# 7.5 – The Quadrature Hybrid

**Reading Assignment:** *pp. 333-336*

There are **two** different types of ideal **4-port 3dB** couplers: the **symmetric** solution and the **anti-symmetric** solution. The symmetric solution is called the **Quadrature Hybrid.**

HO: THE QUADRATURE HYBRID

The Quadrature Hybrid possesses **$D_4$ symmetry**—it has **two** planes of bilateral reflection symmetry.

**Q:** *So?*

**A:** This fact leads to circuit analysis procedure that is an extension of odd-even mode analysis. **Instead** of 2 modes (odd-even), the circuit can be expressed as a superposition of **4 modes**!

**Q:** *Four modes?! That's **twice** as many as 2 modes; that sounds like twice as much **work!***

**A:** Nope! It turns out that analyzing each of the four modes is simple and direct—much easier than analyzing the odd and/or even mode. As a result, this 4-mode analysis is much easier than the odd-even mode analysis.

HO: A QUAD-MODE ANALYSIS OF THE QUADRATURE HYBRID

# The 90° Hybrid Coupler

The 90° Hybrid Coupler is a 4-port device, otherwise known as the **quadrature** coupler or **branch-line** coupler. Its scattering matrix (ideally) has the **symmetric** solution for a matched, lossless, reciprocal 4-port device:

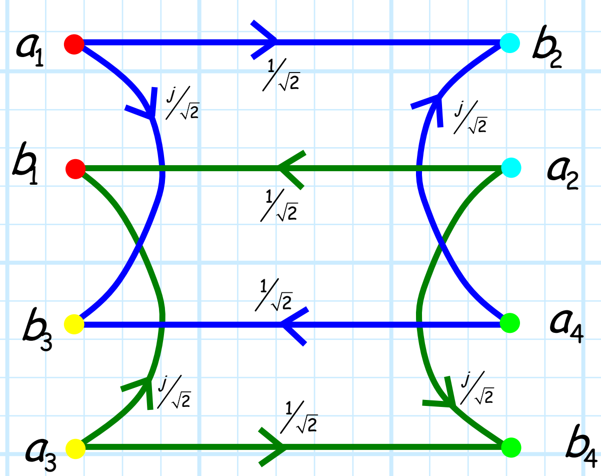
$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

However, for **this** coupler we find that

$$\alpha = \frac{-j}{\sqrt{2}} \qquad\qquad j\beta = \frac{-1}{\sqrt{2}}$$

Therefore, the **scattering matrix** of a quadrature coupler is:

$$\mathcal{S} = \begin{bmatrix} 0 & -j/\sqrt{2} & -1/\sqrt{2} & 0 \\ -j/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & -j/\sqrt{2} \\ 0 & -1/\sqrt{2} & -j/\sqrt{2} & 0 \end{bmatrix}$$



It is evident that, just as with the directional coupler, the ports are **matched** and the device is **lossless**. Note also, that if a signal is incident on one port only, then there will be a port from which **no** power will exit (i.e., an **isolation** port).

**Unlike** the directional coupler, the power that is flows into the input port will be **evenly** divided between the two non-isolated ports.

For example, if 10 mW is incident on port 3 (and all other ports are matched), then 5 mW will flow out of **both** port 1 and port 4, while no power will exit port 2 (the isolated port).

Note however, that the although the **magnitudes** of the signals leaving ports 1 and 4 are **equal**, the relative **phase** of the two signals are separated by **90 degrees** ($e^{j\pi/2} = j$).

We find, therefore, that if in **real** terms the voltage out of port 1 is:

$$v_1(z,t) = \frac{|V_{03}^-|}{\sqrt{2}} \cos(\omega_0 t + \beta z)$$

then the signal form port 4 will be:

$$v_4(z,t) = \frac{|V_{03}^-|}{\sqrt{2}} \sin(\omega_0 t + \beta z)$$

There are **many** useful applications where we require both the **sine** and **cosine** of a signal!

**Q:** *But how do we **construct** this device?*

**A:** Similar to the Wilkinson power divider, we construct a quadrature hybrid with **quarter-wavelength** sections of transmission lines.

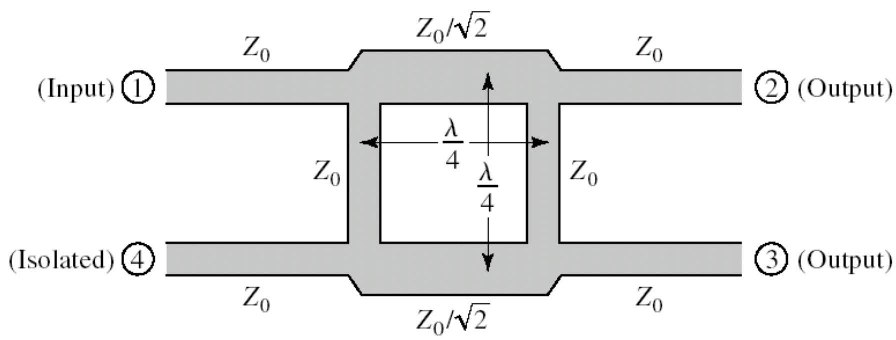



**Figure 7.21 (p. 333)** Geometry of a branch-line coupler.

**Q:** *Wow! How can we **analyze** such a complex circuit?*

**A:** Note that this circuit is **symmetric**—we can use **odd/even mode** analysis!

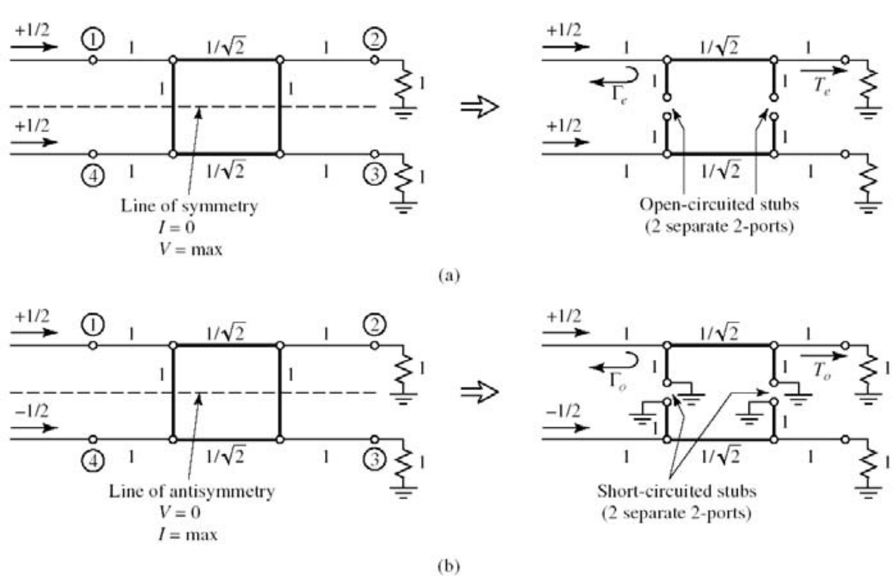



**Figure 7.23 (p. 334)** Decomposition of the branch-line coupler into even- and odd-mode excitations. (*a*) Even mode (*e*). (*b*) Odd mode (*o*).

The **details** of this odd/even mode analysis are provide on pages 333-335 of **your** textbook.

Note that the $\lambda/4$ structures make the quadrature hybrid an inherently **narrow-band** device.

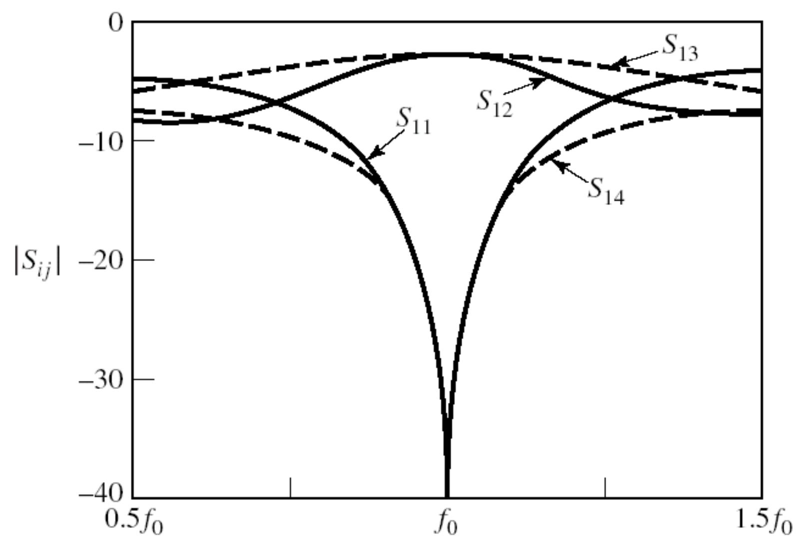
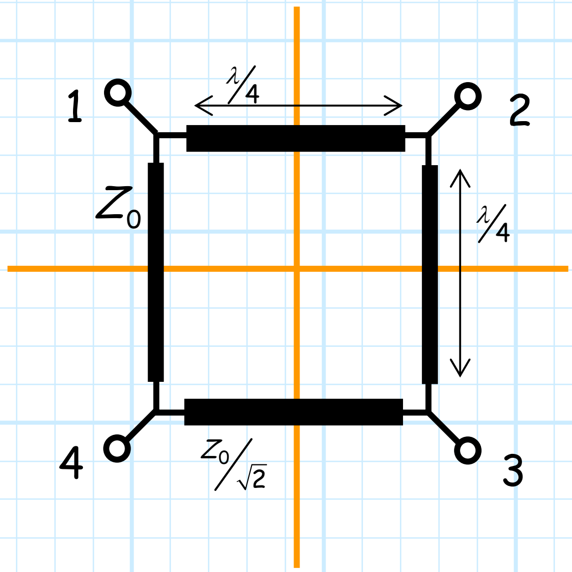


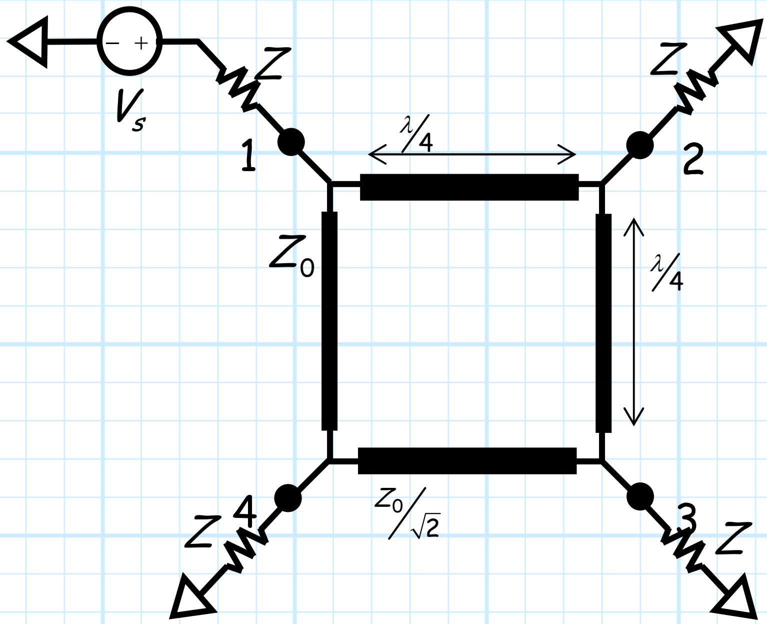

**Figure 7.25 (p. 337)** *S* parameter magnitudes versus frequency for the branch-line coupler of Example 7.5

# A Quad-Mode Analysis of the Quadrature Hybrid

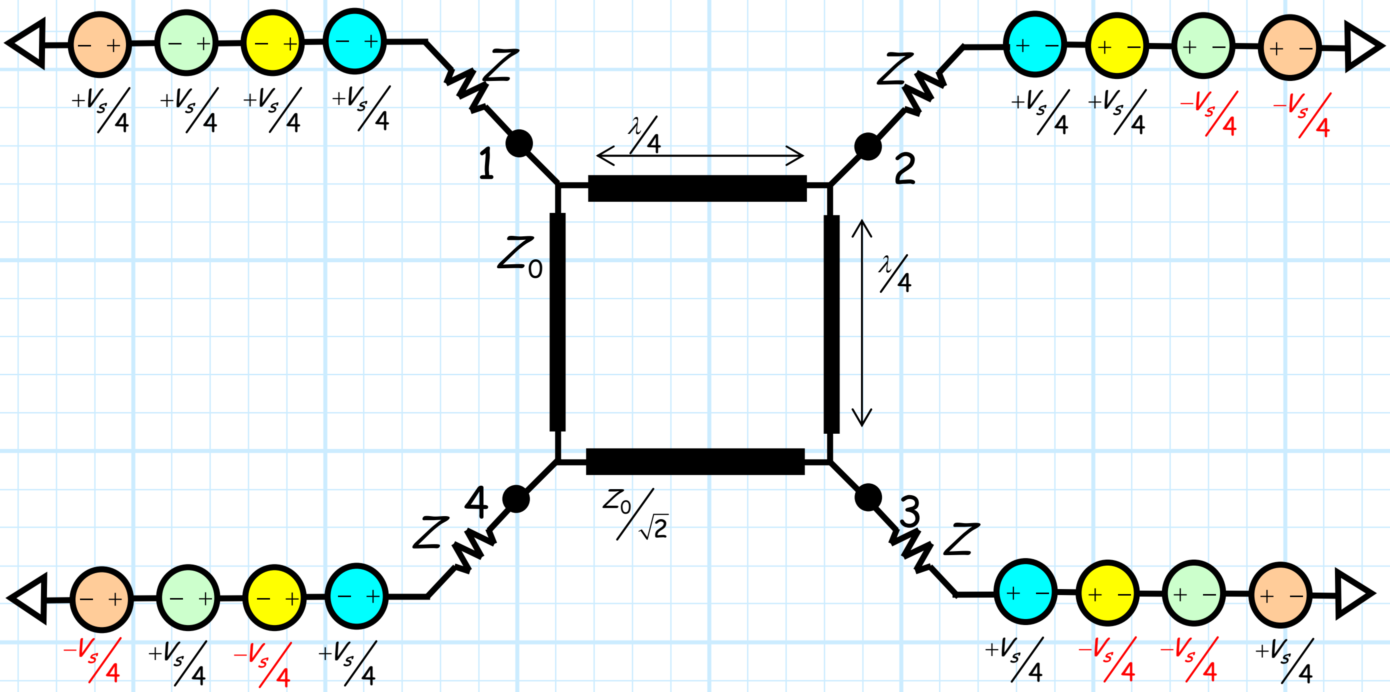
The quadrature hybrid is a matched, lossless, reciprocal four-port network that possesses two planes of bilateral symmetry (i.e. $D_4$ symmetry):




To determine the scattering parameters $S_{11}, S_{21}, S_{31}, S_{41}$, of this network, a matched source is placed on port 1, while matched loads terminate the other 3 ports.



This source destroys both planes of bilateral symmetry in the circuit. We can however recast the circuit above with a precisely equivalent circuit:



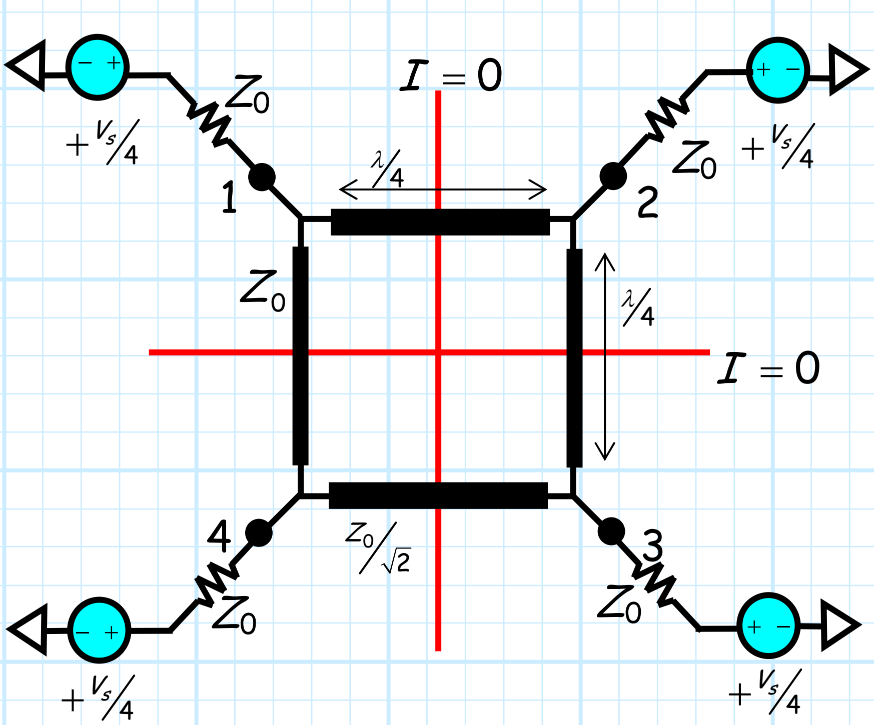

Note that the four series voltage sources on port 1 add to the original value of $V_s$, while the series source at the other four ports add to a value of **zero**—thus providing short circuit from the passive load $Z_0$ to ground.

To analyze this circuit, we can apply **superposition**.

Sequentially turning off all but one source at each of the 4 ports provides us with **four "modes"**. Each of these four modes can be analyzed, and the resulting circuit response is simply a coherent summation of the results of each of the four modes!

The benefit of this procedure is that each of the four modes preserve circuit symmetry. As a result, the planes of bilateral symmetry become virtual shorts and/or virtual opens.

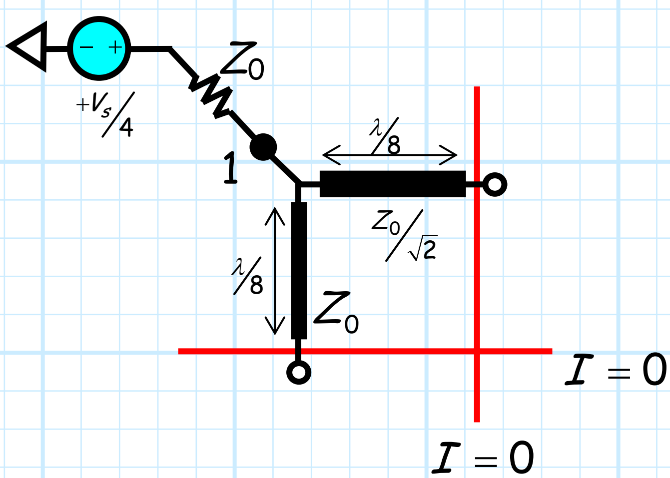
## Mode A




The even symmetry of this circuit is now restored, and so the voltages at each port are identical:

$$V_{1a}^+ = V_{2a}^+ = V_{3a}^+ = V_{4a}^+ = \frac{V_s}{8} \qquad \text{and} \qquad V_{1a}^- = V_{2a}^- = V_{3a}^- = V_{4a}^-$$

The two virtual opens segment this circuit into 4 identical sections. To determine the amplitude $V_{1a}^-$, we need only analyze **one** of these sections:



The circuit has simplified to a 1-port device consisting of the parallel combination of two $\lambda/8$ open-circuited stubs. The admittance of a $\lambda/8$ open-circuit stub is:

$$\begin{aligned} Y_{stub}^{oc} &= jY_0 \cot \beta\ell \\ &= jY_0 \cot \lambda/8 \\ &= jY_0 \end{aligned}$$

As a result, the input admittance is:

$$\begin{aligned} Y_{in}^{d} &= j\sqrt{2}\, Y_0 + j\, Y_0 \\ &= jY_0\left(\sqrt{2}+1\right) \end{aligned}$$

The corresponding reflection coefficient is:

$$\begin{aligned} \Gamma_a &= \frac{Y_0 - Y_{in}^a}{Y_0 + Y_{in}^a} \\ &= \frac{Y_0 - jY_0\left(\sqrt{2}+1\right)}{Y_0 + jY_0\left(\sqrt{2}+1\right)} \\ &= \frac{1 - j\left(\sqrt{2}+1\right)}{1 + j\left(\sqrt{2}+1\right)} \end{aligned}$$

Since the input admittance is purely reactive, the magnitude of this reflection coefficient is $\left|\Gamma_a\right| = 1.0$. The phase of this complex value can be determined from its real and imaginary part:

$$\Gamma_a = \frac{1-j\left(\sqrt{2}+1\right)}{1+j\left(\sqrt{2}+1\right)}\left(\frac{1-j\left(\sqrt{2}+1\right)}{1-j\left(\sqrt{2}+1\right)}\right)$$

$$= \frac{1-j2\left(\sqrt{2}+1\right)-\left(\sqrt{2}+1\right)^2}{1+\left(\sqrt{2}+1\right)^2}$$

$$= \frac{-2\left(\sqrt{2}+1\right)-j2\left(\sqrt{2}+1\right)}{4+2\sqrt{2}}$$

$$= \frac{-\left(\sqrt{2}+1\right)-j\left(\sqrt{2}+1\right)}{\sqrt{2}\left(\sqrt{2}+1\right)}$$

$$= \frac{-1-j}{\sqrt{2}}$$

So that:

$$Re\{\Gamma_a\} = \frac{-1}{\sqrt{2}} \qquad \text{and} \qquad Im\{\Gamma_a\} = \frac{-1}{\sqrt{2}}$$

Thus the reflection coefficient for mode A is:

$$\Gamma_a = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = e^{-j3\pi/4}$$

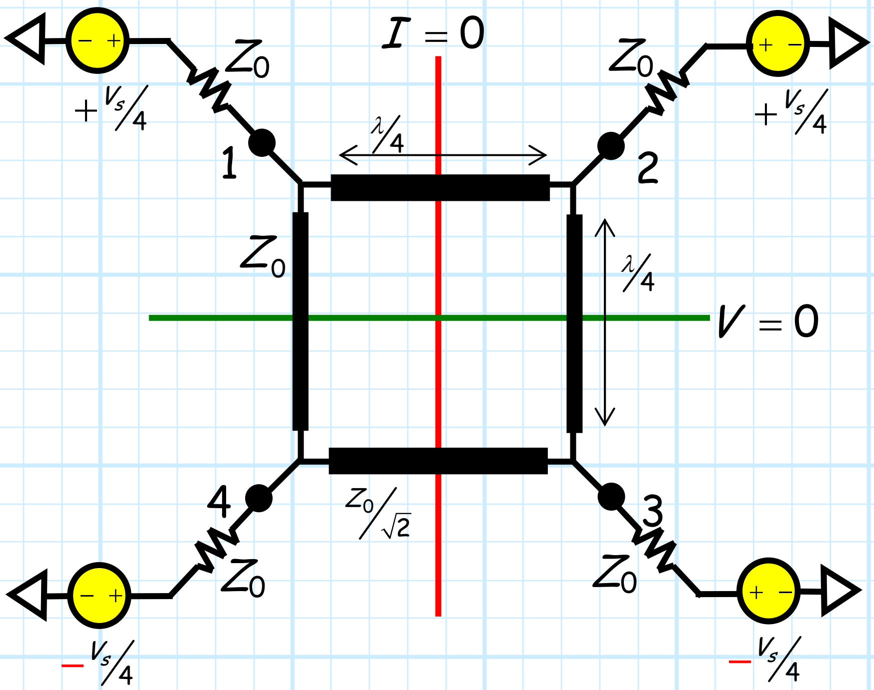
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1a}^- = \Gamma_a V_{1a}^+ = V_s \frac{e^{-j3\pi/4}}{8}$$

And so from the even symmetry of mode A we conclude:

$$V_{1a}^- = V_{2a}^- = V_{3a}^- = V_{4a}^- = V_s \frac{e^{-j3\pi/4}}{8}$$

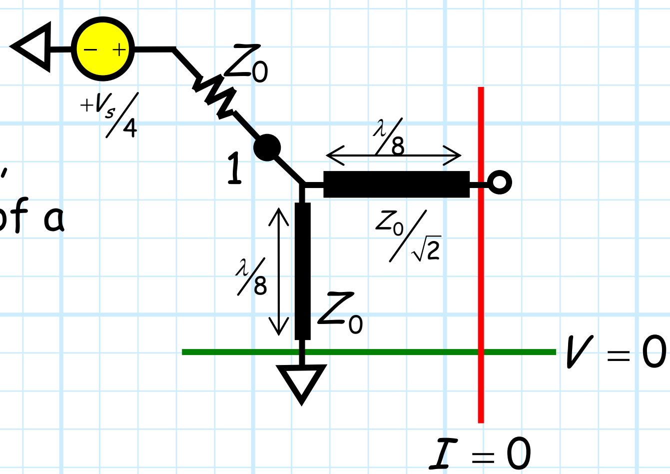
## Mode B




For mode B, the even symmetry exists about the vertical circuit plane, while odd symmetry occurs across the horizontal plane.

$$V_{1b}^+ = V_{2b}^+ = -V_{3b}^+ = -V_{4b}^+ = \frac{V_s}{8} \qquad \text{and} \qquad V_{1b}^- = V_{2b}^- = -V_{3b}^- = -V_{4b}^-$$

The circuit can again be segmented into four sections, with each section consisting of a shorted $\lambda/8$ stub and an open-circuited $\lambda/8$ stub in parallel.



The admittance of a $\lambda/8$ short-circuit stub is:

$$\begin{aligned} Y_{stub}^{sc} &= -jY_0 \tan \beta\ell \\ &= -jY_0 \tan \lambda/8 \\ &= -jY_0 \end{aligned}$$

As a result, the input admittance is:

$$Y_{in}^{b} = j\sqrt{2}\, Y_0 - j\, Y_0 = jY_0\left(\sqrt{2}-1\right)$$

The corresponding reflection coefficient is:

$$\begin{aligned} \Gamma_b &= \frac{1-j\left(\sqrt{2}-1\right)}{1+j\left(\sqrt{2}-1\right)} \\ &= \frac{1-j2\left(\sqrt{2}-1\right)-\left(\sqrt{2}-1\right)^2}{1+\left(\sqrt{2}-1\right)^2} \\ &= \frac{2\left(\sqrt{2}-1\right)-j2\left(\sqrt{2}-1\right)}{2\sqrt{2}\left(\sqrt{2}-1\right)} \\ &= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ &= e^{-j\pi/4} \end{aligned}$$

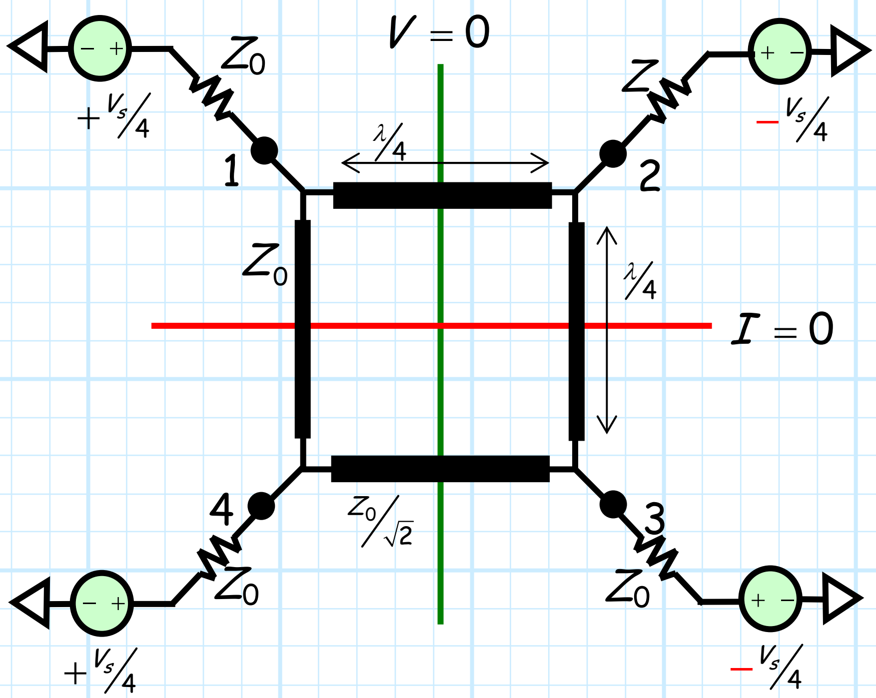
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1b}^{-} = \Gamma_b\, V_{1b}^{+} = V_s\, \frac{e^{-j\pi/4}}{8}$$

And so from the symmetry of mode B we conclude:

$$V_{1b}^- = V_{2b}^- = -V_{3b}^- = -V_{4b}^- = V_s \frac{e^{-j\pi/4}}{8}$$

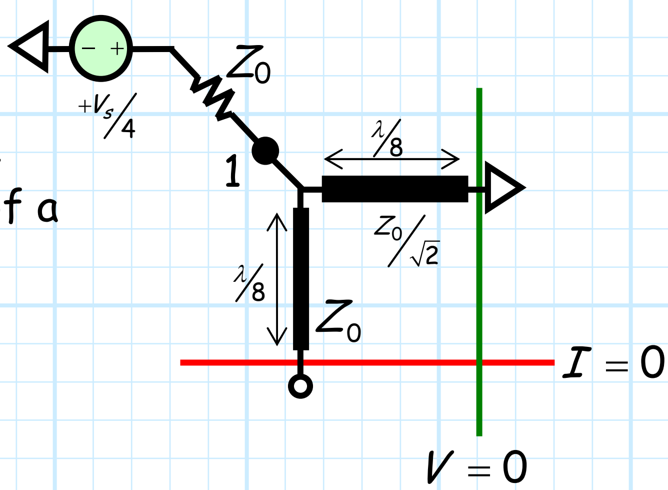
## Mode C




For mode C, odd symmetry exists about the vertical circuit plane, while even symmetry occurs across the horizontal plane.

$$V_{1c}^+ = -V_{2c}^+ = -V_{3c}^+ = V_{4c}^+ = \frac{V_s}{8} \qquad \text{and} \qquad V_{1c}^- = -V_{2c}^- = -V_{3c}^- = V_{4c}^-$$

The circuit can again be segmented into four sections, with each section consisting of a shorted $\lambda/8$ stub and an open-circuited $\lambda/8$ stub in parallel.



As a result, the input admittance is:

$$Y_{in}^{c} = j\sqrt{2}\,Y_0 + j\,Y_0 = jY_0\left(\sqrt{2}-1\right)$$

Note that this result is simply the complex conjugate of $Y_{in}^{b}$, and so we can immediately conclude:

$$\Gamma_c = \Gamma_b^* = e^{+j\pi/4}$$

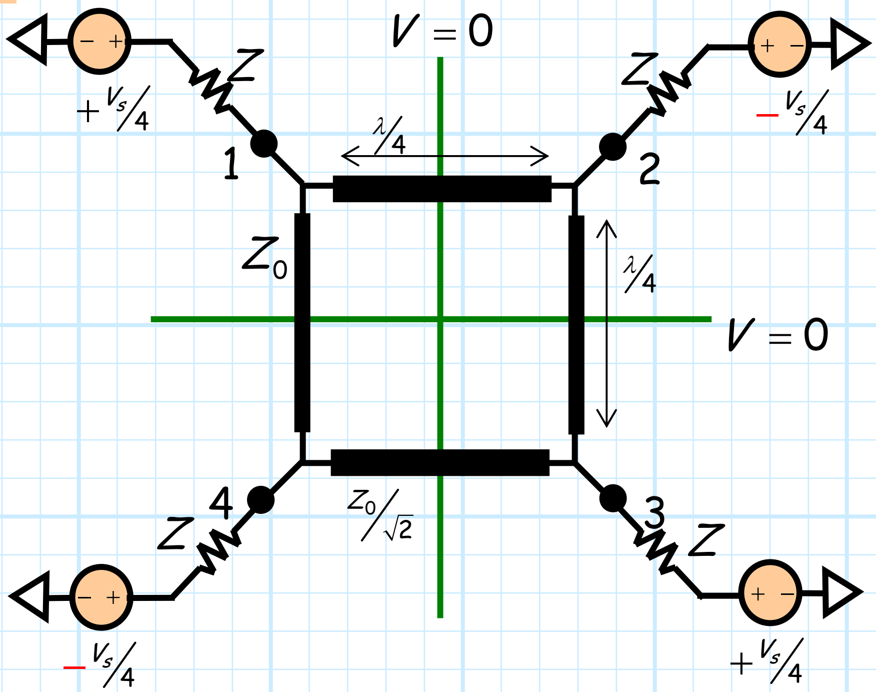
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1c}^{-} = \Gamma_c\,V_{1c}^{+} = V_s\,\frac{e^{+j\pi/4}}{8}$$

And so from the symmetry of mode C we conclude:

$$V_{1c}^{-} = -V_{2c}^{-} = -V_{3c}^{-} = V_{4c}^{-} = V_s\,\frac{e^{+j\pi/4}}{8}$$

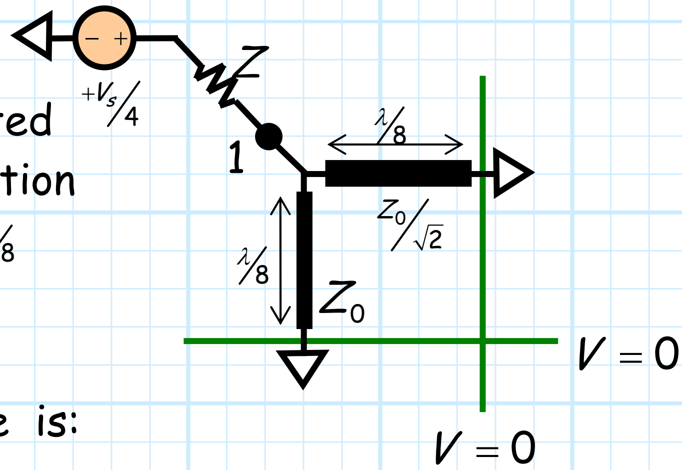
## Mode D



For mode D, odd symmetry exists about both planes of circuit symmetry.

$$+V_{1c}^{+} = -V_{2c}^{+} = +V_{3c}^{+} = -V_{4c}^{+} = \frac{V_s}{8} \quad \text{and} \quad +V_{1c}^{-} = -V_{2c}^{-} = +V_{3c}^{-} = -V_{4c}^{-}$$

The circuit can again be segmented into four sections, with each section consisting two short-circuited $\lambda/8$ stubs in parallel.




As a result, the input admittance is:

$$Y_{in}^{d} = -j\sqrt{2}\, Y_0 - j\, Y_0 = -jY_0\left(\sqrt{2}+1\right)$$

Note that this result is simply the complex conjugate of $Y_{in}^{a}$, and so we can immediately conclude:

$$\Gamma_d = \Gamma_a^{*} = e^{+j3\pi/4}$$

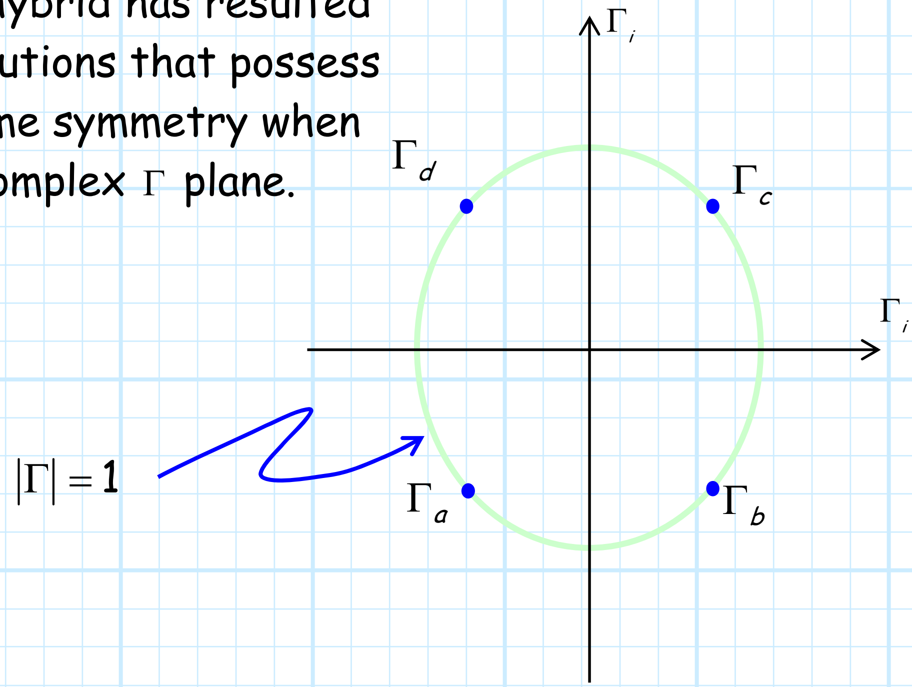
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1d}^{-} = \Gamma_d\, V_{1d}^{+} = V_s\, \frac{e^{+j3\pi/4}}{8}$$

And so from the symmetry of mode D we find:

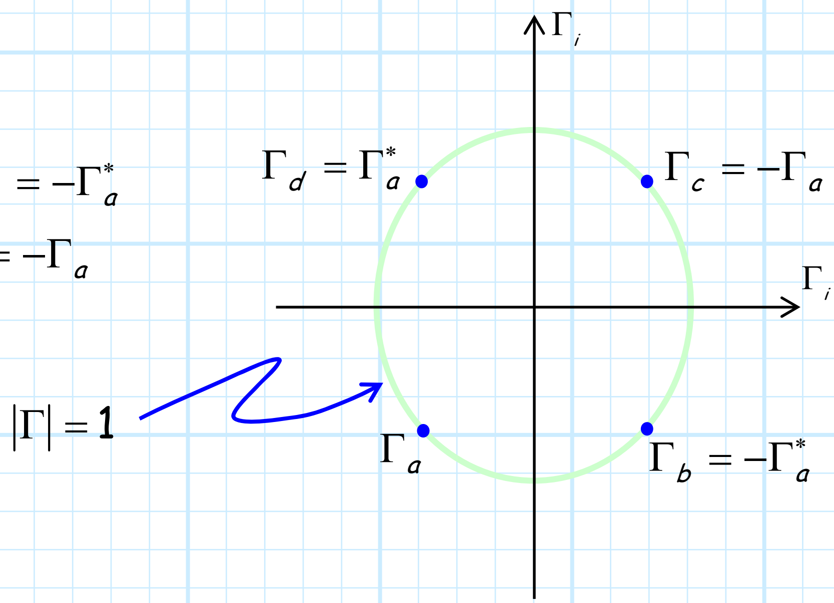
$$+V_{1c}^{-} = -V_{2c}^{-} = +V_{3c}^{-} = -V_{4c}^{-} = V_s\, \frac{e^{+j3\pi/4}}{8}$$

Not surprisingly, the symmetry of the quadrature hybrid has resulted in four modal solutions that possess precisely the same symmetry when plotted on the complex $\Gamma$ plane.




The modal solutions associated with the other three ports are simply symmetric permutations of the port 1 solutions:

$$\Gamma_d = \Gamma_a^*$$
$$\Gamma_b = -\Gamma_d = -\Gamma_a^*$$
$$\Gamma_c = \Gamma_b^* = -\Gamma_a$$



Since our circuit is linear, we can determine the solution to our original circuit as a superposition of our four modal solutions:

$$
\begin{aligned}
V_{01}^{-} &= V_{1a}^{-} + V_{1b}^{-} + V_{1c}^{-} + V_{1d}^{-} \\
&= \frac{V_s}{8} e^{-j3\pi/4} + \frac{V_s}{8} e^{-j\pi/4} + \frac{V_s}{8} e^{+j\pi/4} + \frac{V_s}{8} e^{+j3\pi/4} \\
&= \left( e^{-j3\pi/4} + e^{-j\pi/4} + e^{+j\pi/4} + e^{+j3\pi/4} \right) \frac{V_s}{8} \\
&= \left( 2\cos 3\pi/4 + 2\cos \pi/4 \right) \frac{V_s}{8} \\
&= \left( -2\cos \pi/4 + 2\cos \pi/4 \right) \frac{V_s}{8} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
V_{02}^{-} &= V_{2a}^{-} + V_{2b}^{-} + V_{2c}^{-} + V_{2d}^{-} \\
&= \frac{V_s}{8} e^{-j3\pi/4} + \frac{V_s}{8} e^{-j\pi/4} - \frac{V_s}{8} e^{+j\pi/4} - \frac{V_s}{8} e^{+j3\pi/4} \\
&= \left( e^{-j3\pi/4} + e^{-j\pi/4} - e^{+j\pi/4} - e^{+j3\pi/4} \right) \frac{V_s}{8} \\
&= \left( -j2\sin 3\pi/4 - j2\sin \pi/4 \right) \frac{V_s}{8} \\
&= \left( -j2\cos \pi/4 - j2\cos \pi/4 \right) \frac{V_s}{8} \\
&= V_s \frac{-j}{2\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
V_{03}^{-} &= V_{3a}^{-} + V_{3b}^{-} + V_{3c}^{-} + V_{3d}^{-} \\
&= \frac{V_s}{8} e^{-j 3\pi/4} - \frac{V_s}{8} e^{j\pi/4} - \frac{V_s}{8} e^{-j\pi/4} + \frac{V_s}{8} e^{-j 3\pi/4} \\
&= \left( e^{-j 3\pi/4} - e^{-j\pi/4} - e^{+j\pi/4} + e^{+j 3\pi/4} \right) \frac{V_s}{8} \\
&= \left( 2\cos 3\pi/4 - 2\cos \pi/4 \right) \frac{V_s}{8} \\
&= \left( -2\cos \pi/4 - 2\cos \pi/4 \right) \frac{V_s}{8} \\
&= V_s \frac{-1}{2\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
V_{04}^{-} &= V_{4a}^{-} + V_{4b}^{-} + V_{4c}^{-} + V_{4d}^{-} \\
&= \frac{V_s}{8} e^{-j 3\pi/4} - \frac{V_s}{8} e^{j\pi/4} + \frac{V_s}{8} e^{-j\pi/4} - \frac{V_s}{8} e^{-j 3\pi/4} \\
&= \left( e^{-j 3\pi/4} - e^{j\pi/4} + e^{-j\pi/4} - e^{-j 3\pi/4} \right) \frac{V_s}{8} \\
&= \left( j2\sin 3\pi/4 - j2\sin \pi/4 \right) \frac{V_s}{8} \\
&= \left( j2\sin \pi/4 - j2\sin \pi/4 \right) \frac{V_s}{8} \\
&= 0
\end{aligned}
$$

From these results we can determine the scattering parameters $S_{11}, S_{21}, S_{31}, S_{41}$:

$$
S_{11} = \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{2}{V_s} 0 = 0
$$

$$S_{21} = \frac{V_{02}^-}{V_{01}^+} = \frac{2}{V_s}\frac{-jV_s}{2\sqrt{2}} = \frac{-j}{\sqrt{2}}$$

$$S_{31} = \frac{V_{03}^-}{V_{01}^+} = \frac{2}{V_s}\frac{-V_s}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$S_{41} = \frac{V_{04}^-}{V_{01}^+} = \frac{2}{V_s}0 = 0$$

Given the symmetry of the device, we can extend these four results to determine the entire scattering matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & {}^{-j}\!/\!\sqrt{2} & {}^{-1}\!/\!\sqrt{2} & 0 \\ {}^{-j}\!/\!\sqrt{2} & 0 & 0 & {}^{-1}\!/\!\sqrt{2} \\ {}^{-1}\!/\!\sqrt{2} & 0 & 0 & {}^{-j}\!/\!\sqrt{2} \\ 0 & {}^{-1}\!/\!\sqrt{2} & {}^{-j}\!/\!\sqrt{2} & 0 \end{bmatrix}$$

Precisely the **correct result**!