Chapter - 7 Power Dividers and Couplers

One of the most fundamental problems in microwave engineering is how to efficiently divide signal power.

The simplest microwave problem would seemingly be to equally divide signal power in two:

However, building these ideal 3-port devices is more difficult than you might think!
7.1 - Basic Properties of Dividers and Couplers

Reading Assignment: pp. 308-315

Ideally, a coupler would be matched, lossless, and reciprocal. Let's see if we can build such a 3-port device.

**HO: THE 3-PORT COUPLER**

**Q:** Is a matched, lossless, reciprocal 4-port coupler impossible as well?

**A:** **HO: THE 4-PORT COUPLER**

Four-port networks are called directional couplers, and have some fundamental values that characterize them.

**HO: THE DIRECTIONAL COUPLER**
The 3-Port Coupler

Say we desire a matched and lossless 3-port Coupler. Such a device would have a scattering matrix:

\[ S = \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{bmatrix} \]

Assuming the device is passive and made of simple (isotropic) materials, the device will be reciprocal, so that:

\[ S_{21} = S_{12}, \quad S_{31} = S_{13}, \quad S_{23} = S_{32} \]

Likewise, if it is matched, we know that:

\[ S_{11} = S_{22} = S_{33} = 0 \]

As a result, a lossless, reciprocal coupler would have a scattering matrix of the form:

\[ S = \begin{bmatrix}
0 & S_{21} & S_{31} \\
S_{21} & 0 & S_{32} \\
S_{31} & S_{32} & 0
\end{bmatrix} \]

Just 3 non-zero scattering parameters define the entire matrix!
Likewise, if we wish for this coupler to be lossless, the scattering matrix must be unitary, and therefore:

\[
|S_{21}|^2 + |S_{31}|^2 = 1 \quad S_{31}^* S_{32} = 0 \\
|S_{21}|^2 + |S_{32}|^2 = 1 \quad S_{21}^* S_{32} = 0 \\
|S_{31}|^2 + |S_{32}|^2 = 1 \quad S_{21}^* S_{31} = 0
\]

Since each complex value \( S \) is represented by two real numbers (i.e., real and imaginary parts), the equations above result in 9 real equations. The problem is, the 3 complex values \( S_{21}, S_{31}, \) and \( S_{32} \) are represented by only 6 real unknowns.

We have over constrained our problem! There are no solutions to these equations!

As unlikely as it might seem, this means that a matched, lossless, reciprocal 3-port device of any kind is a physical impossibility!
For example, the following 3 port coupler is lossless, but not matched:

\[ S = \begin{bmatrix}
-\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}
\end{bmatrix} \]

Since:

\[ S_{11} = S_{22} = S_{33} = -\frac{1}{3} \neq 0 \]

the coupler is not matched! However, the matrix is unitary, and therefore this design is lossless.

Alternatively, we might try this 3-port device:

\[ S = \begin{bmatrix}
0 & \frac{3}{5} & \frac{3}{5} \\
\frac{3}{5} & 0 & \frac{3}{5} \\
\frac{3}{5} & \frac{3}{5} & 0
\end{bmatrix} \]

\[ R = \frac{Z_0}{3} \]
For this design, the ports are matched! However, the resistors make the device lossy:

\[ |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 0 + \frac{9}{25} + \frac{9}{25} = \frac{18}{25} < 1 \]

Oh sure, maybe you can make a lossless reciprocal 3-port coupler, or a matched reciprocal 3-port coupler, or even a matched, lossless (but non-reciprocal) 3-port coupler. But try as you might, you **cannot** make a lossless, matched, and reciprocal three port coupler!
The 4-Port Coupler

Guess what! I have determined that—unlike a 3-port device—a matched, lossless, reciprocal 4-port device is physically possible! In fact, I’ve found two general solutions!

The first solution is referred to as the symmetric solution:

\[
S = \begin{bmatrix}
0 & \alpha & j\beta & 0 \\
\alpha & 0 & 0 & j\beta \\
j\beta & 0 & 0 & \alpha \\
0 & j\beta & \alpha & 0
\end{bmatrix}
\]

Note for this symmetric solution, every row and every column of the scattering matrix has the same four values (i.e., \(\alpha\), \(j\beta\), and two zeros)!

The second solution is referred to as the anti-symmetric solution:

\[
S = \begin{bmatrix}
0 & \alpha & \beta & 0 \\
\alpha & 0 & 0 & -\beta \\
\beta & 0 & 0 & \alpha \\
0 & -\beta & \alpha & 0
\end{bmatrix}
\]
Note that for this anti-symmetric solution, two rows and two columns have the same four values (i.e., $\alpha$, $\beta$, and two zeros), while the other two row and columns have (slightly) different values ($\alpha$, $-\beta$, and two zeros).

It is quite evident that each of these solutions are matched and reciprocal. However, to ensure that the solutions are indeed lossless, we must place an additional constraint on the values of $\alpha$, $\beta$. Recall that a necessary condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n$$

Applying this to the symmetric case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

Likewise, for the anti-symmetric case, we also get

$$|\alpha|^2 + |\beta|^2 = 1$$

It is evident that if the scattering matrix is unitary (i.e., lossless), the values $\alpha$ and $\beta$ cannot be independent, but must related as:

$$|\alpha|^2 + |\beta|^2 = 1$$
Generally speaking, we will find that \(|\alpha| \geq |\beta|\). Given the constraint on these two values, we can thus conclude that:

\[ 0 \leq |\beta| \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} \leq |\alpha| \leq 1 \]
A lossless, reciprocal, matched 4-port directional coupler will have a scattering matrix of the form:

\[
S = \begin{bmatrix}
0 & \alpha & j\beta & 0 \\
\alpha & 0 & 0 & j\beta \\
j\beta & 0 & 0 & \alpha \\
0 & j\beta & \alpha & 0
\end{bmatrix}
\]

This ideal coupler is completely characterized by the coupling coefficient \(c\), where we find:

\[
\begin{bmatrix}
0 & \sqrt{1-c^2} & jc & 0 \\
\sqrt{1-c^2} & 0 & 0 & jc \\
jc & 0 & 0 & \sqrt{1-c^2} \\
0 & jcc & \sqrt{1-c^2} & 0
\end{bmatrix}
\]

In other words:

\[
\beta = c \quad \text{and} \quad \alpha = \sqrt{1 - \beta^2} = \sqrt{1 - c^2}
\]

Additionally, for a directional coupler, the coupling coefficient \(c\) will be less than \(1/\sqrt{2}\) always. Therefore, we find that:

\[
0 \leq c \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} \leq \sqrt{1 - c^2} \leq 1
\]
Let's see what this means in terms of the **physical behavior** of a directional coupler. First, consider the case where some signal is incident on **port 1**, with power $P_1^+$. 

If all other ports are matched, we find that the power flowing out of **port 1** is:

$$P_1^- = |S_{11}|^2 P_1^+ = 0$$

while the power out of **port 2** is:

$$P_2^- = |S_{21}|^2 P_1^+ = (1 - c^2) P_1^+$$

and the power out of **port 3** is:

$$P_3^- = |S_{31}|^2 P_1^+ = c^2 P_1^+$$

Finally, we find there is **no power** flowing out of **port 4**:

$$P_4^- = |S_{41}|^2 P_1^+ = 0$$

In the terminology of the directional coupler, we say that port 1 is the **input** port, port 2 is the **through** port, port 3 is the **coupled** port, and port 4 is the **isolation** port.
Note however, that any of the coupler ports can be an input, with a different through, coupled and isolation port for each case.

For example, if a signal is incident on port 2, while all other ports are matched, we find that:

Thus, from the scattering matrix of a directional coupler, we can form the following table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Through</th>
<th>Coupled</th>
<th>Isolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Port 1</td>
<td>Port 2</td>
<td>Port 3</td>
<td>Port 4</td>
</tr>
<tr>
<td>Port 2</td>
<td>Port 1</td>
<td>Port 4</td>
<td>Port 3</td>
</tr>
<tr>
<td>Port 3</td>
<td>Port 4</td>
<td>Port 1</td>
<td>Port 2</td>
</tr>
<tr>
<td>Port 4</td>
<td>Port 3</td>
<td>Port 2</td>
<td>Port 1</td>
</tr>
</tbody>
</table>

Typically, the coupling coefficients for a directional coupler are in the range of approximately:

\[ 0.25 > c^2 > 0.0001 \]
As a result, we find that $\sqrt{1 - c^2} \approx 1$. What this means is that the power out of the **through** port is just **slightly smaller** (typically) than the power incident on the input port.

Likewise, the power out of the **coupling** port is typically a **small fraction** of the power incident on the input port.

**A:** A directional coupler is often used for **sampling** a small portion of the signal power. For example, we might **measure** the output power of the **coupled** port (e.g., $P_{3^-}$) and then we can determine the amount of signal power flowing through the device (e.g., $P_{1^+} = P_{3^-}/c^2$)

Unfortunately, the **ideal** directional coupler **cannot** be built! For example, the input match is never **perfect**, so that the diagonal elements of the scattering matrix, although **very small**, are not zero.

**Q:** Pfft! Just a **small fraction** of the input power! What is the use in doing that??
Likewise, the isolation port is never perfectly isolated, so that the values $S_{41}$, $S_{32}$, $S_{23}$ and $S_{14}$ are also non-zero—some small amount of power leaks out!

As a result, the through port will be slightly less than the value $\sqrt{1-c^2}$. The scattering matrix for a non-ideal coupler would therefore be:

$$
S = \begin{bmatrix}
S_{11} & S_{21} & jc & S_{41} \\
S_{21} & S_{11} & S_{41} & jc \\
jc & S_{41} & S_{11} & S_{21} \\
S_{41} & jc & S_{21} & S_{11}
\end{bmatrix}
$$

From this scattering matrix, we can extract some important parameters about directional couplers:

**Coupling C**

The coupling value is the ratio of the coupled output power to the input power, in dB:

$$
C \text{ (dB)} = 10 \log_{10} \left( \frac{P^+}{P^-} \right) = -10 \log_{10} |jc|^2
$$

This is the primary specification of a directional coupler!
**Directivity $D$**

The directivity is the ratio of the power out of the coupling port to the power out of the isolation port, in dB. This value indicates how effective the device is in "directing" the coupled energy into the correct port. The higher the directivity, the better.

$$D (dB) = 10 \log_{10} \left( \frac{P_3^-}{P_4^-} \right) = 10 \log_{10} \left( \frac{|jc|^2}{|S_{41}|^2} \right)$$

**Isolation $I$**

Isolation is the ratio of the input power to the power out of the isolation port, in dB. This value indicates how "isolated" the isolation port actually is. The higher the isolation, the better.

$$I (dB) = 10 \log_{10} \left( \frac{P_1^+}{P_4^-} \right) = -10 \log_{10} \left( |S_{41}|^2 \right)$$

Note that isolation, directivity, and coupling are **not** independent values! You should be able to quickly show that:

$$I (dB) = C (dB) + D (dB)$$

**Mainline Loss $ML$**

The mainline loss is the ratio of the input power to the power out of the through port, in dB. It indicates how much power the signal loses as it travels from the input to the through port.
\[ ML(dB) = 10 \log_{10} \left( \frac{P_1}{P_2} \right) = -10 \log_{10} |S_{21}|^2 \]

**Coupling Loss ML**

The coupling loss indicates the portion of the mainline loss that is due to coupling some of the input power into the coupling port. Conservation of energy makes this loss is **unavoidable**.

\[ CL(dB) = -10 \log_{10} \left( 1 - |jc|^2 \right) \]

**Insertion Loss IL**

The coupling loss indicates the portion of the mainline loss that is **not** due to coupling some of the input power into the coupling port. This loss is **avoidable**, and thus the **smaller** the insertion loss, the better.

\[ IL(dB) = ML(dB) - CL(dB) \]

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