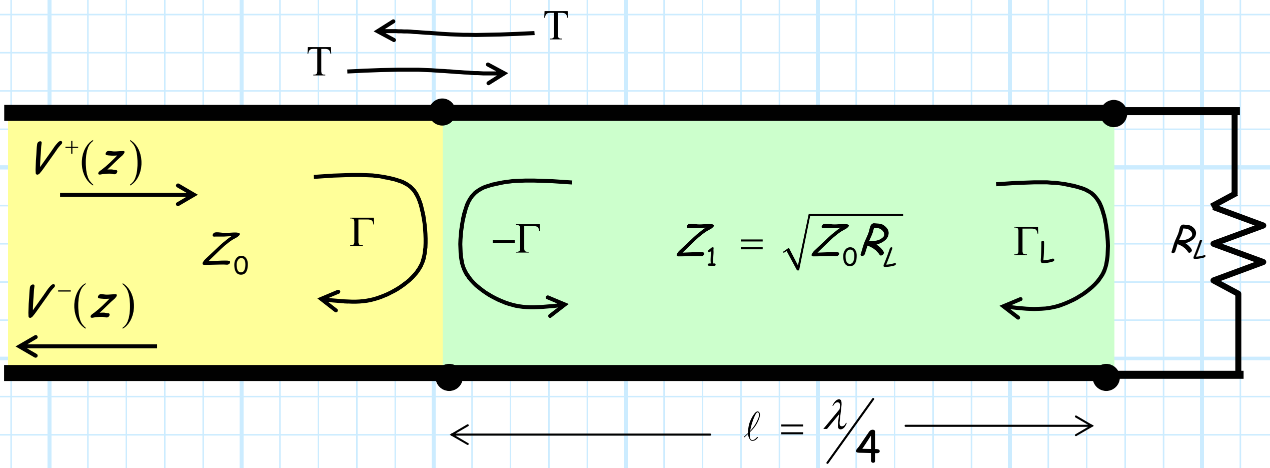
The **quarter-wave** transformer brings up an interesting question in μ-wave engineering.



Q: *Why is there **no** reflection at $z=-\ell$? It appears that the line is **mismatched** at both $z=0$ and $z=-\ell$.*

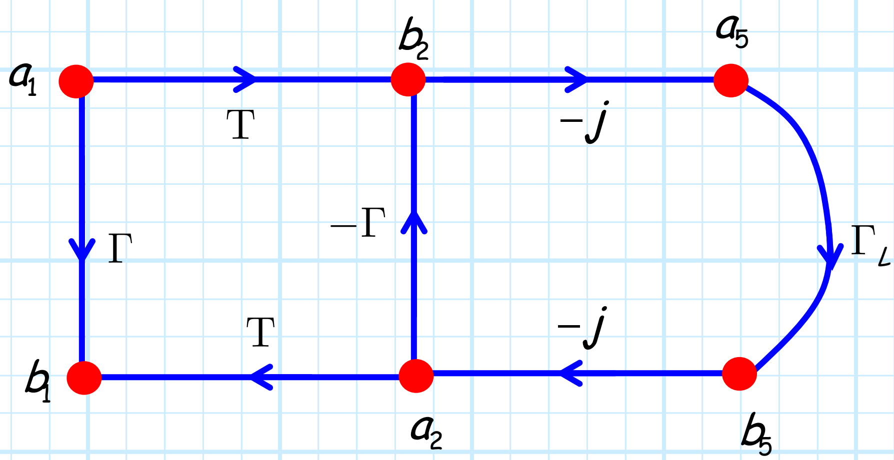
A: In fact there **are** reflections at these mismatched interfaces—an **infinite** number of them!

We can use our **signal flow graph** to in fact determine all the **propagation paths** through the quarter-wave transformer.

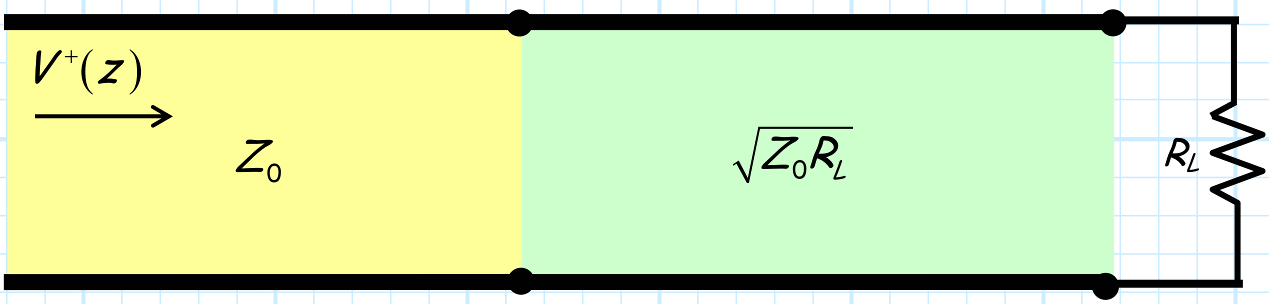


$$a_1 = V^+(z = -\ell)$$

$$b_1 = V^-(z = -\ell)$$

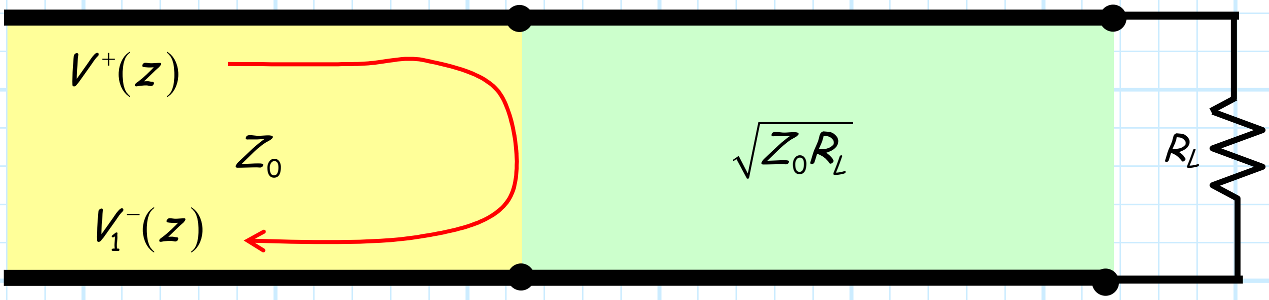


Now let's try to interpret what **physically** happens when the ***incident*** voltage wave:

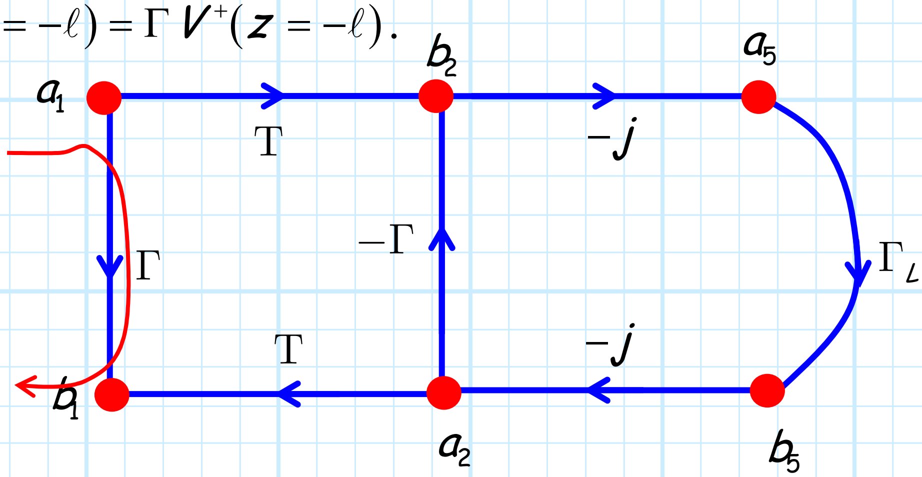


reaches the interface at $z = -\ell$.

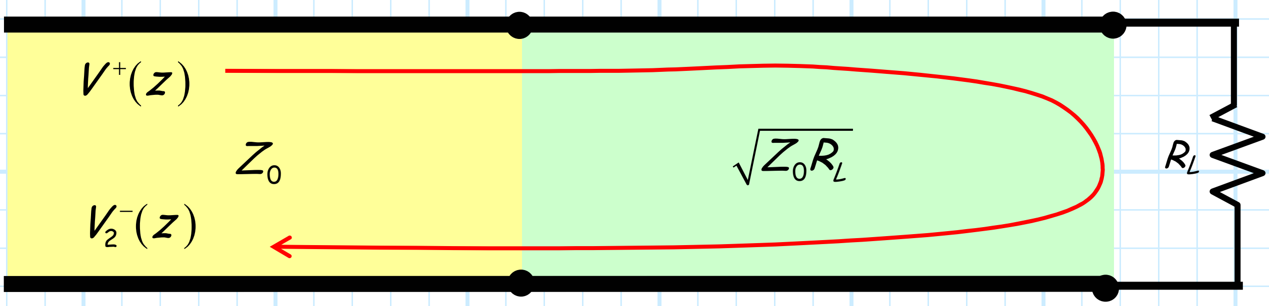
1. At $z = -\ell$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave, a wave that we shall call $V_1^-(z)$:



so $V_1^-(z=-\ell) = \Gamma\, V^+(z=-\ell)$.

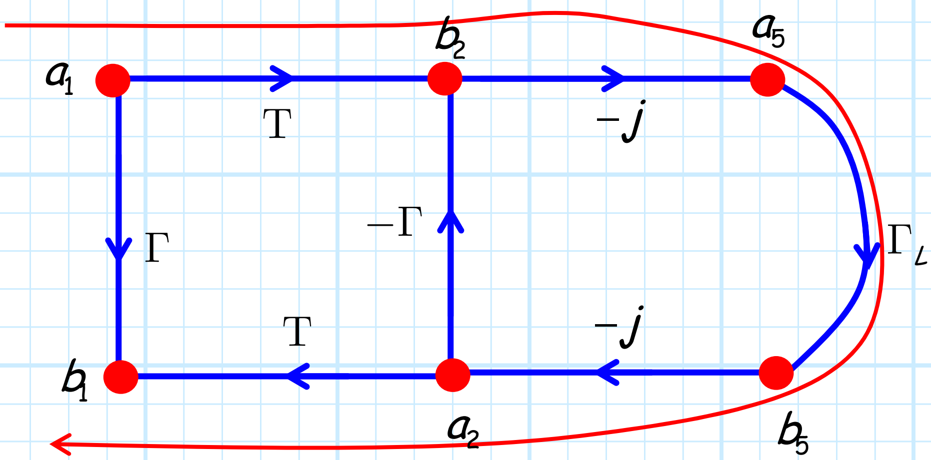


2. However, a **portion** of the incident wave is transmitted (T) across the interface at $z = -\ell$, this wave travels a distance of $\beta\ell = 90^\circ$ to the load at $z = 0$, where a portion of it is reflected (Γ_L). This wave travels back $\beta\ell = 90^\circ$ to the interface at $z = -\ell$, where a portion is again transmitted (T) across into the Z_0 transmission line—**another** reflected wave ($V_2^-(z)$)!

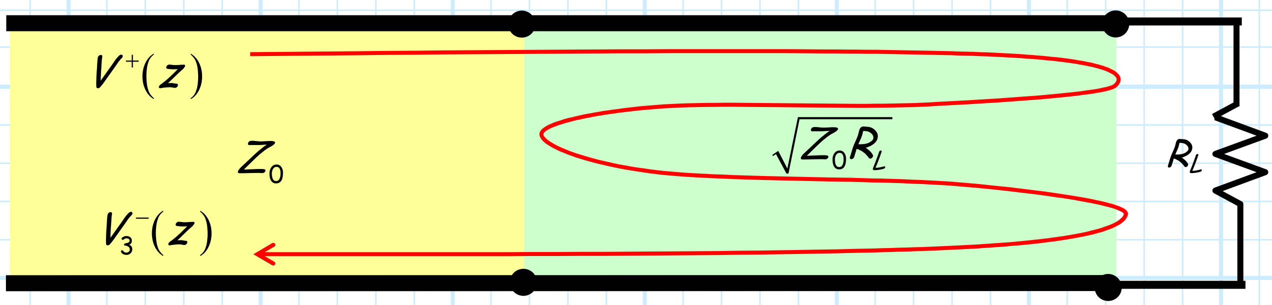


where we have found that traveling $2\beta\ell = 180^\circ$ has produced a **minus** sign in our result:

$$V_2^-(z=-\ell) = \mathrm{T}\ e^{-j90^\circ}\ \Gamma_L\, e^{-j90^\circ}\mathrm{T}\ V^+(z=-\ell)$$
$$= -\mathrm{T}^2\,\Gamma_L\, V^+(z=-\ell)$$

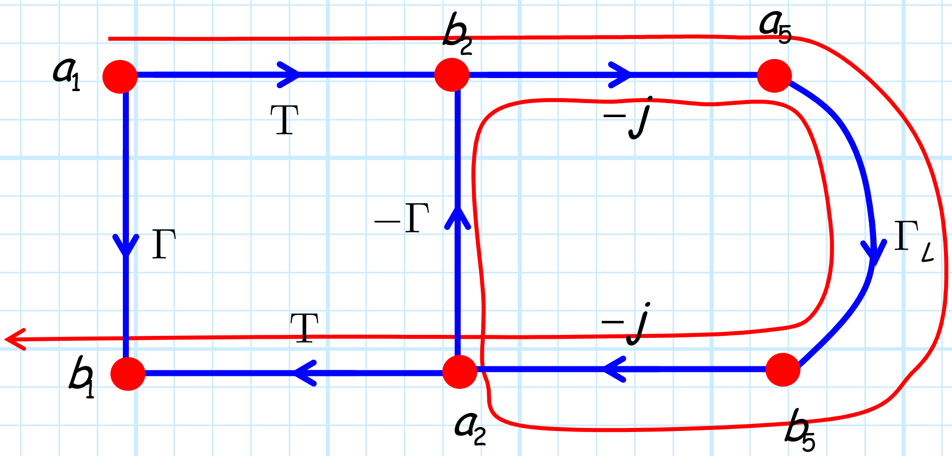


3. However, a **portion** of this **second** wave is also **reflected** (Γ) back into the Z_1 transmission line at $z=-\ell$, where it again travels to $\beta\ell = 90^\circ$ the load, is partially reflected (Γ_L), travels $\beta\ell = 90^\circ$ back to $z=-\ell$, and is partially transmitted into Z_0 (T)—our **third** reflected wave!



where:

$$V_3^-(z=-\ell) = \mathrm{T}\, e^{-j90^\circ}\ \Gamma_L\, e^{-j90^\circ}\ (-\Gamma)\, e^{-j90^\circ}\ \Gamma_L\, e^{-j90^\circ}\mathrm{T}\, V^+(z=-\ell)$$
$$= -\mathrm{T}^2\,(\Gamma_L)^2\,\Gamma\, V^+(z=-\ell)$$



n. We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Q: *But, why then is* $\Gamma = 0$ *?*

A: Each reflected wave $V_n^-(z)$ is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—a operation easily performed since we have expressed our waves with **complex** notation:

$$V^-(z) = \sum_{n=1}^{\infty} V_n^-(z)$$

It can be shown that this infinite series **converges**, with the result:

$$V^{-}(z=-\ell)=\left(\frac{\Gamma-\Gamma^2\,\Gamma_L-\mathrm{T}^2\,\Gamma_L}{1-\Gamma^2}\right)V^{+}(z=-\ell)$$

Thus, the **total** reflection coefficient is:

$$\Gamma_{in}=\frac{\Gamma-\Gamma^2\,\Gamma_L-\mathrm{T}^2\,\Gamma_L}{1-\Gamma^2}$$

Using our definitions, it can likewise be shown that the **numerator** of the above expression is:

$$\Gamma-\Gamma^2\,\Gamma_L-\mathrm{T}^2\,\Gamma_L=\frac{2\left(Z_1^2-Z_0\,R_L\right)}{(Z_1+Z_0)(R_L+Z_1)}$$

It is evident that the numerator (and therefore Γ) will be **zero** if:

$$Z_1^2-Z_0\,R_L \qquad \Rightarrow \qquad Z_1=\sqrt{Z_0\,R_L}$$

Just as we expected!

Physically, this results insures that all the reflected waves add coherently together to produce a **zero value**!

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. Note this signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue on forever.

In other words, in steady-state analysis, **all** the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is **zero**!