<u>5.7 – Chebyshev Multi-section Matching Transformer</u>

Reading Assignment: ***pp. 250-255***

We can also build a multisection matching network such that the function $\Gamma(f)$ is a **Chebyshev** function.

Chebyshev functions **maximize bandwidth**, albeit at the cost of **pass-band ripple**.

<u>HO: The Chebyshev Multi-section Matching Transformer</u>

The Chebyshev Matching Transformer

An **alternative** to Binomial (Maximally Flat) functions (and there are **many** such alternatives!) are **Chebyshev** polynomials.

Chebyshev solutions can provide functions $\Gamma(\omega)$ with **wider bandwidth** than the Binomial case—albeit at the "expense" of **passband ripple**.

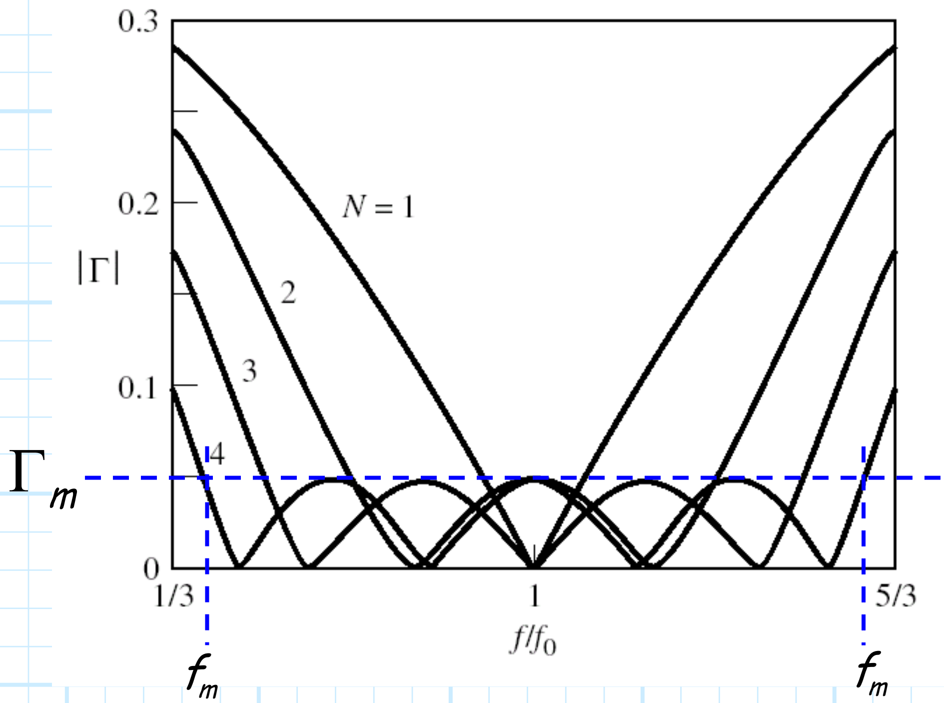

Figure 5.17 (p. 255)
Reflection coefficient magnitude versus frequency for the Chebyshev multisection matching transformers of Example 5.7.

It is evident from the plot above that the Chebychev response is **far** from maximally **flat**! Instead, a Chebyshev matching network exhibits a "**ripple**" in its passband. Note the magnitude of this ripple never exceeds some **maximum** value Γ_m (within the **pass-band**).

The two frequencies at which the value $|\Gamma(\omega)|$ **does** increase beyond Γ_m define the **bandwidth** of the matching network. We denote these frequencies ω_m (the plot above shows the locations of the frequencies for ***N* =4**).

Chebyshev transformers are **symmetric**, i.e.:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N-1}, \text{ etc.}$$

Recall we can express the multi-section function $\Gamma(\theta)$ (where $\theta = \omega T = \beta\ell$) in a **simpler form** when the transformer is symmetric:

$$\Gamma(\theta) = 2\, e^{-jN\theta}\Big[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + G(\theta)\Big]$$

where:

$$G(\theta) = \begin{cases} \dfrac{1}{2}\Gamma_{N/2} & \text{for } N \text{ even} \\ \Gamma_{(N-1)/2}\cos\theta & \text{for } N \text{ odd} \end{cases}$$

Now, the reflection coefficient of a **Chebyshev** matching network has the form:

$$\begin{aligned}\Gamma(\theta) &= A\, e^{-jN\theta}\, T_N\left(\frac{\cos\theta}{\cos\theta_m}\right)\\ &= A\, e^{-jN\theta}\, T_N\left(\cos\theta \sec\theta_m\right)\end{aligned}$$

where $\theta_m = \omega_m T$

The function $T_N\left(\cos\theta \sec\theta_m\right)$ is a **Chebyshev polynomial** of order N.

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

We can determine **higher**-order Chebyshev polynomials using the **recursive formula**:

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

Inserting the **substitution**:

$$x = \cos\theta \sec\theta_m$$

into the Chebyshev polynomials above (and then applying a few **trig identities**) gives the results shown in equations 5.60 on page 252 of **your** book:

$$T_1(\cos\theta \sec\theta_m) = \cos\theta \sec\theta_m$$

$$\begin{aligned} T_2(\cos\theta \sec\theta_m) &= \sec^2\theta_m (1+\cos 2\theta) - 1 \\ &= \sec^2\theta_m \cos 2\theta + (\sec^2\theta_m - 1) \end{aligned}$$

$$\begin{aligned} T_3(\cos\theta \sec\theta_m) &= \sec^3\theta_m (\cos 3\theta + 3\cos\theta) - 3\sec\theta_m \cos\theta \\ &= \sec^3\theta_m \cos 3\theta + (3\sec^2\theta_m - 3)\sec\theta_m \cos\theta \end{aligned}$$

$$\begin{aligned} T_4(\cos\theta \sec\theta_m) &= \sec^4\theta_m (\cos 4\theta + 4\cos 2\theta + 3) \\ &\quad -4\sec^2\theta_m (\cos 2\theta + 1) + 1 \\ &= \sec^4\theta_m \cos 4\theta \\ &\quad + 4\sec^2\theta_m (\sec^2\theta_m - 1)\cos 2\theta \\ &\quad + (3\sec^4\theta_m - 4\sec^2\theta_m + 1) \end{aligned}$$

Note that these polynomials have a $\cos N\theta$ term, a $\cos(N-2)\theta$ term, $\cos(N-4)\theta$ term, etc.—**just** like the **symmetric** multi-section transformer funtion!

For example, a 4-section **Chebyshev** matching network will have the form:

$$\begin{aligned} \Gamma_4(\theta) &= A e^{-j4\theta} T_4(\cos\theta \sec\theta_m) \\ &= A e^{-j4\theta} \big[\sec^4\theta_m \cos 4\theta \\ &\quad + 4\sec^2\theta_m (\sec^2\theta_m - 1)\cos 2\theta \\ &\quad + (3\sec^4\theta_m - 4\sec^2\theta_m + 1) \big] \end{aligned}$$

While the **general form** of a 4-section matching transformer is a polynomial with these **same** terms:

$$\Gamma_4(\theta) = 2\, e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \frac{1}{2}\Gamma_2\right]\Big|_{N=4}$$

$$= 2\, e^{-j4\theta}\left[\Gamma_0 \cos 4\theta + \Gamma_1 \cos(4-2)\theta + \frac{1}{2}\Gamma_2\right]$$

$$= 2\, e^{-j4\theta}\left[\Gamma_0 \cos 4\theta + \Gamma_1 \cos 2\theta + \frac{1}{2}\Gamma_2\right]$$

Thus, we can **determine** the values of marginal reflection coefficients $\Gamma_0, \Gamma_1, \Gamma_2$ by simply **equating** the **3 terms** of the two previous expressions:

$$2\, e^{-j4\theta}\Gamma_0 \cos 4\theta = A\, e^{-j4\theta} \sec^4\theta_m \cos 4\theta$$

$$\Gamma_0 = \frac{1}{2} A \sec^4\theta_m$$

$$2\, e^{-j4\theta}\Gamma_1 \cos 2\theta = A\, e^{-j4\theta}\, 4\sec^2\theta_m\left(\sec^2\theta_m - 1\right)\cos 2\theta$$

$$\Gamma_1 = A\, 2\sec^2\theta_m\left(\sec^2\theta_m - 1\right)$$

$$2\, e^{-j4\theta}\frac{1}{2}\Gamma_2 = A\, e^{-j4\theta}\left(3\sec^4\theta_m - 4\sec^2\theta_m + 1\right)$$

$$\Gamma_2 = A\left(3\sec^4\theta_m - 4\sec^2\theta_m + 1\right)$$

And because it's **symmetric**, we also know that $\Gamma_3 = \Gamma_1$ and $\Gamma_4 = \Gamma_0$.

Now, we can **again** determine the values of characteristic impedance Z_n, using the **iterative** (approximate) relationship:

$$\Gamma_n = \frac{1}{2} ln \frac{Z_{n+1}}{Z_n}$$

Q: *But what about the value of **A** ?*

A: Using the same **boundary condition** analysis that we used for the binomial function, we find from our **transmission line** knowledge that for **any** multi-section matching network, at $\theta = 0$:

$$\Gamma(\theta = 0) = \frac{R_L - Z_0}{R_L + Z_0}$$

Likewise, a **Chebyshev** matching network will have the specific value at $\theta = 0$ of:

$$\begin{aligned}\Gamma(\theta = 0) &= A\, e^{-jN(0)}\, T_N\left(sec\,\theta_m\, cos(0)\right) \\ &= A\, T_N\left(sec\,\theta_m\right)\end{aligned}$$

These two results must of course be **equal**, and equating them allows us to solve for A :

$$A = \frac{R_L - Z_0}{R_L + Z_0} \frac{1}{T_N\left(sec\,\theta_m\right)}$$

Q: *OK, so $\theta_m = \omega_m T$ defines the **bandwidth** of the Chebychev matching network. What about the **ripple value** Γ_m ? How can we **determine** what its value is? Can we **specify** it as a design parameter?*

A: So **many** questions! There is a definite **relationship** between bandwidth θ_m and ripple magnitude Γ_m. In fact, we will find that the two values represent a **design trade**: **increase** the matching network bandwidth (good!), and you will likewise **increase** the ripple value Γ_m (bad!); **decrease** Γ_m(good!), and you will **decrease** the bandwidth(bad!)!

Typically, the ripple magnitude Γ_m is **specified** by the designer, and then the resulting bandwidth frequencies θ_m are **calculated**.

Q: *What happens if the **bandwidth** is too **small**; are we **forced** to increase ripple Γ_m??*

A: Not necessarily! We have **one** other option—**increase** the number of matching **sections** N. This can increase bandwidth **without** modifying ripple magnitude Γ_m.

Now, let's **determine** explicitly the relationship between θ_m and Γ_m. We know that the magnitude of the reflection coefficient is, by **definition**, equal to Γ_m at $\theta = \theta_m$:

$$
\begin{aligned}
\Gamma_m &= \left|\Gamma\left(\theta = \theta_m\right)\right| \\
&= \left|A\, e^{-jN\theta_m}\, T_N\left(\sec\theta_m \cos\theta_m\right)\right| \\
&= \left|A\right| \left|e^{-jN\theta_m}\right| \left|T_N\left(\sec\theta_m \cos\theta_m\right)\right| \\
&= \left|A\right| \left|T_N\left(1\right)\right|
\end{aligned}
$$

One **property** of Chebyshev polynomials is that:

$$\left|T_N\left(1\right)\right| = 1 \quad \text{for **all** values } N$$

Therefore, we **conclude**:

$$\Gamma_m = \left|\Gamma\left(\theta = \theta_m\right)\right| = \left|A\right|$$

But **recall**:

$$\left|A\right| = \left|\frac{R_L - Z_0}{R_L + Z_0}\right| \frac{1}{\left|T_N\left(\sec\theta_m\right)\right|}$$

And **thus**:

$$
\begin{aligned}
\left|T_N\left(\sec\theta_m\right)\right| &= \left|\frac{R_L - Z_0}{R_L + Z_0}\right| \frac{1}{\left|A\right|} \\
&= \left|\frac{R_L - Z_0}{R_L + Z_0}\right| \frac{1}{\Gamma_m}
\end{aligned}
$$

Look at the above expression! It **explicitly** relates the values θ_m and Γ_m.

We can **approximate** this relationship as:

$$\left|T_N\left(\sec\theta_m\right)\right| = \frac{1}{\Gamma_m}\left|\frac{R_L - Z_0}{R_L + Z_0}\right| \approx \frac{1}{2\Gamma_m}\left|\ln\frac{R_L}{Z_0}\right|$$

Making **sense** of the above relationship now requires a little "**hand-waving**"!

It can be shown that (← a phrase professors use while in hand-waving mode!) the general form of the Chebyshev polynomial can **also** be written as:

$$T_N\left(\sec\theta_m \cos\theta\right) = \cosh\left[N\cosh^{-1}\left(\sec\theta_m \cos\theta\right)\right]$$

for all values θ **outside the passband** of the matching network.

Note the value $\theta = 0$ is most definitely **outside** the passband, and thus according to the **above** expression:

$$\begin{aligned} T_N\left(\sec\theta_m \cos\left(\theta=0\right)\right) &= T_N\left(\sec\theta_m\right) \\ &= \cosh\left[N\cosh^{-1}\left(\sec\theta_m\right)\right] \end{aligned}$$

Inserting this result into the expression on the **top** of this page, we find:

$$\left|T_N\left(\sec\theta_m\right)\right| = \frac{1}{2\Gamma_m}\left|\ln\frac{R_L}{Z_0}\right| = \left|\cosh\left[N\cosh^{-1}\left(\sec\theta_m\right)\right]\right|$$

Thus, for a given value of θ_m (as well as N), we find that:

$$\Gamma_m = \frac{1}{2}\left|ln\frac{R_L}{Z_0}\right| \frac{1}{\left|cosh\left[N\,cosh^{-1}\left(sec\,\theta_m\right)\right]\right|}$$

Or, we can rearrange this equation and determine θ_m for a specified value of Γ_m (and N):

$$sec\,\theta_m = \pm cosh\left[\frac{1}{N}cosh^{-1}\left(\frac{1}{2\Gamma_m}\left|ln\frac{R_L}{Z_0}\right|\right)\right]$$

Note that there are **two** solutions θ_m for this equation—**one** value of θ_m will be **less** than $\pi/2$ (defining the **lower** passband frequency), while the **other** will be **greater** than $\pi/2$ (defining the **upper** passband frequency).

Moreover, we find that the **two** values of θ_m will be **symmetric** about the value $\pi/2$! For example, if the **lower** value of θ_m is $\pi/2 - \pi/10$, then the **upper** value of θ_m will be $\pi/2 + \pi/10$.

Q: *So??*

A: This means that the **center** of the passband will be defined by the value $\theta = \pi/2$—and the center of the passband is our **design frequency** ω_0! In other words, since $\omega_0 T = \pi/2$:

$$\omega_0 = \frac{\pi}{2}\frac{1}{T} = \frac{\pi}{2}\frac{v_p}{\ell}$$

Thus, we set the center (i.e., design) frequency by **selecting** the proper value of **section length** ℓ. Note the above expression is **precisely** the same result obtained for the **Binomial** matching network, and thus we have precisely the **same** design rule!

That **design rule** is, set the section lengths ℓ such that they are a **quarter wavelength** at the **design frequency** ω_0:

$$\ell = \frac{\lambda_0}{4}$$

where $\lambda_0 = v_p/\omega_0$.

Summarizing, the Chebyshev matching network design **procedure** is:

1. **Determine** the value N required to meet the bandwidth and ripple Γ_m requirements.

2. Determine the **Chebychev function** $\Gamma(\theta) = A\, e^{-jN\theta}\, T_N\left(\cos\theta \sec\theta_m\right)$.

3. Determine all Γ_n by **equating terms** with the symmetric multisection transformer expression:

$$\Gamma(\theta) = 2\, e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + G(\theta)\right]$$

4. **Calculate** all Z_n using the approximation:

$$\Gamma_n = \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n}$$

5. Determine section **length** $\ell = \lambda_0/4$.