5.5 – The Theory of Small Reflections

Reading Assignment: ***pp. 244-246***

An important and useful **approximation** when considering multi-section matching networks is the **Theory of Small Reflections**.

HO: THE THEORY OF SMALL REFLECTIONS

EXAMPLE: THE THEORY OF SMALL REFLECTIONS

The Theory of Small Reflections provides a simpler mathematical form for analyzing the **frequency response** of many microwave devices.

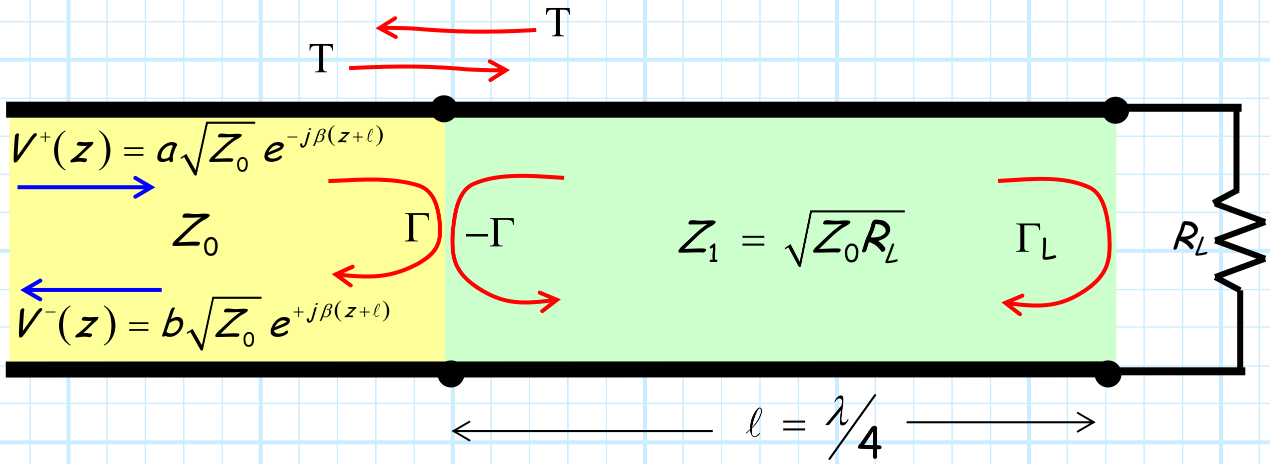
HO: THE FREQUENCY RESPONSE OF THE QUARTER-WAVE MATCHING NETWORK

We can also use the Theory of Small Reflections to provide an **approximate** analysis of a **multi-section** impedance transformer (i.e., multi-section matching network).

HO: THE MULTI-SECTION TRANSFORMER

The Theory of Small Reflections

Recall that we analyzed a **quarter-wave** transformer using the multiple reflection view point.

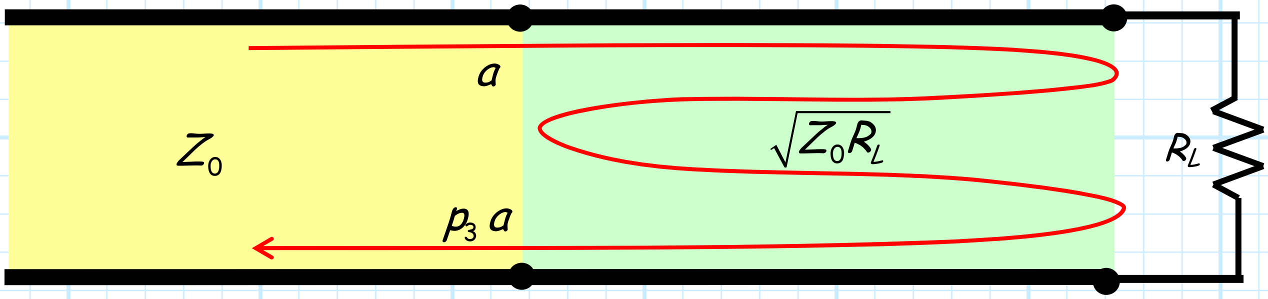


We found that the solution could thus be written as an **infinite** summation of terms (the **propagation series**):

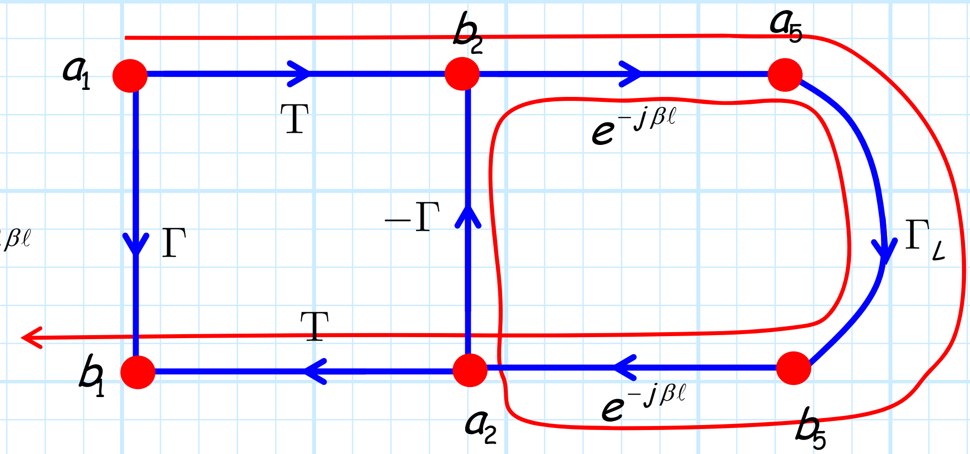
$$b = a\sum_{n=1}^{\infty} p_n$$

where each term had a specific **physical** interpretation, in terms of reflections, transmissions, and propagations.

For example, the **third** term was **path**:



$$p_3 = \mathrm{T}^2 \left(\Gamma_L\right)^2 \Gamma \, e^{-j2\beta\ell}$$



Now let's consider the **magnitude** of this path:

$$\begin{aligned} \left|p_3\right| &= \left|\mathrm{T}\right|^2 \left|\Gamma_L\right|^2 \left|\Gamma\right| \left|e^{-j2\beta\ell}\right| \\ &= \left|\mathrm{T}\right|^2 \left|\Gamma_L\right|^2 \left|\Gamma\right| \end{aligned}$$

Recall that $\Gamma = \Gamma_L$ for a **properly designed** quarter-wave transformer :

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L$$

and so:

$$\left|p_3\right| = \left|\mathrm{T}\right|^2 \left|\Gamma_L\right|^2 \left|\Gamma\right| = \left|\mathrm{T}\right|^2 \left|\Gamma_L\right|^3$$

For the case where values R_L and Z_1 are numerically **"close"** in —i.e., when:

$$\left|R_L - Z_1\right| \ll \left|R_L + Z_1\right|$$

we find that the magnitude of the reflection coefficient will be **very** small:

$$\left|\Gamma_L\right| = \left|\frac{R_L - Z_1}{R_L + Z_1}\right| \ll 1.0$$

As a result, the value $\left|\Gamma_L\right|^3$ will be **very**, **very**, **very** small.

Moreover, we know (since the connector is **lossless**) that:

$$1 = |\Gamma|^2 + |\mathrm{T}|^2 = |\Gamma_L|^2 + |\mathrm{T}|^2$$

and so:

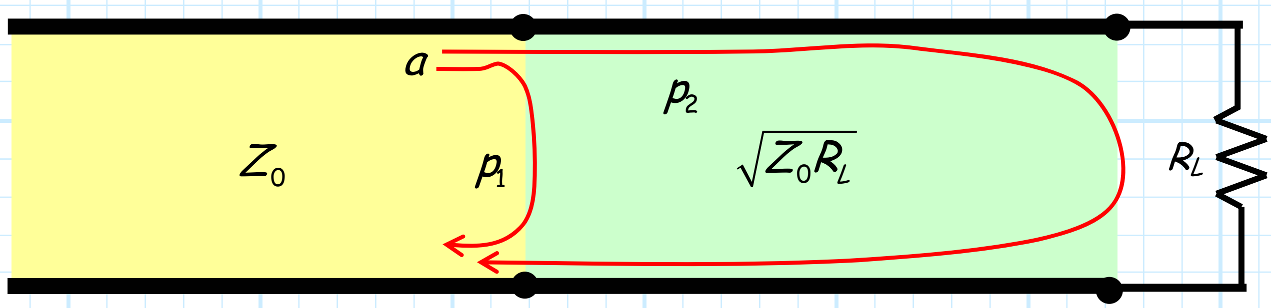
$$|\mathrm{T}|^2 = 1 - |\Gamma_L|^2 \approx 1$$

We can thus conclude that the **magnitude** of path p_3 is likewise **very, very, very** small:

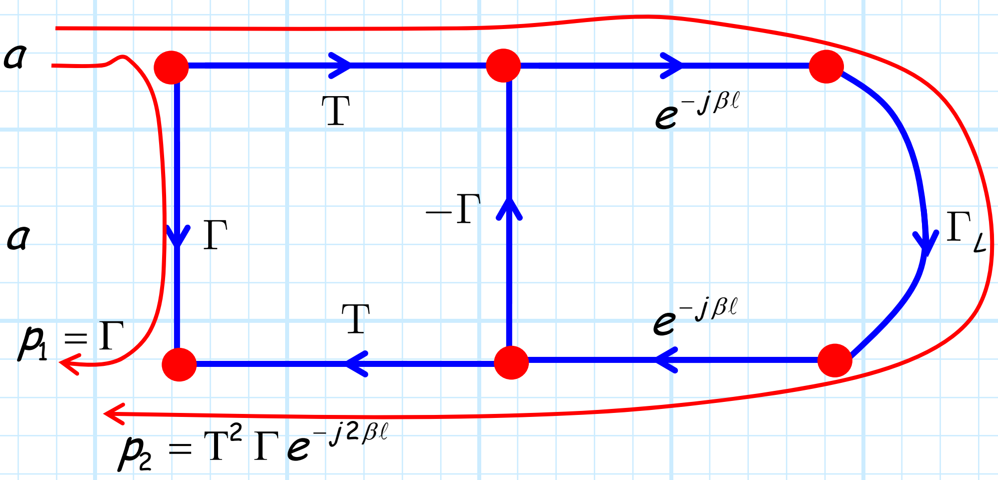
$$|p_3| = |\mathrm{T}|^2 |\Gamma_L|^3 \approx |\Gamma_L|^3 \ll 1$$

This is a **classic case** where we can approximate the propagation series using only the **forward paths**!!

Recall there are **two** forward paths:



$$b \simeq (p_1 + p_2)\, a$$
$$= \left(\Gamma + \mathrm{T}^2\, \Gamma_L\, e^{j2\beta\ell}\right) a$$



Therefore **IF** Z_0 and R_L are very **close** in value, we find that we can **approximate** the reflected wave using only the **direct paths** of the infinite series:

$$\begin{aligned} b &\simeq (p_1 + p_2)\, a \\ &= \left(\Gamma + T^2\, \Gamma_L\, e^{j2\beta\ell}\right) a \end{aligned}$$

Therefore:

$$\begin{aligned} V^-(z) &= b\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \\ &\cong \left(\Gamma + T^2\, \Gamma_L\, e^{j2\beta\ell}\right) a\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \end{aligned}$$

Now, if we likewise apply the **approximation** that $|T| \simeq 1.0$, we conclude for this quarter wave transformer (at the design frequency):

$$\begin{aligned} b &\simeq (p_1 + p_2)\, a \\ &= \left(\Gamma + \Gamma_L\, e^{j2\beta\ell}\right) a \end{aligned}$$

Therefore:

$$\begin{aligned} V^-(z) &= b\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \\ &\cong \left(\Gamma + \Gamma_L\, e^{j2\beta\ell}\right) a\sqrt{Z_0}\; e^{+j\beta(z+\ell)} \end{aligned}$$

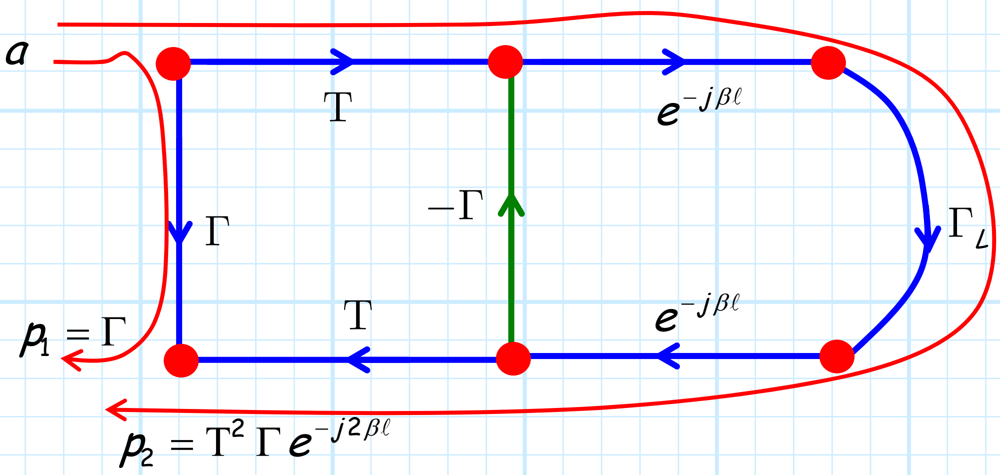
This **approximation**, where we:

1. use only the **direct paths** to calculate the propagation series,

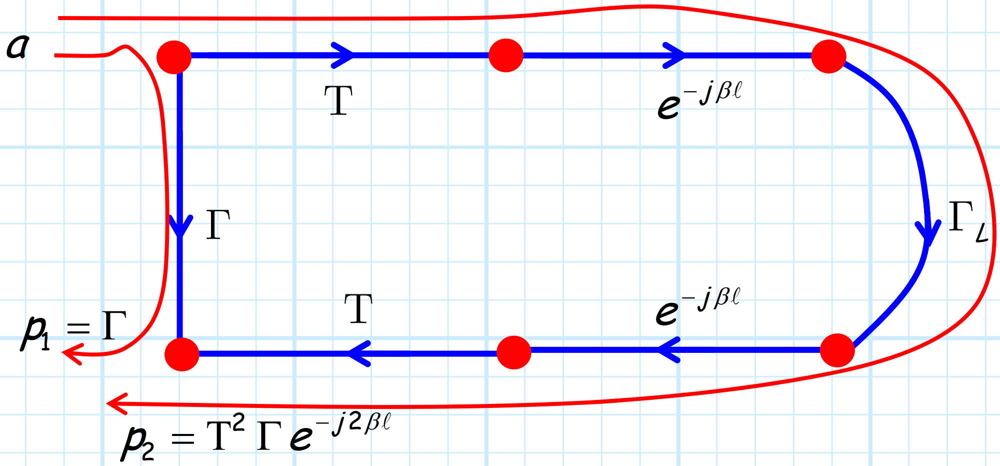
2. approximate the **transmission** coefficients as **one** (i.e., $T = 1$).

is known as the **Theory of Small Reflections**, and allows us to use the propagation series as an **analysis** tool (we don't have to consider an **infinite** number of terms!).

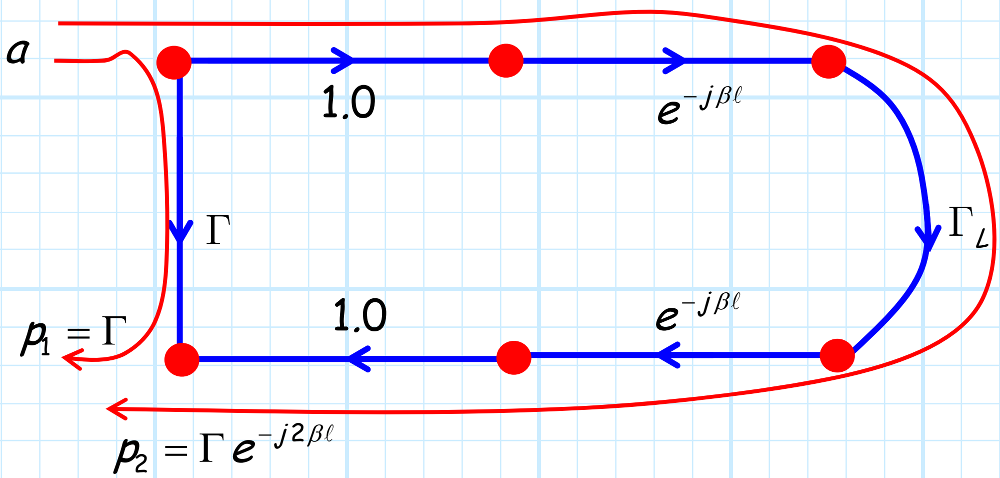
Consider again the quarter-wave matching network *SFG*. Note there is **one branch** ($-\Gamma = S_{22}$ of the connector), that is **not included** in either **direct path**.



With respect to the theory of small reflections (where **only** direct paths are considered), this branch can be **removed** from the *SFG* **without affect**.

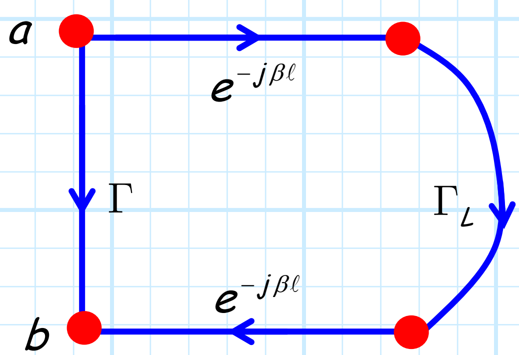


Moreover, the theory of small reflections implements the **approximation** $\mathrm{T} = 1$, so that the *SFG* becomes:



Reducing this SFG by combining the 1.0 branch and the $e^{-j\beta\ell}$ branch via the **series rule**, we get the following **approximate** SFG:

$$\Gamma_{in} = \frac{b}{a}$$
$$= \Gamma + \Gamma_L\, e^{j2\beta\ell}$$



The approximate SFG when applying the theory of small reflections!

Note this **approximate** SFG provides **precisely** the results of the theory of small reflections!

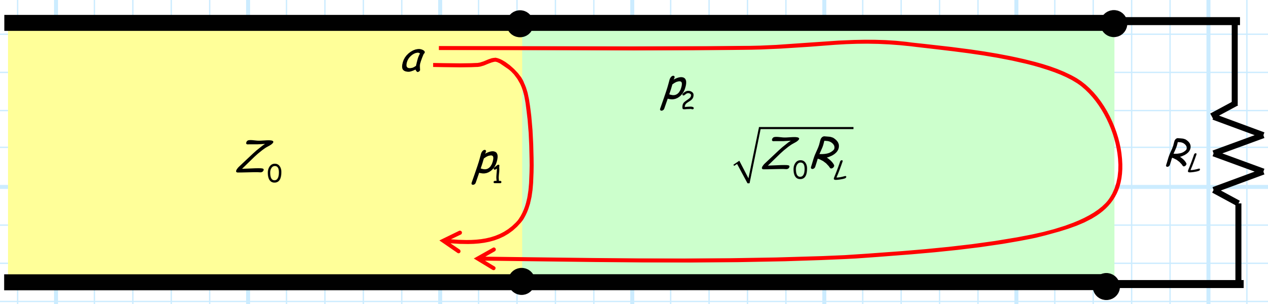
Q: *Why is that?*

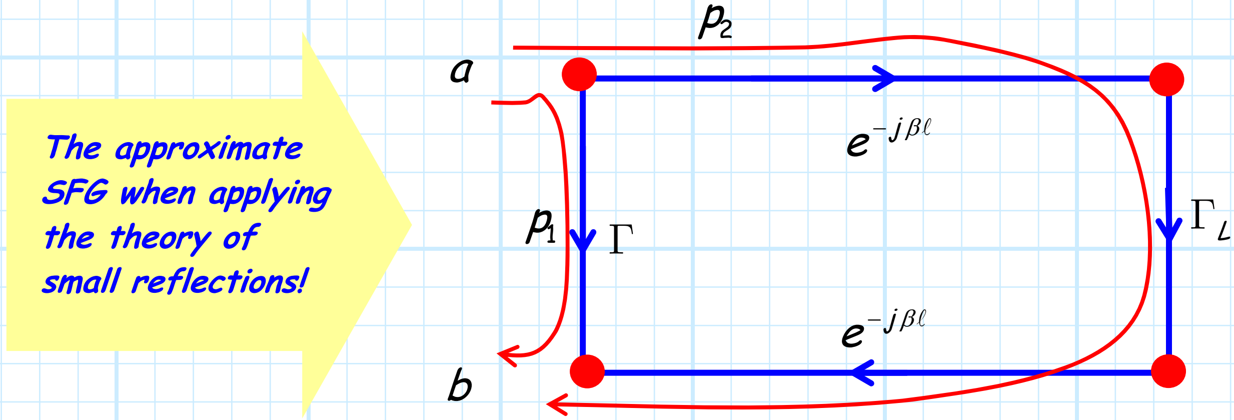
A: The approximate "theory of small reflections SFG" Contains all of the **significant physical propagation mechanisms** of the two *forward paths*, and **only** the two significant propagation mechanisms of the two forward paths.

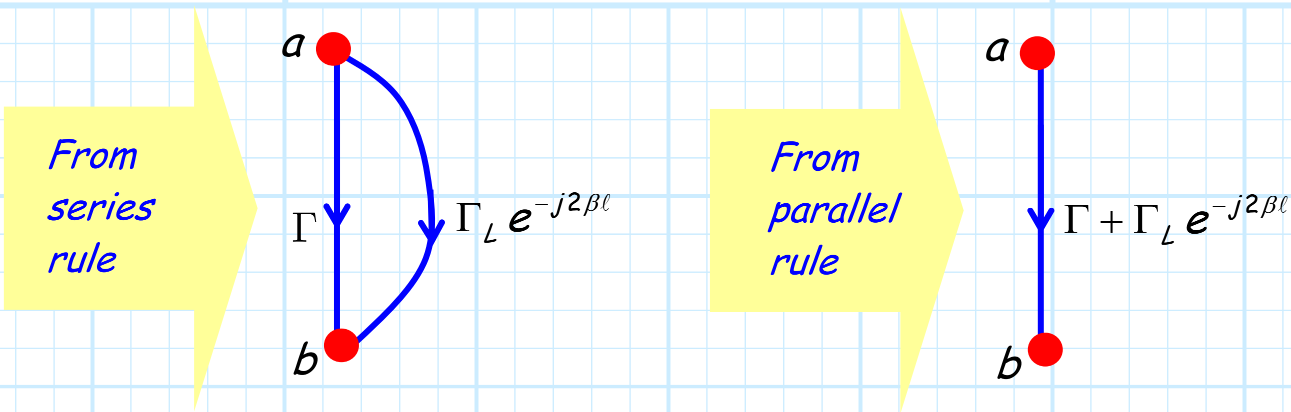
Namely:

1. The **reflection** at the connector (i.e., Γ).

2. The **propagation down** the quarter-wave transmission line ($e^{-j\beta\ell}$), the **reflection** off the load (Γ_L), and the **propagation back up** the quarter-wave transmission line ($e^{-j\beta\ell}$).







Q: *But wait! The quarter-wave transformer is a **matching** network, therefore $\Gamma_{in} = 0$. The **theory of small reflections**, however, provides the **approximate result**:*

$$\Gamma_{in} \approx \Gamma + \Gamma_L e^{-j2\beta\ell}$$

*Is this **approximation** very **accurate**? How **close** is this **approximate** value to the correct answer of $\Gamma_{in} = 0$?*

A: Let's find out!

Recall that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, and so:

$$\begin{aligned}\Gamma_{in} &\approx \Gamma + \Gamma_L e^{-j2\beta\ell} \\ &= \Gamma_L \left(1 + e^{-j2\beta\ell}\right)\end{aligned}$$

Likewise, $\ell = \lambda/4$ (but **only** at the design frequency!) so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = 2\pi/\lambda$!

Thus:

$$\begin{aligned}\Gamma_{in} &\approx \Gamma_L\left(1+e^{-j2\beta\ell}\right)\\ &=\Gamma_L\left(1+e^{-j\pi}\right)\\ &=\Gamma_L\left(1-1\right)\\ &=0 \quad !!!\end{aligned}$$

Q: ***Wow!*** *The theory of small reflections appears to be a* ***perfect*** *approximation—* ***no error*** *at all!?!*

A: Not so fast.

The **theory of small reflections** most definitely provides an **approximate** solution (e.g., it **ignores** most of the terms of the propagation series, and it **approximates** connector transmission as $T=1$, when in fact $T\neq 1$).

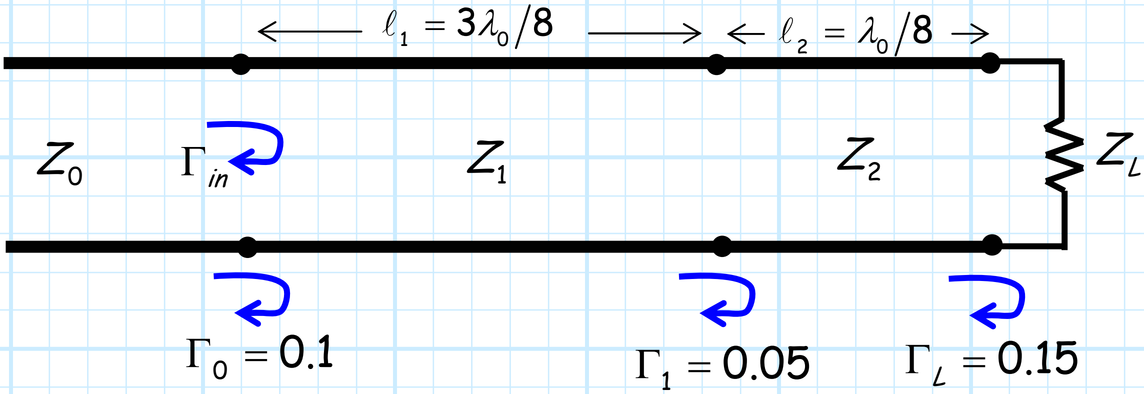
As a result, the solutions derived using the **theory of small reflections** will—generally speaking—exhibit **some** (hopefully small) **error**.

We just got a bit **"lucky"** for the quarter-wave matching network; the "approximate" result $\Gamma_{in}=0$ was exact for this one case!

→ The **theory of small reflections** is an **approximate** analysis tool!

Example: The Theory of Small Reflections

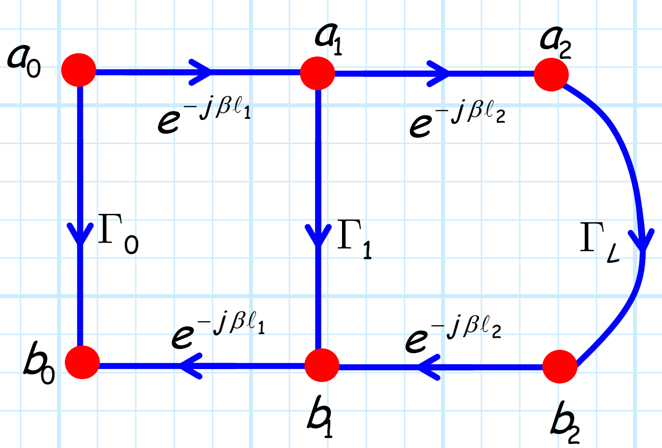
Use the **theory of small reflections** to determine a **numeric** value for the **input** reflection coefficient Γ_{in}, at the design frequency ω_0.



Note that the transmission line sections have **different lengths**!

Solution

Applying the theory of small reflections, the **approximate signal flow graph** of the structure becomes:

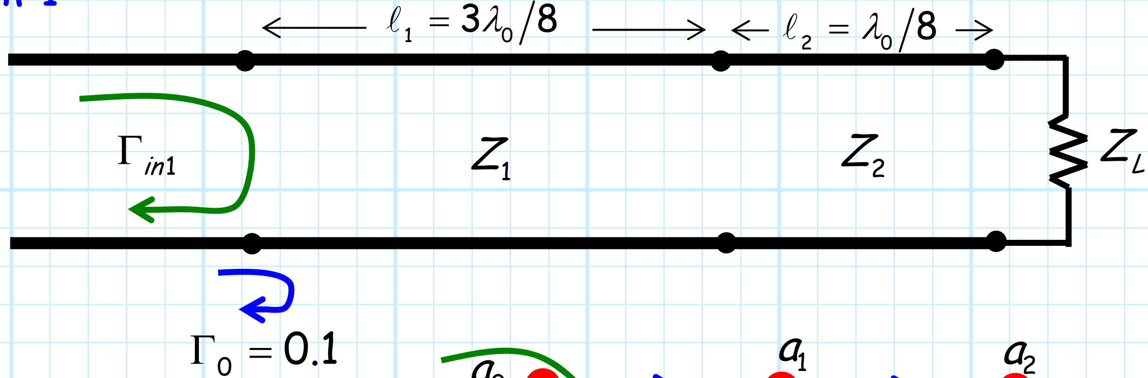


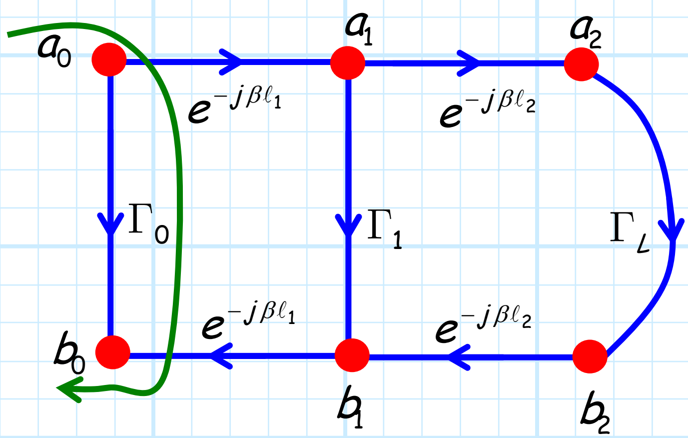
$$e^{-j\beta\ell_1} = e^{-j\left(\frac{2\pi}{\lambda}\frac{3\lambda}{8}\right)} = e^{-j\left(\frac{3\pi}{4}\right)}$$

$$e^{-j\beta\ell_2} = e^{-j\left(\frac{2\pi}{\lambda}\frac{\lambda}{8}\right)} = e^{-j\left(\frac{\pi}{4}\right)}$$

Note there are three **direct** propagation paths:

Path 1

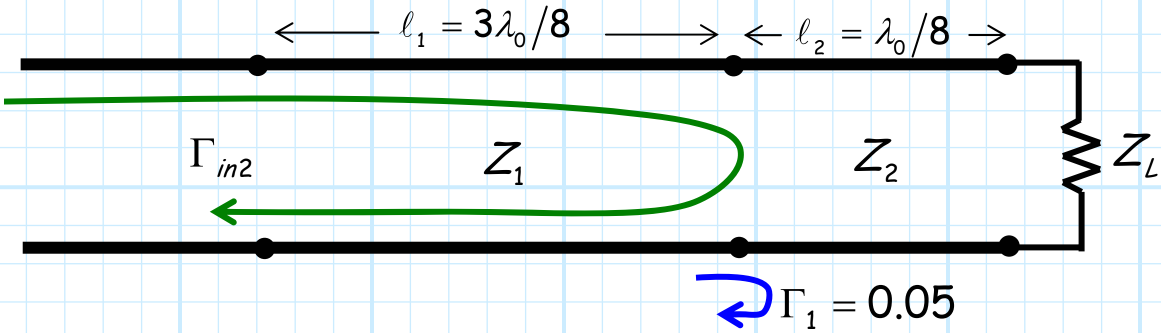




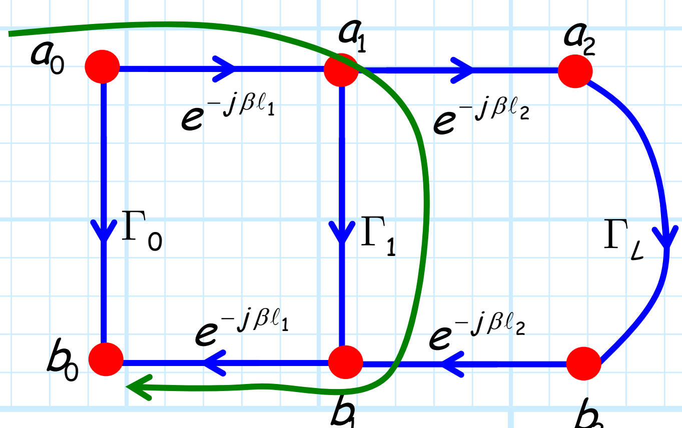
$$\therefore \quad p_1 = \Gamma_0 = 0.1$$

Path 2

This path includes propagation **down** and **back** a transmission line length ℓ_1 !

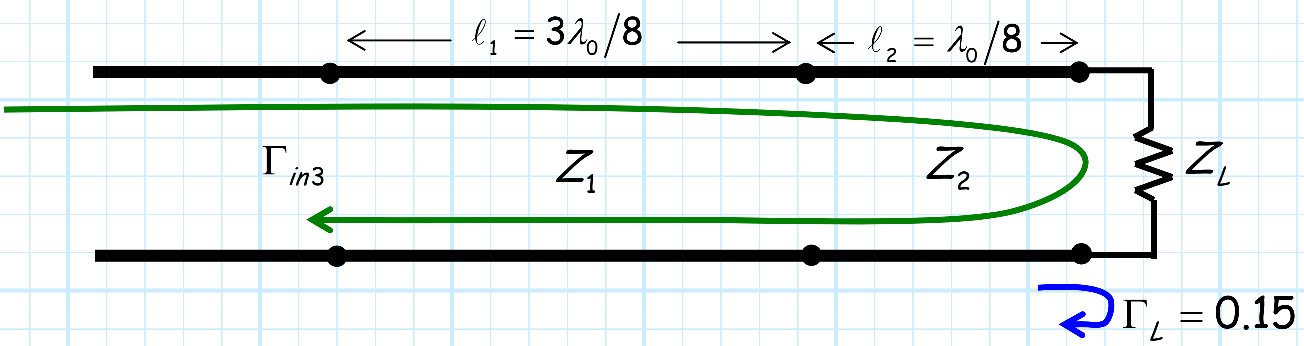


$$\begin{aligned} p_2 &= e^{-j\beta\ell_1}\,\Gamma_1\,e^{-j\beta\ell_1} \\ &= e^{-j^{3\pi}/4}\,0.05e^{-j^{3\pi}/4} \\ &= e^{-j^{3\pi}/2}\,0.05 \\ &= +j0.05 \end{aligned}$$

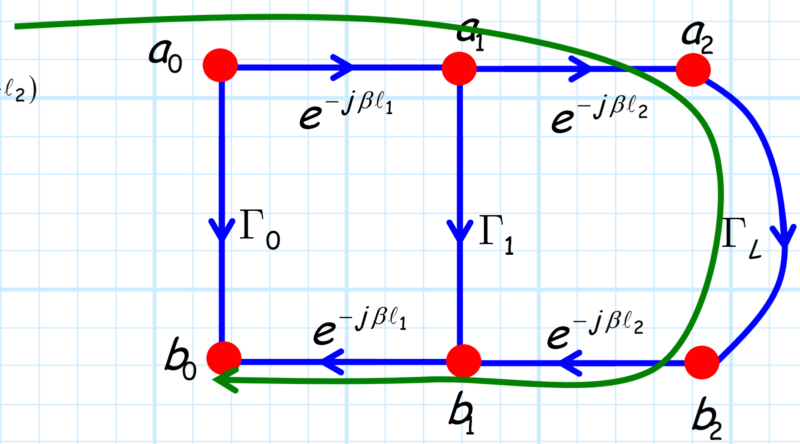


Path 3

This path includes propagation **down** and **back** transmission line lengths of $\ell_1 + \ell_2$!



$$
\begin{aligned}
p_3 &= e^{-j\beta(\ell_1+\ell_2)}\,\Gamma_L\, e^{-j\beta(\ell_1+\ell_2)} \\
&= e^{-j\pi}\ 0.15\ e^{-j\pi} \\
&= e^{-j2\pi}\ 0.15 \\
&= 0.15
\end{aligned}
$$

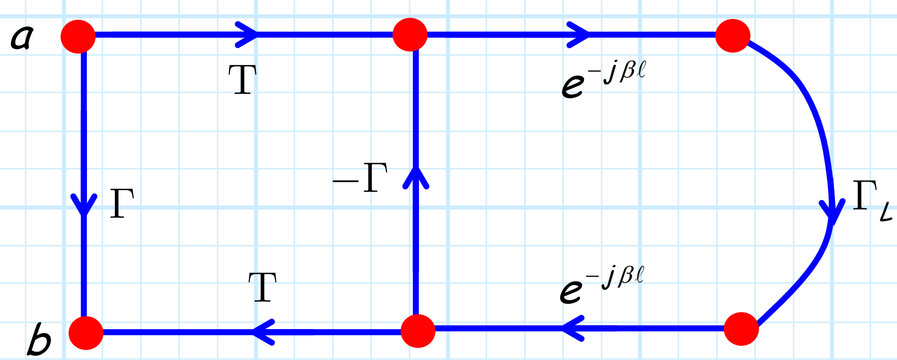


Thus, using the **theory of small reflections** we can determine approximately the input reflection coefficient:

$$
\begin{aligned}
\Gamma_{in} &= \frac{b_0}{a_0} \\
&= p_1 + p_2 + p_3 \\
&= 0.1 + j0.05 + 0.15 \\
&= \underline{\underline{0.25 + j0.05}}
\end{aligned}
$$

The Frequency Response of a Quarter-Wave Matching Network

Q: *You have once again provided us with **confusing** and perhaps useless information. The quarter-wave matching network has an **exact** SFG of:*

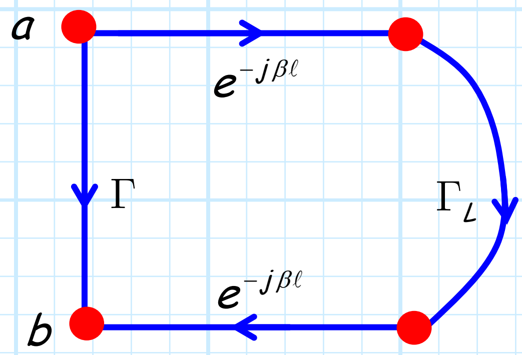


*Using our **reduction rules**, we can **quickly** conclude that:*

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \frac{\mathrm{T}^2\, \Gamma_L\, e^{-j2\beta\ell}}{1-\Gamma\,\Gamma_L}$$

*You could have left this **simple** and **precise** analysis **alone**— BUT **NOOO!!***

You** had to foist upon us a long, **rambling** discussion of "the propagation series" and "direct paths" and "the theory of* *small reflections", culminating with the* ***approximate *(i.e., less accurate!) SFG:*



From which we were able to conclude the ***approximate*** *(i.e., less accurate!) result:*

$$\Gamma_{in} \doteq \frac{b}{a} = \Gamma + \Gamma_L e^{-j2\beta\ell}$$

The ***exact*** *result was* ***simple****—and* ***exact****!* ***Why*** *did you make us determine this* ***approximate*** *result?*

A: In a word: frequency response*****.

Although the exact analysis is **about** as simple to determine as the approximation provided by the theory of small reflections, the **mathematical form** of the result is much simpler to **analyze** and/or **evaluate** (e.g., no **fractional** terms!).

Q: *What exactly would we be analyzing and/or evaluating?*

A: The **frequency response** of the matching network, for one thing.

* OK, **two** words.

Remember, all matching networks must be **lossless**, and so must be made of **reactive** elements (e.g., lossless transmission lines). The impedance of every reactive element is a **function of frequency**, and so too then is Γ_{in}.

Say we wish to determine this **function** $\Gamma_{in}(\omega)$.

Q: *Isn't* $\Gamma_{in}(\omega) = 0$ *for a quarter wave matching network?*

A: Oh my gosh **no**! A properly designed matching network will typically result in a perfect match (i.e., $\Gamma_{in} = 0$) at **one frequency** (i.e., the design frequency). However, if the signal frequency is **different** from this design frequency, then no match will occur (i.e., $\Gamma_{in} \neq 0$).

Recall we discussed this behavior **before**:

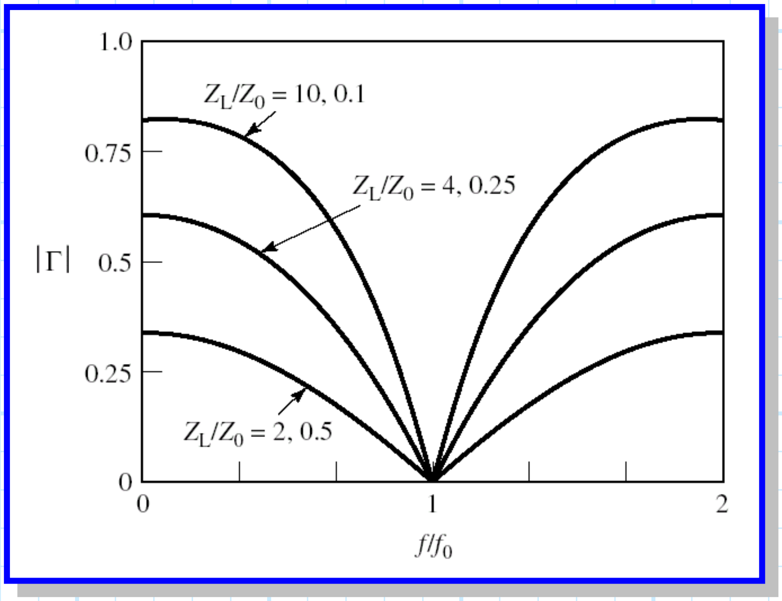


Figure 5.12 (p. 243)
Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

Q: *But **why** is the result:*

$$\Gamma_{in} = \Gamma + \frac{T^2 \Gamma_L e^{-j2\beta\ell}}{1 - \Gamma \Gamma_L}$$

or its approximate form:

$$\Gamma_{in} = \Gamma + \Gamma_L e^{-j2\beta\ell}$$

*dependent on **frequency**? I don't **see** frequency variable ω anywhere in these results!*

A: Look **closer**!

Remember that the value of spatial frequency β (in radians/meter) is dependent on the frequency ω of our eigen function (aka "the signal"):

$$\beta = \left(\frac{1}{v_p}\right)\omega$$

where **you** will recall that v_p is the propagation velocity of a wave moving along a transmission line. This velocity is a constant (i.e., $v_p = 1/\sqrt{LC}$), and so the spatial frequency β is directly proportional to the temporal frequency ω.

Thus, we can rewrite:

$$\beta \ell = \frac{\omega \ell}{v_p} = \omega T$$

Where $T = \ell / v_p$ is the **time** required for the wave to **propagate** a distance ℓ down a transmission line.

As a result, we can write the input reflection coefficient as a function of **spatial frequency** β:

$$\Gamma_{in}(\beta) = \Gamma + \Gamma_L \, e^{-j2\beta\ell}$$

Or equivalently as a function of **temporal frequency** ω:

$$\Gamma_{in}(\omega) = \Gamma + \Gamma_L \, e^{-j2\omega T}$$

Frequently, the reflection coefficient is simply written in terms of the **electrical length** θ of the transmission line, which is simply the **difference in relative phase** between the wave at the beginning and end of the length ℓ of the transmission line.

$$\beta\ell = \theta = \omega T$$

So that:

$$\Gamma_{in}(\theta) = \Gamma + \Gamma_L \, e^{-j2\theta}$$

Note we can simply insert the value $\theta = \beta\ell$ into the expression above to get $\Gamma_{in}(\beta)$, or insert $\theta = \omega T$ into the expression to get $\Gamma_{in}(\omega)$.

Now, we know that $\Gamma = \Gamma_L$ for a properly designed quarter-wave matching network, so the reflection coefficient function can be written as:

$$\Gamma_{in}(\theta) = \Gamma_L \left(1 + e^{-j2\theta}\right)$$

Note that: $$1 = e^{j0} = e^{-j(\theta-\theta)} = e^{-j\theta}\, e^{+j\theta}$$

And that: $$e^{-j2\theta} = e^{-j(\theta+\theta)} = e^{-j\theta}\, e^{-j\theta}$$

And so:

$$\begin{aligned}\Gamma_{in}(\theta) &= \Gamma_L \left(1 + e^{-j2\theta}\right) \\ &= \Gamma_L \left(e^{-j\theta}\, e^{+j\theta} + e^{-j\theta}\, e^{-j\theta}\right) \\ &= \Gamma_L\, e^{-j\theta} \left(e^{+j\theta} + e^{-j\theta}\right) \\ &= \Gamma_L\, e^{-j\theta} \left(2\cos\theta\right)\end{aligned}$$

Where we have used **Euler's equation** to determine that:

$$e^{+j\theta} + e^{-j\theta} = 2\cos\theta$$

Now, let's determine the **magnitude** of our result:

$$\left|\Gamma_{in}(\theta)\right| = \left|\Gamma_L\right| \left|e^{-j\theta}\right| 2 \left|\cos\theta\right| = 2\left|\Gamma_L\right| \left|\cos\theta\right|$$

Note that $\left|\Gamma_{in}(\theta)\right|$ is **zero-valued** only when $\cos\theta = 0$. This of course occurs when $\theta = \pi/2$:

$$\left|\Gamma_{in}(\theta)\right|_{\theta=\pi/2} = 2\left|\Gamma_L\right| \left|\cos \pi/2\right| = 0$$

In other words, a **perfect match** occurs when $\theta = \pi/2$!!

Q: *What the heck does this mean?*

A: Remember, $\theta = \beta\ell$. Thus if $\theta = \pi/2$:

$$\ell = \frac{\theta}{\beta} = \frac{\pi/2}{2\pi/\lambda} = \frac{\lambda}{4} \quad !!$$

As we (should have) suspected, the match occurs at the frequency whose wavelength is equal to **four times** the matching (Z_1) transmission line length, i.e. $\lambda = 4\ell$.

In other words, a perfect match occurs at the **frequency** where $\ell = \lambda/4$.

Note the **physical** length ℓ of the transmission line does **not** change with frequency, but the signal **wavelength** does:

$$\lambda = \frac{v_p}{f}$$

Q: *So, at precisely what **frequency** does a quarter-wave transformer with length ℓ provide a **perfect** match?*

A: Recall also that $\theta = \omega T$, where $T = \ell/v_p$. Thus, for $\theta = \pi/2$:

$$\theta = \frac{\pi}{2} = \omega T \qquad \Rightarrow \qquad \omega = \frac{\pi}{2}\frac{1}{T} = \frac{\pi}{2}\frac{v_p}{\ell}$$

This frequency is called the **design frequency** of the matching network—it's the frequency where a **perfect** match occurs. We denote this as frequency ω_0, which has wavelength λ_0, i.e.:

$$\omega_0 = \frac{\pi}{2T} = \pi \frac{v_p}{2\ell} \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{4T} = \frac{v_p}{4\,\ell} \qquad \lambda_0 = \frac{v_p}{f_0} = 4 v_p T = 4\ell$$

Given this, yet **another way** of expressing $\theta = \beta\ell$ is:

$$\theta = \beta\ell = \frac{\omega}{v_p}\left(\pi \frac{v_p}{2\omega_0}\right) = \pi \frac{\omega}{2\omega_0} = \pi \frac{f}{2f_0}$$

Thus, we conclude:

$$\left|\Gamma_{in}(f)\right| = 2\left|\Gamma_L\right|\left|\cos\left(\pi \tfrac{f}{2f_0}\right)\right|$$

From this result we can determine (approximately) the **bandwidth** of the quarter-wave transformer!

First, we must **define** what we mean by bandwidth. Say the **maximum** acceptable level of the reflection coefficient is value Γ_m. This is an arbitrary value, set by **you** the microwave engineer (typical values of Γ_m range from 0.05 to 0.2).

We will denote the frequencies where this maximum value Γ_m occurs f_m. In other words:

$$\left|\Gamma_{in}(f = f_m)\right| = \Gamma_m = 2\left|\Gamma_L\right|\left|\cos\left(\pi \tfrac{f_m}{2f_0}\right)\right|$$

There are **two solutions** to this equation, the first is:

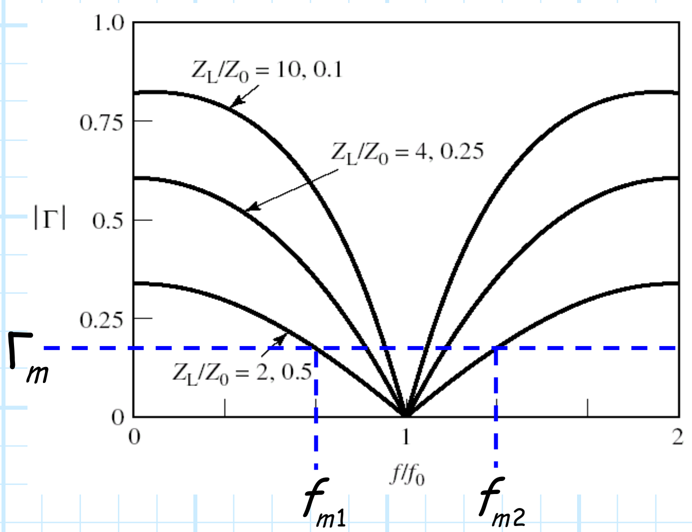
$$f_{m1} = \frac{2f_0}{\pi} \cos^{-1}\left(\frac{\Gamma_m}{2\left|\Gamma_L\right|} \right)$$

And the second:

$$f_{m2} = \frac{2f_0}{\pi} \cos^{-1}\left(-\frac{\Gamma_m}{2\left|\Gamma_L\right|} \right)$$

Important note! Make sure $\cos^{-1} x$ is expressed in **radians**!

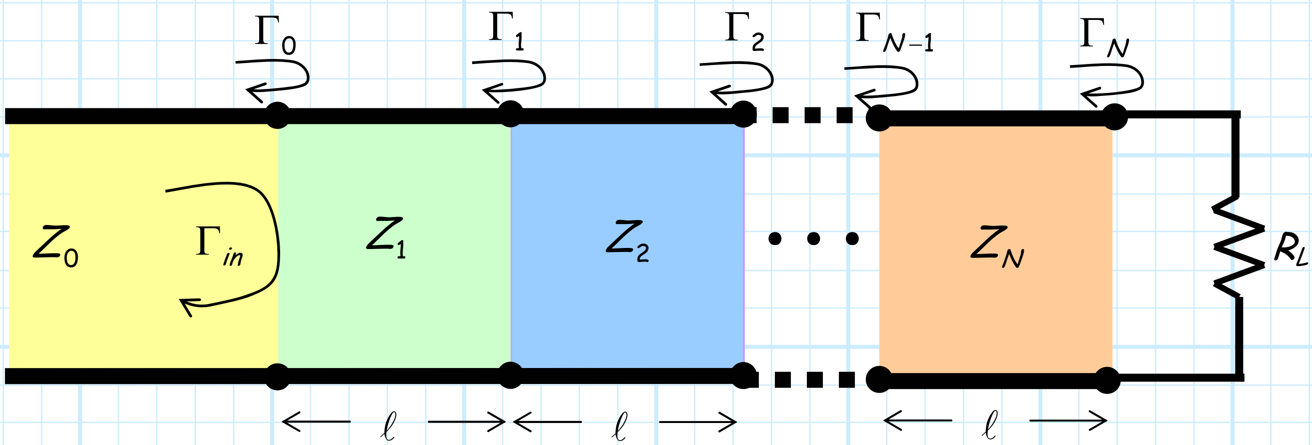
You will find that $f_{m1} < f_0 < f_{m2}$ so, the values f_{m1} and f_{m2} define the **lower** and **upper** limits on matching network **bandwidth**.



All this analysis was brought to you by the **"simple" mathematical form** of $\Gamma_{in}(f)$ that resulted from the theory of small reflections!

The Multi-section Transformer

Consider a sequence of N transmission line **sections**; each section has **equal length** ℓ, but **dissimilar** characteristic impedances:



Where the marginal reflection coefficients are:

$$\Gamma_0 \doteq \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad \Gamma_n \doteq \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \qquad \Gamma_N \doteq \frac{R_L - Z_N}{R_L + Z_N}$$

If the load resistance R_L is **less** than Z_0, then we should design the transformer such that:

$$Z_0 > Z_1 > Z_2 > Z_3 \cdots > Z_N > R_L$$

Conversely, if R_L is **greater** than Z_0, then we will design the transformer such that:

$$Z_0 < Z_1 < Z_2 < Z_3 \cdots < Z_N < R_L$$

In other words, we **gradually transition** from Z_0 to R_L !

Note that since R_L is **real**, and since we assume **lossless** transmission lines, all Γ_n will be **real** (this is important!).

Likewise, since we **gradually** transition from one section to another, each value:

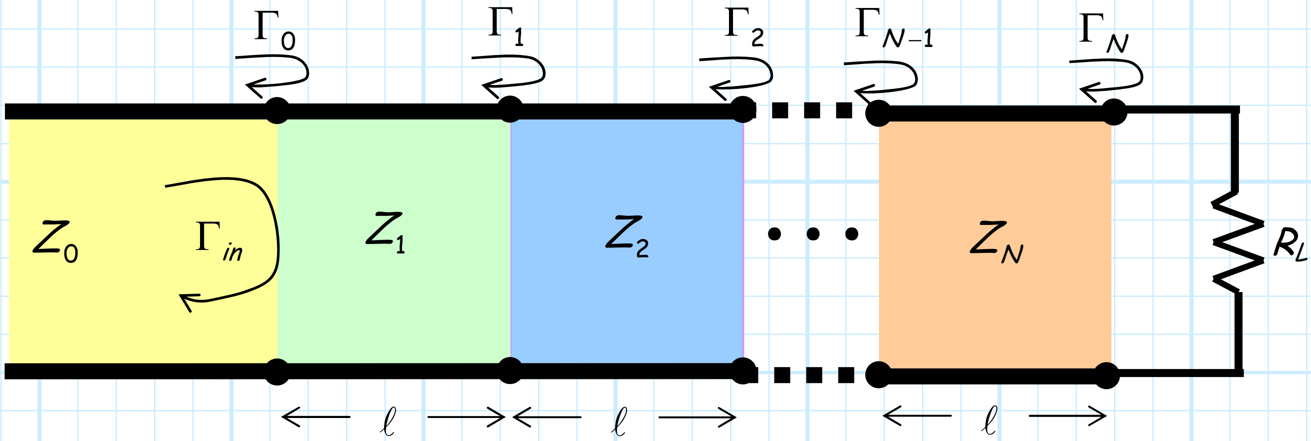
$$Z_{n+1} - Z_n$$

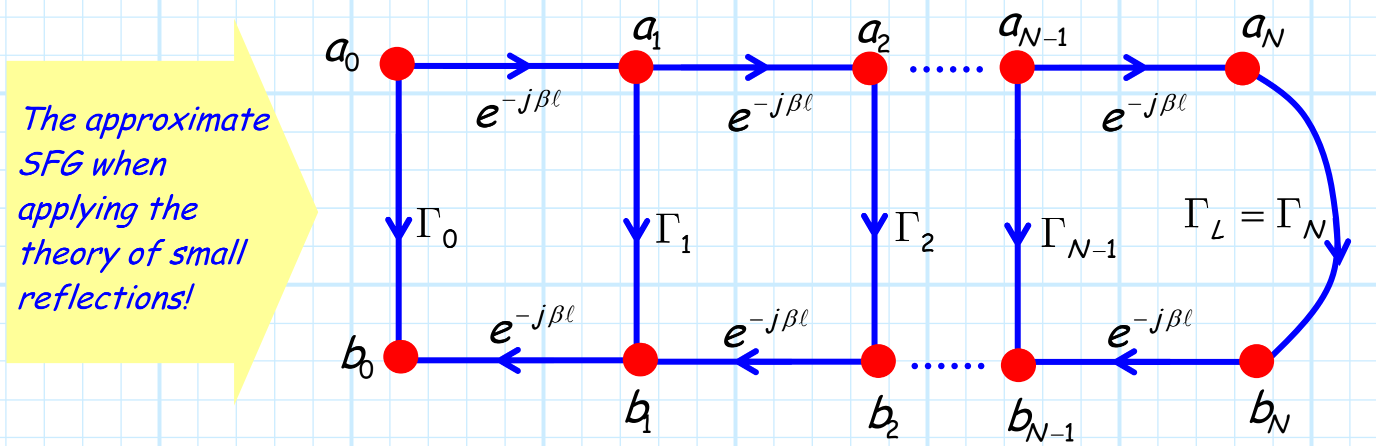
will be **small**.

As a result, each marginal reflection coefficient Γ_n will be **real** and have a **small** magnitude.

This is also **important**, as it means that we can apply the "**theory of small reflections**" to analyze this multi-section transformer!

The theory of small reflections allows us to **approximate** the input reflection coefficient of the transformer as:





$$
\begin{aligned}
\frac{b_0}{a_0} &= \Gamma_{in}(\beta) \\
&\simeq \Gamma_0 + \Gamma_1\, e^{-j2\beta\ell} + \Gamma_2\, e^{-j4\beta\ell} + \cdots + \Gamma_N\, e^{-j2N\beta\ell} \\
&= \sum_{n=0}^{N} \Gamma_n\, e^{-j2n\beta\ell}
\end{aligned}
$$

We can alternatively express the input reflection coefficient as a function of **frequency** ($\beta\ell = \omega T$):

$$
\begin{aligned}
\Gamma_{in}(\omega) &= \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T} \\
&= \sum_{n=0}^{N} \Gamma_n\, e^{-j(2nT)\omega}
\end{aligned}
$$

where:

$$T \doteq \frac{\ell}{v_p} = \text{propagation time through 1 section}$$

We see that the function $\Gamma_{in}(\omega)$ is expressed as a **weighted** set of N **basis functions**! I.E.,

$$\Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \Psi(\omega)$$

where:

$$c_n = \Gamma_n \qquad \textit{and} \qquad \Psi(\omega) = e^{-j(2nT)\omega}$$

We find, therefore, that by **selecting** the proper values of basis weights c_n (i.e., the proper values of reflection coefficients Γ_n), we can **synthesize** any function $\Gamma_{in}(\omega)$ of frequency ω, provided that:

1. $\Gamma_{in}(\omega)$ is **periodic** in $\omega = 1/2T$

2. we have sufficient **number** of sections N.

Q: *What function **should** we synthesize?*

A: **Ideally**, we would want to make $\Gamma_{in}(\omega) = 0$ (i.e., the reflection coefficient is zero for all frequencies).

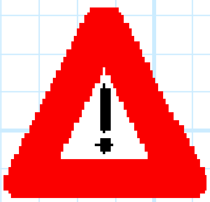
Bad news: this **ideal** function $\Gamma_{in}(\omega) = 0$ would require an **infinite** number of sections (i.e., $N = \infty$)!

Instead, we seek to find an "**optimal**" function for $\Gamma_{in}(\omega)$, given a **finite** number of N elements.

Once we determine these optimal functions, we can find the values of coefficients Γ_n (or equivalently, Z_n) that will result in a matching transformer that exhibits this **optimal** frequency response.

To **simplify** this process, we can make the transformer **symmetrical**, such that:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \quad \Gamma_2 = \Gamma_{N-2}, \quad \cdots\cdots$$



Note that this **does** NOT mean that:

$$Z_0 = Z_N, \quad Z_1 = Z_{N-1}, \quad Z_2 = Z_{N-2}, \quad \cdots\cdots$$

We find then that:

$$\Gamma(\omega) = e^{-jN\omega T}\left[\Gamma_0\left(e^{jN\omega T} + e^{-jN\omega T}\right) + \Gamma_1\left(e^{j(N-2)\omega T} + e^{-j(N-2)\omega T}\right)\right.$$
$$\left. + \Gamma_2\left(e^{j(N-4)\omega T} + e^{-j(N-4)\omega T}\right) + \cdots\right]$$

and since:

$$e^{jx} + e^{-jx} = 2\cos(x)$$

we can write for N **even**:

$$\Gamma(\omega) = 2\,e^{-jN\omega T}\Big[\Gamma_0 \cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N-2n)\omega T + \cdots + \frac{1}{2}\Gamma_{N/2}\Big]$$

whereas for N **odd**:

$$\Gamma(\omega) = 2\,e^{-jN\omega T}\Big[\Gamma_0 \cos N\omega T + \Gamma_1 \cos(N-2)\omega T + \cdots + \Gamma_n \cos(N-2n)\omega T + \cdots + \Gamma_{(N-1)/2} \cos \omega T\Big]$$

The remaining **question** then is this: *given an optimal and realizable function* $\Gamma_{in}(\omega)$, **how** do we determine the necessary number of **sections** N, and **how** do we determine the **values** of all reflection coefficients Γ_n??