5.5 - The Theory of Small Reflections

Reading Assignment: pp. 244-246

An important and useful approximation when considering multi-section matching networks is the **Theory of Small Reflections**.

**HO: THE THEORY OF SMALL REFLECTIONS**

**EXAMPLE: THE THEORY OF SMALL REFLECTIONS**

The Theory of Small Reflections provides a simpler mathematical form for analyzing the **frequency response** of many microwave devices.

**HO: THE FREQUENCY RESPONSE OF THE QUARTER-WAVE MATCHING NETWORK**

We can also use the Theory of Small Reflections to provide an approximate analysis of a multi-section impedance transformer (i.e., multi-section matching network).

**HO: THE MULTI-SECTION TRANSFORMER**
The Theory of Small Reflections

Recall that we analyzed a quarter-wave transformer using the multiple reflection viewpoint.

We found that the solution could thus be written as an infinite summation of terms (the propagation series):

\[ b = a \sum_{n=1}^{\infty} p_n \]

where each term had a specific physical interpretation, in terms of reflections, transmissions, and propagations.

For example, the third term was path:
Now let's consider the magnitude of this path:

\[ |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| |e^{-j2\beta\ell}| \]
\[ = |T|^2 |\Gamma_L|^2 |\Gamma| \]

Recall that \( \Gamma = \Gamma_L \) for a properly designed quarter-wave transformer:

\[ \Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \]

and so:

\[ |p_3| = |T|^2 |\Gamma_L|^2 |\Gamma| = |T|^2 |\Gamma_L|^3 \]

For the case where values \( R_L \) and \( Z_1 \) are numerically "close" in —i.e., when:

\[ |R_L - Z_1| \ll |R_L + Z_1| \]

we find that the magnitude of the reflection coefficient will be very small:

\[ |\Gamma_L| = \left| \frac{R_L - Z_1}{R_L + Z_1} \right| \ll 1.0 \]

As a result, the value \( |\Gamma_L|^3 \) will be very, very, very small.
Moreover, we know (since the connector is lossless) that:

\[ 1 = |\Gamma|^2 + |T|^2 = |\Gamma_L|^2 + |T|^2 \]

and so:

\[ |T|^2 = 1 - |\Gamma_L|^2 \approx 1 \]

We can thus conclude that the magnitude of path \( p_3 \) is likewise very, very, very small:

\[ |p_3| = |T|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1 \]

This is a classic case where we can approximate the propagation series using only the forward paths!!

Recall there are two forward paths:
Therefore **IF** $Z_0$ and $R_L$ are very close in value, we find that we can **approximate** the reflected wave using only the **direct paths** of the infinite series:

$$b = (p_1 + p_2) a$$
$$= (\Gamma + \Gamma^2 e^{j2\beta \ell}) a$$

Therefore:

$$V^-(z) = b\sqrt{Z_0} e^{+j\beta(z + \ell)}$$
$$\approx (\Gamma + \Gamma^2 \Gamma e^{j2\beta \ell}) a\sqrt{Z_0} e^{+j\beta(z + \ell)}$$

Now, if we likewise apply the **approximation** that $|\Gamma| \approx 1.0$, we conclude for this quarter wave transformer (at the design frequency):

$$b = (p_1 + p_2) a$$
$$= (\Gamma + \Gamma e^{j2\beta \ell}) a$$

Therefore:

$$V^-(z) = b\sqrt{Z_0} e^{+j\beta(z + \ell)}$$
$$\approx (\Gamma + \Gamma e^{j2\beta \ell}) a\sqrt{Z_0} e^{+j\beta(z + \ell)}$$
This approximation, where we:

1. use only the direct paths to calculate the propagation series,

2. approximate the transmission coefficients as one (i.e., \( T = 1 \)).

is known as the Theory of Small Reflections, and allows us to use the propagation series as an analysis tool (we don't have to consider an infinite number of terms!).

Consider again the quarter-wave matching network \( SFG \). Note there is one branch \((-\Gamma = S_{22} \text{ of the connector})\), that is not included in either direct path.
With respect to the theory of small reflections (where only direct paths are considered), this branch can be removed from the SFG without affect.

Moreover, the theory of small reflections implements the approximation $T = 1$, so that the SFG becomes:

Reducing this SFG by combining the 1.0 branch and the $e^{-j\beta\ell}$ branch via the series rule, we get the following approximate SFG:

$$\Gamma_m = \frac{b}{a} = \Gamma + \Gamma_L e^{j2\beta\ell}$$

The approximate SFG when applying the theory of small reflections!
Note this **approximate** SFG provides **precisely** the results of the theory of small reflections!

**Q:** Why is that?

**A:** The approximate “theory of small reflections SFG”
Contains all of the **significant physical propagation mechanisms** of the two *forward paths*, and only the two significant propagation mechanisms of the two forward paths.

Namely:

1. The **reflection** at the connector (i.e., $\Gamma$).

2. The **propagation down** the quarter-wave transmission line ($e^{-j\beta t}$), the **reflection** off the load ($\Gamma_L$), and the **propagation back up** the quarter-wave transmission line ($e^{+j\beta t}$).
Q: But wait! The quarter-wave transformer is a matching network, therefore $\Gamma_{in} = 0$. The theory of small reflections, however, provides the approximate result:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell}$$

Is this approximation very accurate? How close is this approximate value to the correct answer of $\Gamma_{in} = 0$?

A: Let's find out!

Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell}$$

$$= \Gamma_L \left(1 + e^{-j2\beta\ell}\right)$$

Likewise, $\ell = \frac{3}{4}$ (but only at the design frequency!) so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where you of course recall that $\beta = \frac{2\pi}{\lambda}$!
Thus:

\[ \Gamma_{in} \approx \Gamma_L \left( 1 + e^{-j2\beta L} \right) \]
\[ = \Gamma_L \left( 1 + e^{-j\pi} \right) \]
\[ = \Gamma_L (1 - 1) \]
\[ = 0 \quad !!! \]

**Q:** Wow! The theory of small reflections appears to be a perfect approximation—no error at all?!

**A:** Not so fast.

The theory of small reflections most definitely provides an approximate solution (e.g., it ignores most of the terms of the propagation series, and it approximates connector transmission as \( T = 1 \), when in fact \( T \neq 1 \)).

As a result, the solutions derived using the theory of small reflections will—generally speaking—exhibit some (hopefully small) error.

We just got a bit “lucky” for the quarter-wave matching network; the “approximate” result \( \Gamma_{in} = 0 \) was exact for this one case!

→ The theory of small reflections is an approximate analysis tool!
Example: The Theory of Small Reflections

Use the theory of small reflections to determine a numeric value for the input reflection coefficient $\Gamma_{in}$, at the design frequency $\omega_0$.

Note that the transmission line sections have different lengths!

Solution

Applying the theory of small reflections, the approximate signal flow graph of the structure becomes:
Note there are three direct propagation paths:

**Path 1**

\[ \ell_1 = \frac{3\lambda_0}{8} \quad \ell_2 = \frac{\lambda_0}{8} \]

\[ \Gamma_{in1} \]

\[ \Gamma_0 = 0.1 \]

\[ p_1 = \Gamma_0 = 0.1 \]

**Path 2**

This path includes propagation down and back a transmission line length \( \ell_1 \)!

\[ \ell_1 = \frac{3\lambda_0}{8} \quad \ell_2 = \frac{\lambda_0}{8} \]

\[ \Gamma_{in2} \]

\[ p_2 = e^{-j\beta_1} \Gamma_1 e^{-j\beta_1} \]

\[ = e^{-j\frac{3\pi}{4}} 0.05 e^{-j\frac{3\pi}{4}} \]

\[ = e^{-j\frac{3\pi}{2}} 0.05 \]

\[ = +j0.05 \]
Path 3

This path includes propagation down and back transmission line lengths of $\ell_1 + \ell_2$!

\[
p_3 = e^{-j \beta (\ell_1 + \ell_2)} \Gamma_L e^{-j \beta (\ell_1 + \ell_2)}
= e^{-j \pi} 0.15 e^{-j \pi}
= e^{-j 2\pi} 0.15
= 0.15
\]

Thus, using the theory of small reflections we can determine approximately the input reflection coefficient:

\[
\Gamma_{in} = \frac{b_0}{a_0} = p_1 + p_2 + p_3 = 0.1 + j0.05 + 0.15 = 0.25 + j0.05
\]
The Frequency Response of a Quarter-Wave Matching Network

**Q:** You have once again provided us with confusing and perhaps useless information. The quarter-wave matching network has an exact SFG of:

\[
\Gamma_{in} \triangleq \frac{b}{a} = \Gamma + \frac{T^2 \Gamma L e^{-j2\beta L}}{1 - \Gamma \Gamma_L}
\]

You could have left this simple and precise analysis alone—but NOOO!!

You had to foist upon us a long, rambling discussion of “the propagation series” and “direct paths” and “the theory of
small reflections", culminating with the approximate (i.e., less accurate!) SFG:

\[ e^{-j\beta L} \]

\[ e^{-j\beta L} \]

\[ \Gamma \]

\[ \Gamma \]

From which we were able to conclude the approximate (i.e., less accurate!) result:

\[ \Gamma_{in} = \frac{b}{a} = \Gamma + \Gamma_L e^{-j2\beta L} \]

The exact result was simple—and exact! Why did you make us determine this approximate result?

A: In a word: frequency response*.

Although the exact analysis is about as simple to determine as the approximation provided by the theory of small reflections, the mathematical form of the result is much simpler to analyze and/or evaluate (e.g., no fractional terms!).

Q: What exactly would we be analyzing and/or evaluating?

A: The frequency response of the matching network, for one thing.

* OK, two words.
Remember, all matching networks must be **lossless**, and so must be made of **reactive** elements (e.g., lossless transmission lines). The impedance of every reactive element is a **function of frequency**, and so too then is $\Gamma_{in}$.

Say we wish to determine this function $\Gamma_{in}(\omega)$.

**Q:** Isn’t $\Gamma_{in}(\omega) = 0$ for a quarter wave matching network?

**A:** Oh my gosh **no**! A properly designed matching network will typically result in a perfect match (i.e., $\Gamma_{in} = 0$) at one **frequency** (i.e., the design frequency). However, if the signal frequency is **different** from this design frequency, then no match will occur (i.e., $\Gamma_{in} \neq 0$).

Recall we discussed this behavior **before**:

![Figure 5.12 (p. 243)](image)

Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.
**Q:** But why is the result:

\[ \Gamma_{in} = \Gamma + \frac{T^2 \Gamma L e^{-j2\beta \ell}}{1 - \Gamma \Gamma_L} \]

or its approximate form:

\[ \Gamma_{in} = \Gamma + \Gamma_L e^{-j2\beta \ell} \]

dependent on frequency? I don’t see frequency variable \( \omega \) anywhere in these results!

**A:** Look closer!

Remember that the value of spatial frequency \( \beta \) (in radians/meter) is dependent on the frequency \( \omega \) of our eigenfunction (aka “the signal”):

\[ \beta = \left( \frac{1}{\nu_p} \right) \omega \]

where you will recall that \( \nu_p \) is the propagation velocity of a wave moving along a transmission line. This velocity is a constant (i.e., \( \nu_p = 1/\sqrt{LC} \)), and so the spatial frequency \( \beta \) is directly proportional to the temporal frequency \( \omega \).

Thus, we can rewrite:
\[ \beta \ell = \frac{\omega \ell}{v_p} = \omega T \]

Where \( T = \ell/v_p \) is the time required for the wave to propagate a distance \( \ell \) down a transmission line.

As a result, we can write the input reflection coefficient as a function of spatial frequency \( \beta \):

\[ \Gamma_{in}(\beta) = \Gamma + \Gamma_L e^{-j2\beta \ell} \]

Or equivalently as a function of temporal frequency \( \omega \):

\[ \Gamma_{in}(\omega) = \Gamma + \Gamma_L e^{-j2\omega T} \]

Frequently, the reflection coefficient is simply written in terms of the electrical length \( \theta \) of the transmission line, which is simply the difference in relative phase between the wave at the beginning and end of the length \( \ell \) of the transmission line.

\[ \beta \ell = \theta = \omega T \]

So that:

\[ \Gamma_{in}(\theta) = \Gamma + \Gamma_L e^{-j2\theta} \]

Note we can simply insert the value \( \theta = \beta \ell \) into the expression above to get \( \Gamma_{in}(\beta) \), or insert \( \theta = \omega T \) into the expression to get \( \Gamma_{in}(\omega) \).
Now, we know that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, so the reflection coefficient function can be written as:

$$\Gamma_{in}(\theta) = \Gamma_L \left( 1 + e^{-j2\theta} \right)$$

Note that:

$$1 = e^{j0} = e^{-j(\theta - \theta)} = e^{-j\theta} e^{+j\theta}$$

And that:

$$e^{-j2\theta} = e^{-j(\theta+\theta)} = e^{-j\theta} e^{-j\theta}$$

And so:

$$\Gamma_{in}(\theta) = \Gamma_L \left( 1 + e^{-j2\theta} \right)$$

$$= \Gamma_L \left( e^{-j\theta} e^{+j\theta} + e^{-j\theta} e^{-j\theta} \right)$$

$$= \Gamma_L e^{-j\theta} \left( e^{+j\theta} + e^{-j\theta} \right)$$

$$= \Gamma_L e^{-j\theta} (2\cos\theta)$$

Where we have used Euler’s equation to determine that:

$$e^{+j\theta} + e^{-j\theta} = 2\cos\theta$$

Now, let’s determine the magnitude of our result:

$$|\Gamma_{in}(\theta)| = |\Gamma_L| |e^{-j\theta}| \left| \begin{array}{c} 2 \cos\theta \end{array} \right| = 2 |\Gamma_L| |\cos\theta|$$

Note that $|\Gamma_{in}(\theta)|$ is zero-valued only when $\cos\theta = 0$. This of course occurs when $\theta = \frac{\pi}{2}$:

$$|\Gamma_{in}(\theta)|_{\theta = \frac{\pi}{2}} = 2 |\Gamma_L| \left| \cos\frac{\pi}{2} \right| = 0$$
In other words, a perfect match occurs when $\theta = \frac{\pi}{2}$.

**Q:** What the heck does this mean?

**A:** Remember, $\theta = \beta \ell$. Thus if $\theta = \frac{\pi}{2}$:

$$\ell = \frac{\theta}{\beta} = \frac{\frac{\pi}{2}}{2 \pi / \lambda} = \frac{\lambda}{4}$$

As we (should have) suspected, the match occurs at the frequency whose wavelength is equal to four times the matching ($Z_1$) transmission line length, i.e. $\lambda = 4\ell$.

In other words, a perfect match occurs at the frequency where $\ell = \lambda / 4$.

Note the physical length $\ell$ of the transmission line does not change with frequency, but the signal wavelength does:

$$\lambda = \frac{v_p}{f}$$

**Q:** So, at precisely what frequency does a quarter-wave transformer with length $\ell$ provide a perfect match?

**A:** Recall also that $\theta = \omega T$, where $T = \ell / v_p$. Thus, for $\theta = \frac{\pi}{2}$:

$$\theta = \frac{\pi}{2} = \omega T \quad \Rightarrow \quad \omega = \frac{\pi}{2} \frac{1}{T} = \frac{\pi}{2} \frac{v_p}{\ell}$$
This frequency is called the **design frequency** of the matching network—it’s the frequency where a **perfect** match occurs. We denote this as frequency $\omega_0$, which has wavelength $\lambda_0$, i.e.:

$$\omega_0 = \frac{\pi}{2T} = \frac{\pi v_p}{2\ell} \quad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{4T} = \frac{v_p}{4\ell} \quad \lambda_0 = \frac{v_p}{f_0} = 4v_pT = 4\ell$$

Given this, yet another way of expressing $\theta = \beta \ell$ is:

$$\theta = \beta \ell = \frac{\omega}{v_p} \left( \frac{\pi v_p}{2\omega_0} \right) = \frac{\pi}{2\omega} = \frac{\omega}{2\omega_0} = \frac{f}{2f_0}$$

Thus, we conclude:

$$|\Gamma_{in}(f)| = 2 |\Gamma_L| \left| \cos \left( \pi \frac{f}{2f_0} \right) \right|$$

From this result we can determine (approximately) the **bandwidth** of the quarter-wave transformer!

First, we must define what we mean by bandwidth. Say the **maximum** acceptable level of the reflection coefficient is value $\Gamma_m$. This is an arbitrary value, set by you the microwave engineer (typical values of $\Gamma_m$ range from 0.05 to 0.2).

We will denote the frequencies where this maximum value $\Gamma_m$ occurs $f_m$. In other words:

$$|\Gamma_{in}(f = f_m)| = \Gamma_m = 2 |\Gamma_L| \left| \cos \left( \pi \frac{f_m}{2f_0} \right) \right|$$
There are two solutions to this equation, the first is:

\[ f_{m1} = \frac{2f_0}{\pi} \cos^{-1} \left( \frac{\Gamma_m}{2|\Gamma_L|} \right) \]

And the second:

\[ f_{m2} = \frac{2f_0}{\pi} \cos^{-1} \left( -\frac{\Gamma_m}{2|\Gamma_L|} \right) \]

Important note! Make sure \( \cos^{-1}x \) is expressed in radians!

You will find that \( f_{m1} < f_0 < f_{m2} \) so, the values \( f_{m1} \) and \( f_{m2} \) define the lower and upper limits on matching network bandwidth.

All this analysis was brought to you by the “simple” mathematical form of \( \Gamma_{in}(f) \) that resulted from the theory of small reflections!
Consider a sequence of $N$ transmission line sections; each section has equal length $\ell$, but dissimilar characteristic impedances:

$$
\begin{align*}
\Gamma_0 & = \frac{Z_1 - Z_0}{Z_1 + Z_0} \\
\Gamma_n & = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \\
\Gamma_N & = \frac{R_L - Z_N}{R_L + Z_N}
\end{align*}
$$

Where the marginal reflection coefficients are:

If the load resistance $R_L$ is less than $Z_0$, then we should design the transformer such that:

$$
Z_0 > Z_1 > Z_2 > Z_3 \ldots > Z_N > R_L
$$
Conversely, if $R_L$ is greater than $Z_0$, then we will design the transformer such that:

$$Z_0 < Z_1 < Z_2 < Z_3 \ldots < Z_N < R_L$$

In other words, we gradually transition from $Z_0$ to $R_L$!

Note that since $R_L$ is real, and since we assume lossless transmission lines, all $\Gamma_n$ will be real (this is important!).

Likewise, since we gradually transition from one section to another, each value:

$$Z_{n+1} - Z_n$$

will be small.

As a result, each marginal reflection coefficient $\Gamma_n$ will be real and have a small magnitude.

This is also important, as it means that we can apply the "theory of small reflections" to analyze this multi-section transformer!

The theory of small reflections allows us to approximate the input reflection coefficient of the transformer as:
We can alternatively express the input reflection coefficient as a function of frequency \( \beta \ell = \omega T \):

\[
\Gamma_m(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \ldots + \Gamma_N e^{-j2N\omega T} = \sum_{n=0}^{N} \Gamma_n e^{-j(2nT)\omega}
\]

where:
\[ T = \frac{\ell}{V_p} \text{ propagation time through 1 section} \]

We see that the function \( \Gamma_{in}(\omega) \) is expressed as a **weighted** set of \( N \) basis functions! I.E.,

\[ \Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \Psi(\omega) \]

where:

\[ c_n = \Gamma_n \quad \text{and} \quad \Psi(\omega) = e^{-j(2nT)\omega} \]

We find, therefore, that by **selecting** the proper values of basis weights \( c_n \) (i.e., the proper values of reflection coefficients \( \Gamma_n \)), we can **synthesize** any function \( \Gamma_{in}(\omega) \) of frequency \( \omega \), provided that:

1. \( \Gamma_{in}(\omega) \) is **periodic** in \( \omega = 1/2T \)

2. we have sufficient **number** of sections \( N \).

**Q:** What function **should** we synthesize?

**A:** Ideally, we would want to make \( \Gamma_{in}(\omega) = 0 \) (i.e., the reflection coefficient is zero for all frequencies).

**Bad news:** this **ideal** function \( \Gamma_{in}(\omega) = 0 \) would require an **infinite** number of sections (i.e., \( N = \infty \)).
Instead, we seek to find an “optimal” function for $\Gamma_{in}(\omega)$, given a finite number of $N$ elements.

Once we determine these optimal functions, we can find the values of coefficients $\Gamma_n$ (or equivalently, $Z_n$) that will result in a matching transformer that exhibits this optimal frequency response.

To simplify this process, we can make the transformer symmetrical, such that:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \quad \Gamma_2 = \Gamma_{N-2}, \quad \ldots$$

Note that this does NOT mean that:

$$Z_0 = Z_N, \quad Z_1 = Z_{N-1}, \quad Z_2 = Z_{N-2}, \quad \ldots$$

We find then that:

$$\Gamma(\omega) = e^{-jN\omega T} \left[ \Gamma_0 \left( e^{jN\omega T} + e^{-jN\omega T} \right) + \Gamma_1 \left( e^{j(N-2)\omega T} + e^{-j(N-2)\omega T} \right) + \Gamma_2 \left( e^{j(N-4)\omega T} + e^{-j(N-4)\omega T} \right) + \ldots \right]$$

and since:

$$e^{jx} + e^{-jx} = 2 \cos(x)$$

we can write for $N$ even:
\[ \Gamma(\omega) = 2 e^{-jN\omega T} \left[ \Gamma_0 \cos N\omega T + \Gamma_1 \cos (N-2)\omega T + \cdots + \Gamma_n \cos (N-2n)\omega T + \cdots + \frac{1}{2} \Gamma_{N/2} \right] \]

whereas for \( N \text{ odd} \):

\[ \Gamma(\omega) = 2 e^{-jN\omega T} \left[ \Gamma_0 \cos N\omega T + \Gamma_1 \cos (N-2)\omega T + \cdots + \Gamma_n \cos (N-2n)\omega T + \cdots + \Gamma_{(N-1)/2} \cos \omega T \right] \]

The remaining question then is this: given an optimal and realizable function \( \Gamma_{in}(\omega) \), how do we determine the necessary number of sections \( N \), and how do we determine the values of all reflection coefficients \( \Gamma_n \)?