5.5 – The Theory of Small Reflections

Reading Assignment: ***pp. 244-246***

An important and useful **approximation** when considering multi-section matching networks is the **Theory of Small Reflections**.

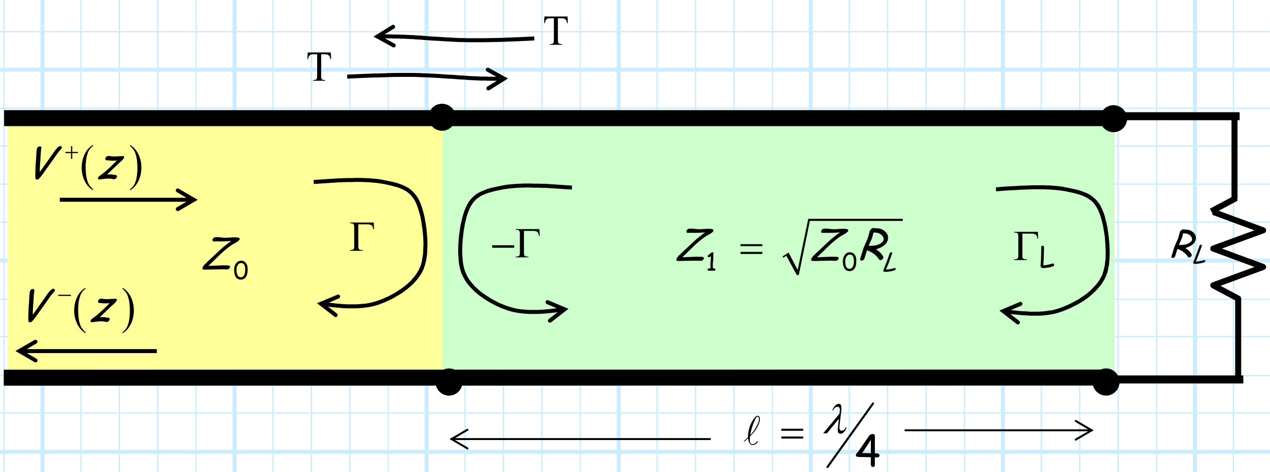
HO: THE THEORY OF SMALL REFLECTIONS

We can use the Theory of Small Reflections to provide an **approximate** analysis of a multi-section impedance transformer (i.e., multi-section matching network).

HO: THE MULTISECTION TRANSFORMER

The Theory of Small Reflections

Recall that we analyzed a **quarter-wave** transformer using the multiple reflection view point.

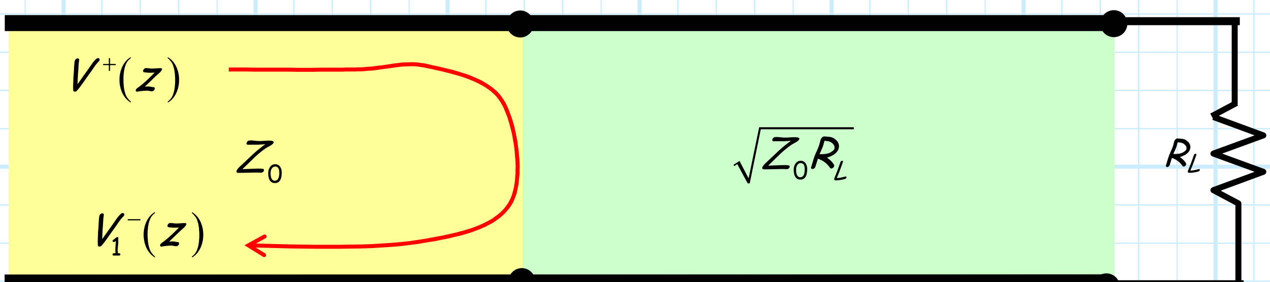


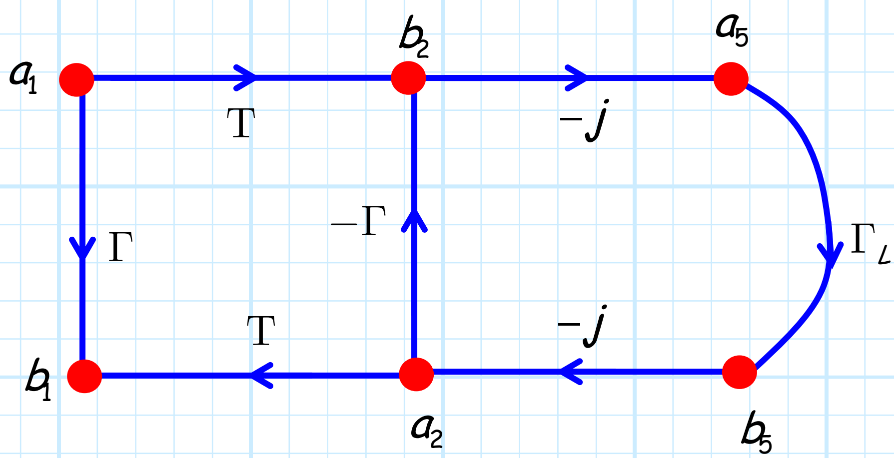
We found that the solution could be written as an **infinite** summation of terms:

$$V^-(z) = \sum_{n=1}^{\infty} V_n^-(z)$$

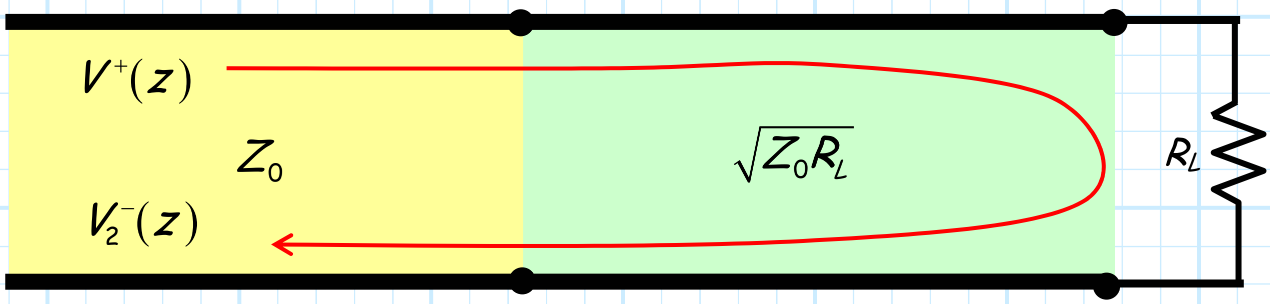
where each term had a specific **physical** interpretation, in terms of reflections, transmissions, and propagations.

For example, the **first** term was $V_1^-(z=-\ell) = \Gamma\, V^+(z=-\ell)$:

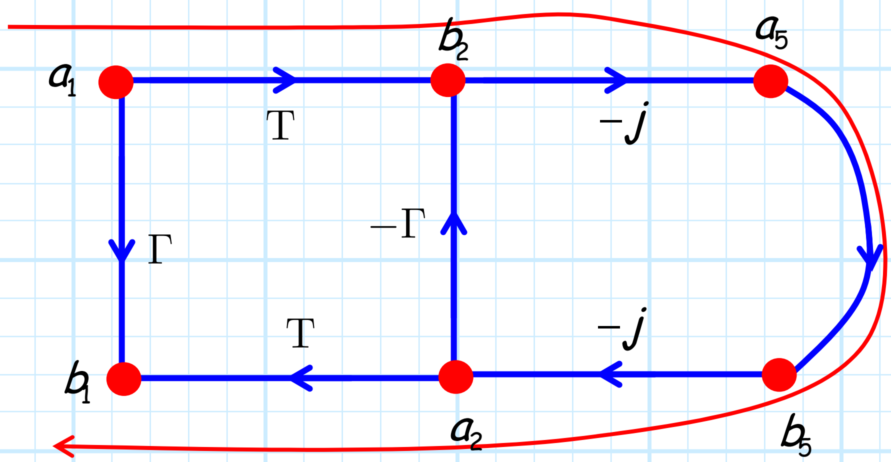




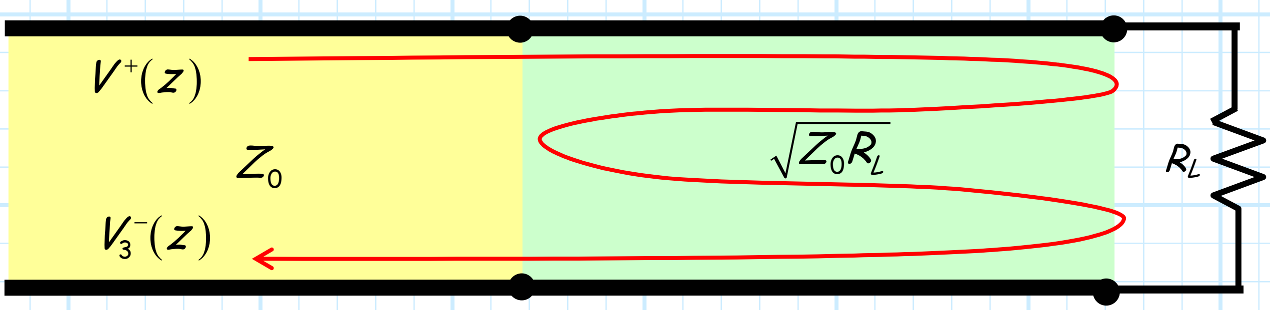
While the **second** term was:



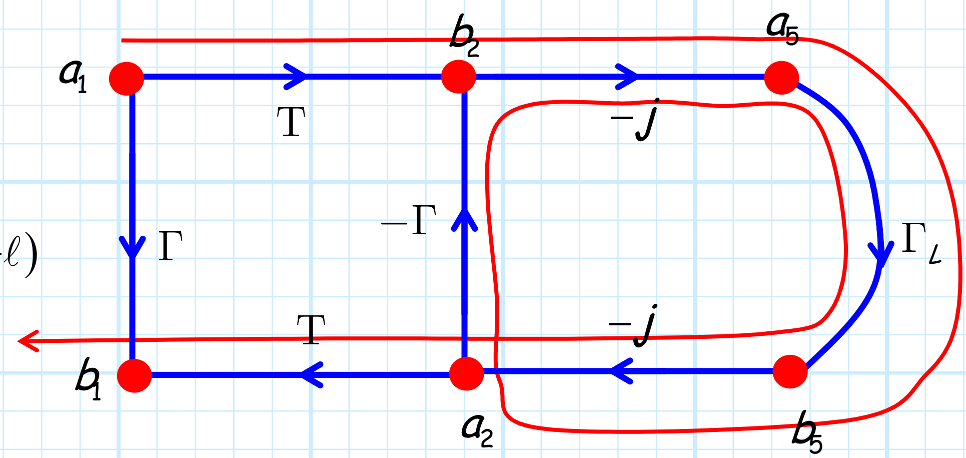
$$V_2^-(z=-\ell) =$$
$$-T^2\,\Gamma_L\,V^+(z=-\ell)$$



Contrast these first two terms with the **third** term:



$$V_3^-(z=-\ell) = -T^2 (\Gamma_L)^2 \Gamma V^+(z=-\ell)$$



This third term is characterized by **three** reflections, and thus there is a **product** of **three** reflection coefficient values Γ in the term. In fact, **all** other terms in this **infinite** series will likewise describe mechanisms with **multiple** reflections.

This is in **contrast** with the first **two** terms, which exhibit just a **single** reflection coefficient in the product.

Q: ***How*** *is this even remotely important?*

A: This is an important observation when we consider the case where Z_0 and R_L are very **close** in value. If this is true, then the value $\sqrt{Z_0 R_L}$ must **likewise** be close in value to both Z_0 and R_L.

As a result, we find that the magnitudes of the marginal **reflection** coefficients Γ and Γ_L will be **very small**:

$$|\Gamma| \ll 1$$

while the magnitudes of the marginal **transmission** coefficient T will be approximately equal to **one**:

$$|T| \simeq 1.0$$

This means that all **higher-order** terms (those with multiple reflections and thus with **products** of multiple reflection coefficients) will be **very, very small**:

$$|\Gamma|^n \approx 0 \quad \text{for } n \geq 2$$

The result is that these higher order terms make **little** difference in the final result—the **first two** (single reflection) terms are the **dominant** terms in the infinite series.

Therefore **IF** Z_0 and R_L are very **close** in value, we find that we can approximate the reflected wave using only the first two terms of the infinite series:

$$\begin{aligned} V^-(z=-\ell) &\simeq \Gamma\, V^+(z=-\ell) - \mathrm{T}^2\,\Gamma_L\, V^+(z=-\ell) \\ &= \left(\Gamma - \mathrm{T}^2\,\Gamma_L\right) V^+(z=-\ell) \end{aligned}$$

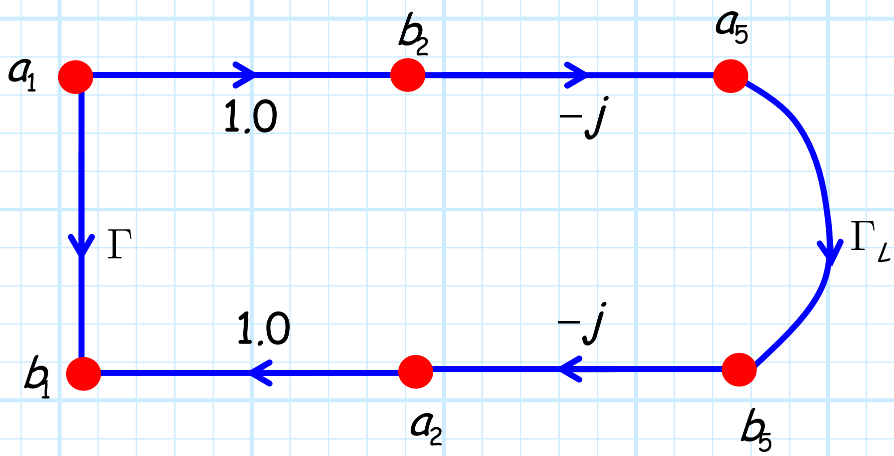
Now, if we likewise apply the approximation that $|\mathrm{T}| \simeq 1.0$, we conclude for this quarter wave transformer (at the design frequency):

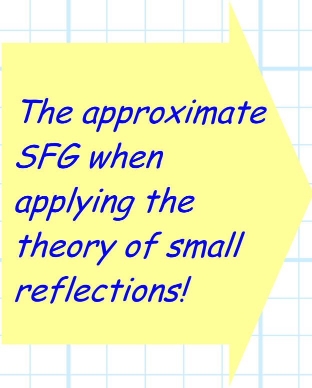
$$\begin{aligned} V^-(z=-\ell) &\simeq \left(\Gamma - \mathrm{T}^2\,\Gamma_L\right) V^+(z=-\ell) \\ &\simeq \left(\Gamma - \Gamma_L\right) V^+(z=-\ell) \end{aligned}$$

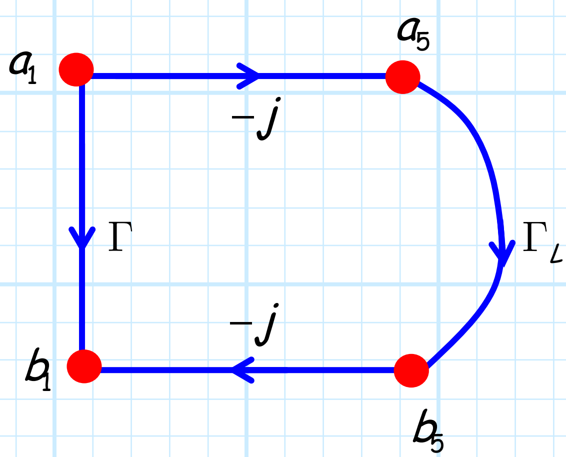
This approximation (using only **single reflection** terms) is known as the **Theory of Small Reflections**, and allows us to use the multiple reflection view point as an **analysis** tool (we don't have to consider an infinite number of terms!).

Note that this result is provided from the signal flow graph:

$$\frac{b_1}{a_1} = \Gamma + (-j)^2\,\Gamma_L$$
$$= \Gamma - \Gamma_L$$



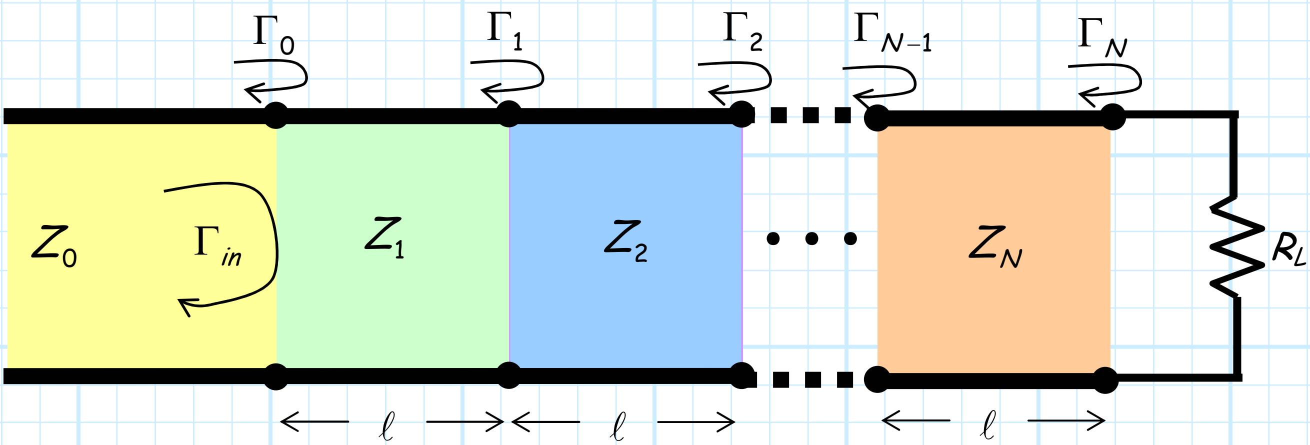




However, we can only use this **approximation** when the marginal reflection coefficient at each transition is **very small** (i.e., the change in impedance is **slight**).

The Multi-section Transformer

Consider a sequence of N transmission line **sections**; each section has **equal length** ℓ, but **dissimilar** characteristic impedances:



Where the marginal reflection coefficients are:

$$\Gamma_0 \doteq \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad \Gamma_n \doteq \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \qquad \Gamma_N \doteq \frac{R_L - Z_N}{R_L + Z_N}$$

If the load resistance R_L is **less** than Z_0, then we should design the transformer such that:

$$Z_0 > Z_1 > Z_2 > Z_3 \cdots > Z_N > R_L$$

Conversely, if R_L is **greater** than Z_0, then we will design the transformer such that:

$$Z_0 < Z_1 < Z_2 < Z_3 \cdots < Z_N < R_L$$

In other words, we **gradually transition** from Z_0 to R_L !

Note that since R_L is **real**, and since we assume **lossless** transmission lines, all Γ_n will be **real** (this is important!).

Likewise, since we **gradually** transition from one section to another, each value:

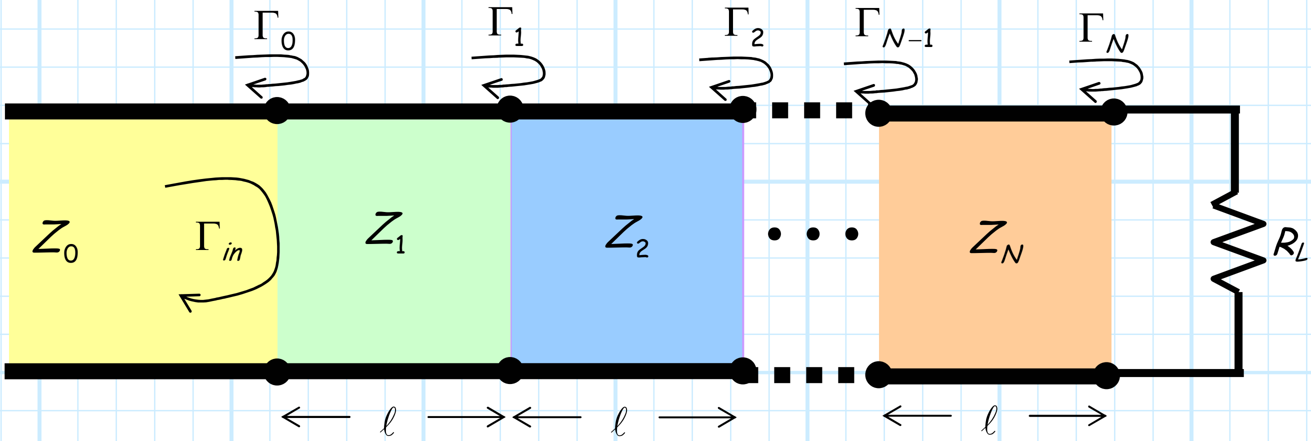
$$Z_{n+1} - Z_n$$

will be **small**.

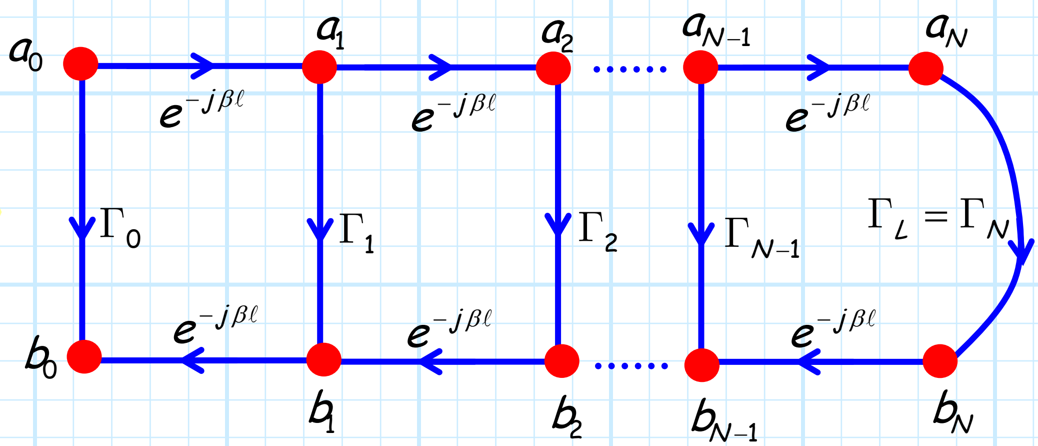
As a result, each marginal reflection coefficient Γ_n will be **real** and have a **small** magnitude.

This is also **important**, as it means that we can apply the "**theory of small reflections**" to analyze this multi-section transformer!

The theory of small reflections allows us to **approximate** the input reflection coefficient of the transformer as:



The approximate SFG when applying the theory of small reflections!



$$
\begin{aligned}
\frac{b_0}{a_0} &= \Gamma_{in}(\beta) \\
&\simeq \Gamma_0 + \Gamma_1\, e^{-j2\beta\ell} + \Gamma_2\, e^{-j4\beta\ell} + \cdots + \Gamma_N\, e^{-j2N\beta\ell} \\
&= \sum_{n=0}^{N} \Gamma_n\, e^{-j2n\beta\ell}
\end{aligned}
$$

Q: *So why is the input reflection coefficient a function of β? Isn't β a constant?*

A: Nope. Recall that for a **lossless** line:

$$\beta = \omega\sqrt{LC}$$

In other words, beta (and thus the electrical length of each transmission line section) is a **function of frequency**! Recall:

$$\beta \ell = \omega \frac{\ell}{v_p} = \omega T$$

where:

$$T \doteq \frac{\ell}{v_p} = \text{propagation time through 1 section}$$

Therefore, we can alternatively express the input reflection coefficient as a function of **frequency**:

$$\begin{aligned}\Gamma_{in}(\omega) &= \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T} \\ &= \sum_{n=0}^{N} \Gamma_n \, e^{-j(2nT)\omega}\end{aligned}$$

We see that the function $\Gamma_{in}(\omega)$ is expressed as a **weighted** set of N **basis functions**! I.E.,

$$\Gamma_{in}(\omega) = \sum_{n=0}^{N} c_n \, \Psi(\omega)$$

where:

$$c_n = \Gamma_n \qquad \textit{and} \qquad \Psi(\omega) = e^{-j(2nT)\omega}$$

We find, therefore, that by **selecting** the proper values of basis weights c_n (i.e., the proper values of reflection coefficients Γ_n), we can **synthesize** any function $\Gamma_{in}(\omega)$ of frequency ω, provided that:

1. $\Gamma_{in}(\omega)$ is **periodic** in $\omega = 1/2T$

2. we have sufficient **number** of sections N.

Q: *What function **should** we synthesize?*

A: **Ideally**, we would want to make $\Gamma_{in}(\omega) = 0$ (i.e., the reflection coefficient is zero for all frequencies).

Bad news: this **ideal** function $\Gamma_{in}(\omega) = 0$ would require an **infinite** number of sections (i.e., $N = \infty$)!

Instead, we seek to find an "**optimal**" function for $\Gamma_{in}(\omega)$, given a **finite** number of N elements.

Once we determine these optimal functions, we can find the values of coefficients Γ_n (or equivalently, Z_n) that will result in a matching transformer that exhibits this **optimal** frequency response.

To **simplify** this process, we can make the transformer **symmetrical**, such that:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \quad \Gamma_2 = \Gamma_{N-2}, \quad \cdots\cdots$$

Note that this **does** NOT mean that:

$$Z_0 = Z_N, \quad Z_1 = Z_{N-1}, \quad Z_2 = Z_{N-2}, \quad \cdots\cdots$$

We find then that:

$$\Gamma(\omega) = e^{-jN\omega T}\Big[\Gamma_0\left(e^{jN\omega T} + e^{-jN\omega T}\right) + \Gamma_1\left(e^{j(N-2)\omega T} + e^{-j(N-2)\omega T}\right) + \Gamma_2\left(e^{j(N-4)\omega T} + e^{-j(N-4)\omega T}\right) + \cdots\Big]$$

and since:

$$e^{jx} + e^{-jx} = 2\cos(x)$$

we can write for N **even**:

$$\Gamma(\omega) = 2\,e^{-jN\omega T}\left[\Gamma_0\cos N\omega T + \Gamma_1\cos(N-2)\omega T + \cdots + \Gamma_n\cos(N-2n)\omega T + \cdots + \frac{1}{2}\Gamma_{N/2}\right]$$

whereas for N **odd**:

$$\Gamma(\omega) = 2\,e^{-jN\omega T}\left[\Gamma_0\cos N\omega T + \Gamma_1\cos(N-2)\omega T + \cdots + \Gamma_n\cos(N-2n)\omega T + \cdots + \Gamma_{(N-1)/2}\cos\omega T\right]$$

The remaining **question** then is this: given an optimal and realizable function $\Gamma_{in}(\omega)$, **how** do we determine the necessary number of **sections** N, and **how** do we determine the **values** of all reflection coefficients Γ_n??