5.6 - Binomial Multi-section Matching Transformer

Reading Assignment: pp. 246-250

One way to maximize bandwidth is to construct a multisection matching network with a function $\Gamma(f)$ that is maximally flat.

Q: Maximally flat? What kind of function is maximally flat?

This function maximizes bandwidth by providing a solution that is maximally flat.

A: **HO: Maximally Flat Functions**

1. We can build a multisection matching network such that the function $\Gamma(f)$ is a binomial function.

2. The binomial function is maximally flat.

Q: Meaning?

A: Meaning the function $\Gamma(f)$ is maximally flat $\Rightarrow$ a wideband solution!

**HO: The Binomial Multi-section Matching Transformer**
Maximally Flat Functions

Consider some function \( f(x) \). Say that we know the value of the function at \( x = 1 \) is 5:

\[
 f(x = 1) = 5
\]

This of course says something about the function \( f(x) \), but it doesn't tell us much!

We can additionally determine the first derivative of this function, and likewise evaluate this derivative at \( x = 1 \). Say that this value turns out to be zero:

\[
 \left. \frac{df(x)}{dx} \right|_{x=1} = 0
\]

Note that this does not mean that the derivative of \( f(x) \) is equal to zero, it merely means that the derivative of \( f(x) \) is zero at the value \( x = 1 \). Presumably, \( df(x)/dx \) is non-zero at other values of \( x \).

So, we now have two pieces of information about the function \( f(x) \). We can add to this list by continuing to take higher-order derivatives and evaluating them at the single point \( x = 1 \).

Let's say that the values of all the derivatives (at \( x = 1 \)) turn out to have a zero value:
\[ \frac{d f^n(x)}{dx^n} \bigg|_{x=1} = 0 \text{ for } n=1,2,3,\ldots,\infty \]

We say that this function is completely flat at the point \(x=1\). Because all the derivatives are zero at \(x=1\), it means that the function cannot change in value from that at \(x=1\).

In other words, if the function has a value of 5 at \(x=1\), (i.e., \(f(x=1)=5\)), then the function must have a value of 5 at all \(x\)!

The function \(f(x)\) thus must be the constant function:

\[ f(x) = 5 \]

Now let's consider the following problem—say some function \(f(x)\) has the following form:

\[ f(x) = ax^3 + bx^2 + cx \]

We wish to determine the values \(a\), \(b\), and \(c\) so that:

\[ f(x=1) = 5 \]

and that the value of the function \(f(x)\) is as close to a value of 5 as possible in the region where \(x=1\).

In other words, we want the function to have the value of 5 at \(x=1\), and to change from that value as slowly as possible as we
“move” from $x=1$.

**Q:** Don’t we simply want the **completely flat function** $f(x) = 5$?

**A:** That would be the **ideal** function for this case, but notice that solution is **not** an option. Note there are no values of $a$, $b$, and $c$ that will make:

$$a x^3 + b x^2 + c x = 5$$

for all values $x$.

**Q:** So what do we do?

**A:** Instead of the completely flat solution, we can find the **maximally flat** solution!

The maximally flat solution comes from determining the values $a$, $b$, and $c$ so that as many derivatives as possible are zero at the point $x=1$.

For example, we wish to make the first derivate equal to zero at $x=1$:

$$0 = \left. \frac{df(x)}{dx} \right|_{x=1} = \left. (3ax^2 + 2bx + c) \right|_{x=1} = 3a + 2b + c$$
Likewise, we wish to make the second derivative equal to zero at $x=1$:

$$0 = \frac{d^2 f(x)}{dx^2} \bigg|_{x=1} = (6ax + 2b) \bigg|_{x=1} = 6a + 2b$$

Here we must **stop** taking derivatives, as our solution only has three degrees of design freedom (i.e., 3 unknowns $a$, $b$, $c$).

**Q:** But we only have taken two derivatives, can’t we take one more?

**A:** No! We already have a third “design” equation: the value of the function must be 5 at $x=1$:

$$5 = f(x=1) = a(1)^3 + b(1)^2 + c(1) = a + b + c$$

So, we have used the **maximally flat** criterion at $x=1$ to generate **three** equations and **three** unknowns:

$$5 = a + b + c$$

$$0 = 3a + 2b + c$$

$$0 = 6a + 2b$$
Solving, we find:

\[ a = 5 \]
\[ b = -15 \]
\[ c = 15 \]

Therefore, the maximally flat function (at \( x = 1 \)) is:

\[ f(x) = 5x^3 - 15x^2 + 15x \]
The Binomial Multi-Section Transformer

Recall that a multi-section matching network can be described using the theory of small reflections as:

$$\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \cdots + \Gamma_N e^{-j2N\omega T}$$

$$= \sum_{n=0}^{N} \Gamma_n e^{-j2n\omega T}$$

where:

$$T = \frac{\ell}{v_p} = \text{propagation time through 1 section}$$

Note that for a multi-section transformer, we have $N$ degrees of design freedom, corresponding to the $N$ characteristic impedance values $Z_n$.

Q: What should the values of $\Gamma_n$ (i.e., $Z_n$) be?

A: We need to define $N$ independent design equations, which we can then use to solve for the $N$ values of characteristic impedance $Z_n$.

First, we start with a single design frequency $\omega_0$, where we wish to achieve a perfect match:
That's just one design equation: we need $N - 1$ more!

These addition equations can be selected using many criteria—one such criterion is to make the function $\Gamma_{in}(\omega)$ maximally flat at the point $\omega = \omega_0$.

To accomplish this, we first consider the Binomial Function:

$$\Gamma(\theta) = A(1 + e^{-j\theta})^N$$

This function has the desirable properties that:

$$\Gamma(\theta = \pi/2) = A(1 + e^{-j\pi})^N = A(1 - 1)^N = 0$$

and that:

$$\frac{d^n \Gamma(\theta)}{d\theta^n} \bigg|_{\theta = \pi/2} = 0 \text{ for } n = 1, 2, 3, \ldots, N - 1$$

In other words, this Binomial Function is maximally flat at the point $\theta = \pi/2$, where it has a value of $\Gamma(\theta = \pi/2) = 0$.

Q: So? What does this have to do with our multi-section matching network?
Let’s expand (multiply out the $N$ identical product terms) of the Binomial Function:

$$\Gamma(\theta) = A (1 + e^{-j2\theta})^N$$

$$= A\left(C_0^N + C_1^N e^{-j2\theta} + C_2^N e^{-j4\theta} + C_3^N e^{-j6\theta} + \ldots + C_N^N e^{-j2N\theta}\right)$$

where:

$$C_n^N = \frac{N!}{(N-n)! n!}$$

Compare this to an $N$-section transformer function:

$$\Gamma_{in}(\omega) = \Gamma_0 + \Gamma_1 e^{-j2\omega T} + \Gamma_2 e^{-j4\omega T} + \ldots + \Gamma_N e^{-j2N\omega T}$$

and it is obvious the two functions have identical forms, provided that:

$$\Gamma_n = A C_n^N \quad \text{and} \quad \omega T = \theta$$

Moreover, we find that this function is very desirable from the standpoint of the a matching network. Recall that $\Gamma(\theta) = 0$ at $\theta = \pi/2$ -- a perfect match!

Additionally, the function is maximally flat at $\theta = \pi/2$, therefore $\Gamma(\theta) \approx 0$ over a wide range around $\theta = \pi/2$ -- a wide bandwidth!
Q: But how does $\theta = \pi/2$ relate to frequency $\omega$?

A: Remember that $\omega T = \theta$, so the value $\theta = \pi/2$ corresponds to the frequency:

$$\omega_0 = \frac{1}{T} \frac{\pi}{2} = \frac{v_p}{\ell} \frac{\pi}{2}$$

This frequency ($\omega_0$) is therefore our design frequency—the frequency where we have a perfect match.

Note that the length $\ell$ has an interesting relationship with this frequency:

$$\ell = \frac{v_p}{\omega_0} \frac{\pi}{2} = \frac{1}{\beta_0} \frac{\pi}{2} = \frac{\lambda_0}{2\pi} = \frac{\lambda_0}{4}$$

In other words, a Binomial Multi-section matching network will have a perfect match at the frequency where the section lengths $\ell$ are a quarter wavelength!

Thus, we have our first design rule:

Set section lengths $\ell$ so that they are a quarter-wavelength ($\lambda_0/4$) at the design frequency $\omega_0$.

Q: I see! And then we select all the values $Z_n$ such that $\Gamma_n = A C_n^N$. But wait! What is the value of $A$??
A: We can determine this value by evaluating a boundary condition!

Specifically, we can easily determine the value of $\Gamma(\omega)$ at $\omega = 0$.

Note as $\omega$ approaches zero, the electrical length $\beta \ell$ of each section will likewise approach zero. Thus, the input impedance $Z_{in}$ will simply be equal to $R_L$ as $\omega \to 0$.

As a result, the input reflection coefficient $\Gamma(\omega = 0)$ must be:

$$\Gamma(\omega = 0) = \frac{Z_{in}(\omega = 0) - Z_0}{Z_{in}(\omega = 0) + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}$$

However, we likewise know that:

$$\Gamma(0) = A \left(1 + e^{-j2(0)}\right)^N = A \left(1 + 1\right)^N = A \cdot 2^N$$
Equating the two expressions:

\[ \Gamma(0) = A 2^N \frac{R_L - Z_0}{R_L + Z_0} \]

And therefore:

\[ A = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0} \quad (A \text{ can be negative!}) \]

We now have a form to calculate the required marginal reflection coefficients \( \Gamma_n \):

\[ \Gamma_n = A C_n^N = \frac{A N!}{(N-n)!n!} \]

Of course, we also know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

\[ \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \]

Equating the two and solving, we find that the section characteristic impedances must satisfy:

\[ Z_{n+1} = Z_n \frac{1 + \Gamma_n}{1 - \Gamma_n} = Z_n \frac{1 + A C_n^N}{1 - A C_n^N} \]
Note this is an iterative result—we determine $Z_i$ from $Z_0$, $Z_2$ from $Z_1$, and so forth.

Q: This result appears to be our second design equation. Is there some reason why you didn't draw a big blue box around it?

A: Alas, there is a big problem with this result.

Note that there are $N+1$ coefficients $\Gamma_n$ (i.e., $n \in \{0, 1, \ldots, N\}$) in the Binomial series, yet there are only $N$ design degrees of freedom (i.e., there are only $N$ transmission line sections!).

Thus, our design is a bit over constrained, a result that manifests itself the finally marginal reflection coefficient $\Gamma_N$.

Note from the iterative solution above, the last transmission line impedance $Z_N$ is selected to satisfy the mathematical requirement of the penultimate reflection coefficient $\Gamma_{N-1}$:

$$\Gamma_{N-1} = \frac{Z_N - Z_{N-1}}{Z_N + Z_{N-1}} = A C_N^{N-1}$$

Thus the last impedance must be:

$$Z_N = Z_{N-1} \frac{1 + A C_N^{N-1}}{1 - A C_N^{N-1}}$$
But there is one more mathematical requirement! The last marginal reflection coefficient must likewise satisfy:

\[ \Gamma_N = A C_N^N = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0} \]

where we have used the fact that \( C_N^N = 1 \).

But, we just selected \( Z_N \) to satisfy the requirement for \( \Gamma_{N-1} \)—we have no physical design parameter to satisfy this last mathematical requirement!

As a result, we find to our great consternation that the last requirement is not satisfied:

\[ \Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq A C_N^N \]

Q: Yikes! Does this mean that the resulting matching network will not have the desired Binomial frequency response?

A: That’s exactly what it means!

Q: You big #%@#$%&!!!! Why did you waste all my time by discussing an over-constrained design problem that can’t be built?

A: Relax; there is a solution to our dilemma—albeit an approximate one.
You undoubtedly have previously used the approximation:

\[
\frac{y - x}{y + x} \approx \frac{1}{2} \ln \left( \frac{y}{x} \right)
\]

An approximation that is especially accurate when \(|y - x|\) is small (i.e., when \(\frac{y}{x} \approx 1\)).

Now, we know that the values of \(Z_{n+1}\) and \(Z_n\) in a multi-section matching network are typically very close, such that \(|Z_{n+1} - Z_n|\) is small. Thus, we use the approximation:

\[
\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2} \ln \left( \frac{Z_{n+1}}{Z_n} \right)
\]
Likewise, we can also apply this approximation (although not as accurately) to the value of $A$:

$$A = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0} \approx 2^{-(N+1)} \ln \left( \frac{R_L}{Z_0} \right)$$

So, let's start over, only this time we'll use these approximations. First, determine $A$:

$$A \approx 2^{-(N+1)} \ln \left( \frac{R_L}{Z_0} \right) \quad (A \text{ can be negative!})$$

Now use this result to calculate the mathematically required marginal reflection coefficients $\Gamma_n$:

$$\Gamma_n = A C_n^N = \frac{A N!}{(N-n)!n!}$$

Of course, we also know that these marginal reflection coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n \approx \frac{1}{2} \ln \left( \frac{Z_{n+1}}{Z_n} \right)$$
Equating the two and solving, we find that the section characteristic impedances must satisfy:

\[ Z_{n+1} = Z_n \exp(2\Gamma_n) \]

Now this is our second design rule. Note it is an iterative rule—we determine \( Z_1 \) from \( Z_0 \), \( Z_2 \) from \( Z_1 \), and so forth.

**Q:** Huh? How is this any better? How does applying approximate math lead to a better design result??

**A:** Applying these approximations help resolve our over-constrained problem. Recall that the over-constraint resulted in:

\[ \Gamma_N = \frac{R_L - Z_N}{R_L + Z_N} \neq AC_N^N \]

But, as it turns out, these approximations leads to the happy situation where:

\[ \Gamma_N \approx \frac{1}{2} \ln \left( \frac{R_L}{Z_N} \right) = AC_N^N \quad \leftarrow \text{Sanity check!!} \]

provided that the value \( A \) is likewise the approximation given above.
Effectively, these approximations couple the results, such that each value of characteristic impedance $Z_n$ approximately satisfies both $\Gamma_n$ and $\Gamma_{n+1}$. Summarizing:

* If you use the “exact” design equations to determine the characteristic impedances $Z_n$, the last value $\Gamma_N$ will exhibit a significant numeric error, and your design will not appear to be maximally flat.

* If you instead use the “approximate” design equations to determine the characteristic impedances $Z_n$, all values $\Gamma_n$ will exhibit a slight error, but the resulting design will appear to be maximally flat, Binomial reflection coefficient function $\Gamma(\omega)$!

Figure 5.15 (p. 250)
Reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 5.6 $Z_L = 50\Omega$ and $Z_0 = 100\Omega$. 
Note that as we increase the number of sections, the matching bandwidth increases.

Q: Can we determine the value of this bandwidth?

A: Sure! But we first must define what we mean by bandwidth.

As we move from the design (perfect match) frequency \( f_0 \) the value \( |\Gamma(f)| \) will increase. At some frequency \( (f_m, \text{say}) \) the magnitude of the reflection coefficient will increase to some unacceptably high value \( (\Gamma_m, \text{say}) \). At that point, we no longer consider the device to be matched.

Note there are two values of frequency \( f_m \) —one value less than design frequency \( f_0 \), and one value greater than design frequency \( f_0 \). These two values define the bandwidth \( \Delta f \) of the matching network:

\[
\Delta f = f_{m2} - f_{m1} = 2(f_0 - f_{m1}) = 2(f_{m2} - f_0)
\]
Q: So what is the numerical value of $\Gamma_m$?

A: I don’t know—it’s up to you to decide!

Every engineer must determine what they consider to be an acceptable match (i.e., decide $\Gamma_m$). This decision depends on the application involved, and the specifications of the overall microwave system being designed.

However, we typically set $\Gamma_m$ to be 0.2 or less.

Q: OK, after we have selected $\Gamma_m$, can we determine the two frequencies $f_m$?

A: Sure! We just have to do a little algebra.

We start by rewriting the Binomial function:

$$\Gamma(\theta) = A \left( 1 + e^{-j2\theta} \right)^N$$

$$= Ae^{-jN\theta} \left( e^{+j\theta} + e^{-j\theta} \right)^N$$

$$= Ae^{-jN\theta} \left( e^{+j\theta} + e^{-j\theta} \right)^N$$

$$= Ae^{-jN\theta} (2 \cos \theta)^N$$

Now, we take the magnitude of this function:

$$|\Gamma(\theta)| = 2^N |A| |e^{-jN\theta}| |\cos \theta|^N$$

$$= 2^N |A| |\cos \theta|^N$$
Now, we define the values $\theta$ where $|\Gamma(\theta)| = \Gamma_m$ as $\theta_m$. I.E.,

$$\Gamma_m = |\Gamma(\theta = \theta_m)| = 2^N |A| \cos \theta_m^N$$

We can now solve for $\theta_m$ (in radians!) in terms of $\Gamma_m$:

$$\theta_{m1} = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right] \quad \theta_{m2} = \cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Note that there are two solutions to the above equation (one less than $\pi/2$ and one greater than $\pi/2$)!

Now, we can convert the values of $\theta_m$ into specific frequencies.

Recall that $\omega T = \theta$, therefore:

$$\omega_m = \frac{1}{T} \theta_m = \frac{v_p}{\ell} \theta_m$$

But recall also that $\ell = \lambda_0/4$, where $\lambda_0$ is the wavelength at the design frequency $f_0$ (not $f_m$!), and where $\lambda_0 = v_p/f_0$.

Thus we can conclude:

$$\omega_m = \frac{v_p}{\ell} \theta_m = \frac{4v_p}{\lambda_0} \theta_m = (4f_0) \theta_m$$
or:

\[ f_m = \frac{1}{2\pi} \nu_p \theta_m = \frac{(4f_0) \theta_m}{2\pi} = \frac{(2f_0) \theta_m}{\pi} \]

where \( \theta_m \) is expressed in radians. Therefore:

\[
\begin{align*}
    f_{m1} &= \frac{2f_0}{\pi} \cos^{-1} \left[ 1 + \frac{1}{2} \left( \frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right] \\
    f_{m2} &= \frac{2f_0}{\pi} \cos^{-1} \left[ -\frac{1}{2} \left( \frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right]
\end{align*}
\]

Thus, the **bandwidth** of the binomial matching network can be determined as:

\[
\Delta f = 2(f_0 - f_{m1}) = 2f_0 - \frac{4f_0}{\pi} \cos^{-1} \left[ 1 + \frac{1}{2} \left( \frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right]
\]

Note that this equation can be used to determine the **bandwidth** of a binomial matching network, given \( \Gamma_m \) and number of sections \( N \).

However, it can likewise be used to determine the **number of sections** \( N \) required to meet a specific bandwidth requirement!
Finally, we can list the **design steps** for a binomial matching network:

1. **Determine** the value $N$ required to meet the bandwidth ($\Delta f$ and $\Gamma_m$) requirements.

2. Determine the approximate value $A$ from $Z_0, R_L$ and $N$.

3. Determine the **marginal reflection coefficients** $\Gamma_n = AC_n^N$ required by the **binomial** function.

4. Determine the characteristic impedance of each section using the **iterative approximation**:

   $$Z_{n+1} = Z_n \exp\left[2\Gamma_n\right]$$

5. Perform the **sanity check**:

   $$\Gamma_N \approx \frac{1}{2} \ln \left( \frac{R_L}{Z_N} \right) = AC_N^N$$

6. Determine section **length** $\ell = \lambda_0/4$ for design frequency $f_0$. 