5.5 - The Theory of Small Reflections

Reading Assignment: pp. 244-246

An important and useful approximation when considering multi-section matching networks is the Theory of Small Reflections.

**HO: The Theory of Small Reflections**

We can use the Theory of Small Reflections to provide an approximate analysis of a multi-section impedance transformer (i.e., multi-section matching network).

**HO: The Multisection Transformer**
Recall that we analyzed a quarter-wave transformer using the multiple reflection viewpoint. We found that the solution could be written as an infinite summation of terms:

\[ V^-(z) = \sum_{n=1}^{\infty} V_n^-(z) \]

where each term had a specific physical interpretation, in terms of reflections, transmissions, and propagations.

For example, the first term was \( V_1^-(z = -\ell) = \Gamma V^+(z = -\ell) \):
While the second term was:

\[ V^+(z) \]

\[ Z_0 \]

\[ V_2^-(z) \]

\[ \sqrt{Z_0 R_L} \]

\[ V_2^-(z = -\ell) = -T^2 \Gamma \bar{L} V^+(z = -\ell) \]

Contrast these first two terms with the third term:

\[ V^+(z) \]

\[ Z_0 \]

\[ V_3^-(z) \]

\[ \sqrt{Z_0 R_L} \]
This third term is characterized by three reflections, and thus there is a product of three reflection coefficient values $\Gamma$ in the term. In fact, all other terms in this infinite series will likewise describe mechanisms with multiple reflections.

This is in contrast with the first two terms, which exhibit just a single reflection coefficient in the product.

Q: How is this even remotely important?

A: This is an important observation when we consider the case where $Z_0$ and $R_L$ are very close in value. If this is true, then the value $\sqrt{Z_0R_L}$ must likewise be close in value to both $Z_0$ and $R_L$.

As a result, we find that the magnitudes of the marginal reflection coefficients $\Gamma$ and $\Gamma_L$ will be very small:

$$|\Gamma| \ll 1$$

while the magnitudes of the marginal transmission coefficient $T$ will be approximately equal to one:

$$|T| \approx 1.0$$
This means that all higher-order terms (those with multiple reflections and thus with products of multiple reflection coefficients) will be very, very small:

\[ |\Gamma|^n \approx 0 \quad \text{for } n \geq 2 \]

The result is that these higher order terms make little difference in the final result—the first two (single reflection) terms are the dominant terms in the infinite series.

Therefore IF \( Z_0 \) and \( R_L \) are very close in value, we find that we can approximate the reflected wave using only the first two terms of the infinite series:

\[
V^{-}(z = -\ell) \approx \Gamma V^{+}(z = -\ell) - T^2 \Gamma_L V^{+}(z = -\ell) \\
= (\Gamma - T^2 \Gamma_L) V^{+}(z = -\ell)
\]

Now, if we likewise apply the approximation that \(|\Gamma| \approx 1.0\), we conclude for this quarter wave transformer (at the design frequency):

\[
V^{-}(z = -\ell) \approx (\Gamma - T^2 \Gamma_L) V^{+}(z = -\ell) \\
= (\Gamma - \Gamma_L) V^{+}(z = -\ell)
\]

This approximation (using only single reflection terms) is known as the Theory of Small Reflections, and allows us to use the multiple reflection viewpoint as an analysis tool (we don't have to consider an infinite number of terms!).
Note that this result is provided from the signal flow graph:

\[
\frac{b_1}{a_1} = \Gamma + (-j)^2 \Gamma_L
\]

\[
= \Gamma - \Gamma_L
\]

However, we can only use this approximation when the marginal reflection coefficient at each transition is very small (i.e., the change in impedance is slight).
The Multi-section Transformer

Consider a sequence of $N$ transmission line sections; each section has equal length $\ell$, but dissimilar characteristic impedances:

Where the marginal reflection coefficients are:

$$
\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0} \quad \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \quad \Gamma_N = \frac{R_L - Z_N}{R_L + Z_N}
$$

If the load resistance $R_L$ is less than $Z_0$, then we should design the transformer such that:

$$Z_0 > Z_1 > Z_2 > Z_3 \ldots > Z_N > R_L$$
Conversely, if \( R_L \) is greater than \( Z_0 \), then we will design the transformer such that:

\[
Z_0 < Z_1 < Z_2 < Z_3 \ldots < Z_N < R_L
\]

In other words, we \textbf{gradually transition} from \( Z_0 \) to \( R_L \)!

Note that since \( R_L \) is \textbf{real}, and since we assume \textbf{lossless} transmission lines, all \( \Gamma_n \) will be \textbf{real} (this is important!).

Likewise, since we \textbf{gradually} transition from one section to another, each value:

\[
Z_{n+1} - Z_n
\]

will be \textbf{small}.

As a result, each marginal reflection coefficient \( \Gamma_n \) will be \textbf{real} and have a \textbf{small} magnitude.

This is also \textbf{important}, as it means that we can apply the "\textbf{theory of small reflections}" to analyze this multi-section transformer!

The theory of small reflections allows us to \textbf{approximate} the input reflection coefficient of the transformer as:
The approximate SFG when applying the theory of small reflections!

\[ \frac{b_0}{a_0} = \Gamma_{in}(\beta) \]
\[ = \Gamma_0 + \Gamma_1 e^{-j2\beta\ell} + \Gamma_2 e^{-j4\beta\ell} + \cdots + \Gamma_N e^{-j2N\beta\ell} \]
\[ = \sum_{n=0}^{N} \Gamma_n e^{-j2n\beta\ell} \]

**Q:** So why is the input reflection coefficient a function of \( \beta \)? Isn’t \( \beta \) a constant?

**A:** Nope. Recall that for a lossless line:

\[ \beta = \omega \sqrt{LC} \]

In other words, beta (and thus the electrical length of each transmission line section) is a function of frequency! Recall:
\[ \beta \ell = \omega \frac{\ell}{v_p} = \omega T \]

where:

\[ T = \frac{\ell}{v_p} = \text{propagation time through 1 section} \]

Therefore, we can alternatively express the input reflection coefficient as a function of frequency:

\[
\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T} \\
= \sum_{n=0}^{N} \Gamma_n e^{-j(2nT)\omega}
\]

We see that the function \( \Gamma_{in}(\omega) \) is expressed as a \textit{weighted} set of \( N \) basis functions! I.E.,

\[ \Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \Psi(\omega) \]

where:

\[ c_n = \Gamma_n \quad \text{and} \quad \Psi(\omega) = e^{-j(2nT)\omega} \]

We find, therefore, that by \textit{selecting} the proper values of basis weights \( c_n \) (i.e., the proper values of reflection coefficients \( \Gamma_n \)), we can \textit{synthesize} any function \( \Gamma_{in}(\omega) \) of frequency \( \omega \), provided that:

1. \( \Gamma_{in}(\omega) \) is \textit{periodic} in \( \omega = 1/2T \)
2. we have sufficient number of sections $N$.

**Q:** What function should we synthesize?

**A:** Ideally, we would want to make $\Gamma_{in}(\omega) = 0$ (i.e., the reflection coefficient is zero for all frequencies).

**Bad news:** this ideal function $\Gamma_{in}(\omega) = 0$ would require an infinite number of sections (i.e., $N = \infty$)!

Instead, we seek to find an “optimal” function for $\Gamma_{in}(\omega)$, given a finite number of $N$ elements.

Once we determine these optimal functions, we can find the values of coefficients $\Gamma_n$ (or equivalently, $Z_n$) that will result in a matching transformer that exhibits this optimal frequency response.

To simplify this process, we can make the transformer symmetrical, such that:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \quad \Gamma_2 = \Gamma_{N-2}, \quad \ldots$$

Note that this does NOT mean that:

$$Z_0 = Z_N, \quad Z_1 = Z_{N-1}, \quad Z_2 = Z_{N-2}, \quad \ldots$$

We find then that:
\[
\Gamma(\omega) = e^{-jN\omega T} \left[ \Gamma_0 \left( e^{jN\omega T} + e^{-jN\omega T} \right) + \Gamma_1 \left( e^{j(N-2)\omega T} + e^{-j(N-2)\omega T} \right) + \Gamma_2 \left( e^{j(N-4)\omega T} + e^{-j(N-4)\omega T} \right) + \ldots \right]
\]

and since:
\[
e^{jx} + e^{-jx} = 2 \cos(x)
\]

we can write for \(N_{\text{even}}\):
\[
\Gamma(\omega) = 2 e^{-jN\omega T} \left[ \Gamma_0 \cosN\omega T + \Gamma_1 \cos(N-2)\omega T + \ldots + \Gamma_n \cos(N-2n)\omega T + \frac{1}{2} \Gamma_{N/2} \right]
\]

whereas for \(N_{\text{odd}}\):
\[
\Gamma(\omega) = 2 e^{-jN\omega T} \left[ \Gamma_0 \cosN\omega T + \Gamma_1 \cos(N-2)\omega T + \ldots + \Gamma_n \cos(N-2n)\omega T + \Gamma_{(N-1)/2} \cos\omega T \right]
\]

The remaining question then is this: given an optimal and realizable function \(\Gamma_{in}(\omega)\), how do we determine the necessary number of sections \(N\), and how do we determine the values of all reflection coefficients \(\Gamma_n\)??